Infrared Divergence of the Angular Momentum of Bremsstrahlung and the Physical Structure of the Electron*

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In a private communication, Biedenharn pointed out that the statistical independence of photon emission in the infrared limit in a bremsstrahlung scattering process may lead to a logarithmically divergent angular momentum for the radiated photons and the scattering system. It is shown that the expectation values of these angular momentum vectors are free of infrared divergence, but the expectation values of their squares are infinite due to an infrared divergence. The physical significance of this result is that an incident electron of arbitrarily large impact parameter can scatter from a finite range potential. This may be ascribed to the physical structure of the electron, namely its photon cloud.

I. INTRODUCTION

IT is known that for a bremsstrahlung scattering process, the probability of emitting n photons ($n=0, 1, 2, \cdots$) of any mode obeys a Poisson distribution function in the limit of soft photon emission. This is usually interpreted to mean that soft photon emission occurs in a statistically independent way.[1] Biedenharn[2] gave a simple intuitive argument that this statistical independence interpretation may lead to a logarithmically divergent angular momentum transfer by the radiated photons. His argument may be summarized as follows: Consider for definiteness electron scattering. For sufficiently low-energy electron scattering, the dipole mode of radiation is dominant. This means that each emitted photon transfers with it one unit of angular momentum independently of its energy. In addition, the number of unobserved soft photons is infinite (being proportional to $\int dk/k$). If it is assumed that the angular momenta add randomly, the average angular momentum transferred by the photons is also infinite (logarithmically divergent). This appears contradictory because an infinite angular momentum reaction on the scattered electron is not observed. Thus he concludes that the statistical independence interpretation of the Poisson distribution may be an oversimplification.

As may be seen, Biedenharn's argument is statistical. It does not take into account any virtual photon effects. Of course, virtual photons cannot contribute directly to the angular momentum of the radiated photons. It is conceivable, however, that the damping effect of virtual photons to the scattering amplitude, which cancels out the infrared divergence of the scattering cross section coming from real photons,[3] may also damp down the average value of the angular momentum transfer by the emitted photons. It is thus of interest to calculate the expectation value of angular momentum in the scattering state. In Sec. II, the expectation values of the electromagnetic field angular momentum and the total angular momentum vectors in the scattering state together with their squares are calculated. The physical meaning of the results of Sec. II is discussed qualitatively in Sec. III, and a summary and some concluding remarks are given in Sec. IV.

II. CALCULATION OF THE ANGULAR MOMENTUM

For definiteness, we confine our attention to the process of electron scattering from a potential. So that the potential itself cannot be responsible for the occurrence of high angular momenta, it will be chosen to have a definite, small spatial extension.

The starting point of our discussion will be a recapitulation of the main results of an analysis of the infrared divergence.[3] For those readers not familiar with a general treatment of the infrared divergence, the most important fact to keep in mind is that infrared effects depend only on the large scale features of the charge-current distributions. This has the consequence that infrared contributions can *always* be factored out and depend only on incident and final momenta of charged particles.

It will be convenient to describe the process in the momentum representation for the electron. The representative of the initial state, $\phi=|\mathbf{p},0\rangle$, is then

$$\langle\mathbf{p}'|\phi\rangle=\delta(\mathbf{p}'-\mathbf{p})|0\rangle, \tag{2.1}$$

corresponding to an electron of momentum $\mathbf{p}$ and no photons; $|0\rangle$ is the photon vacuum. The final state is then $S\phi$, and the representative of the scattered part of the state is

$$\langle\mathbf{p}'|S-1|\phi\rangle=\Omega(\mathbf{p}',\mathbf{p})|0\rangle, \tag{2.2}$$

where Ω is an operator which produces the superposition of photons in the final state. It may be expressed in the form

$$\Omega=e^{\alpha B(\mathbf{p}',\mathbf{p})}\sum_{n=0}^{\infty}\frac{1}{n!}\times\int\cdots\int\prod_{i=1}^{n}\frac{d^3k_i}{(2\omega_i)^{1/2}}(-2\pi i)\delta(\epsilon-\sum_{i=1}^{n}\omega_i)\times\mu_n(\mathbf{k}_1,\cdots\mathbf{k}_n)a^\dagger(\mathbf{k}_1)\cdots a^\dagger(\mathbf{k}_n), \tag{2.3}$$

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[1] R. J. Glauber, Phys. Rev. **84**, 395 (1951).
[2] Professor L. C. Biedenharn (private communications).
[3] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961); further references may be found in that paper.

where

$$B=\frac{i}{(2\pi)^3}\int\frac{d^4k}{k^2-\lambda^2}\left(\frac{2p_\mu'-k_\mu}{2p'\cdot k-k^2}-\frac{2p_\mu-k_\mu}{2p\cdot k-k^2}\right)^2.$$

The meaning of the various factors in (2.3) will now be described. The matrix element for scattering with production of n real photons of specified momenta is $\mu_n e^{\alpha B}$. The factor $\exp(\alpha B)$ is independent of n and contains all the infrared-divergent contributions from virtual photons,[3,4] while virtual photon contributions which are not infrared divergent are contained in μ_n. To keep separate contributions finite we use a photon mass λ; we may then study the infrared divergence by seeing how various quantities behave as $\lambda\to 0$. The $(n=0)$ term in the sum is by definition $(-2\pi i)\delta(\epsilon)\mu_0$. The $a_\mu{}^\dagger(\mathbf{k})$ are creation operators for photons; for simplicity the polarization indices (μ_i) have been omitted from (2.3). These operators have commutation relations

$$[a_\mu(\mathbf{k}),a_\nu{}^\dagger(\mathbf{k}')]=\delta_{\mu\nu}\delta(\mathbf{k}-\mathbf{k}'),\qquad(2.4)$$

where μ, $\nu=1$, 2, the two transverse polarization directions. The $(1/n!)$ occurs in (2.3) because each contribution to the state vector occurs $n!$ times (μ_n is symmetric). Conservation of energy is contained in the factor $(-2\pi i)\delta(\epsilon-\Sigma\omega_i)$, where $\epsilon=E-E'$ is the electron's energy loss and ω_i is the energy of the ith photon. The dependence of μ_n on $\mathbf{p}$, $\mathbf{p}'$, electron spin state, etc., has been suppressed.

The infrared structure of μ_n must now be discussed. As the infrared contributions depend only on the large scale distributions of current, they are readily identified and may be factored out. Carrying out an analysis similar to that given in Sec. 2 of Ref. 3, applied there to $\rho_n(\mathbf{k}_1,\cdots,\mathbf{k}_n)$ and $\tilde{\rho}_n(\mathbf{k}_1,\cdots,\mathbf{k}_n)$, we find for $\mu_n(\mathbf{k}_1,\cdots,\mathbf{k}_n)$ the expansion[5]:

$$\mu_n(\mathbf{k}_1\cdots\mathbf{k}_n)=\sum_{\text{perm}}\sum_{r=0}^{n}\frac{1}{r!(n-r)!}\times\Big[\prod_{i=1}^{r}T(\mathbf{k}_i)\Big]\gamma_{n-r}(\mathbf{k}_{r+1}\cdots\mathbf{k}_n).\qquad(2.5)$$

$\sum_{\text{perm}}$ is the sum over all permutations of the momentum vectors $\mathbf{k}_i$, $i=1,\cdots,n$. Upon integration, the square of $T(\mathbf{k}_i)$ leads to an infrared divergence; for photon polarization $\mathbf{e}$, T is given by

$$T(\mathbf{k})=\frac{e}{(2\pi)^{3/2}}\left(\frac{\mathbf{e}\cdot\mathbf{p}'}{k\cdot p'}-\frac{\mathbf{e}\cdot\mathbf{p}}{k\cdot p}\right),\qquad(2.6)$$

where $k\cdot p=\omega E-\mathbf{k}\cdot\mathbf{p}$. Integrals containing factors of γ_j are not infrared divergent. We may regard (2.5) as a decomposition of μ_n into contributions where r photons are infrared and $n-r$ are not.

To avoid the complication of the energy-conserving δ function inside the sum of (2.3), we make the formal substitution

$$\delta(\epsilon-\sum\omega_i)\to\delta(\epsilon-H_{\text{em}}),\qquad(2.7)$$

where H_{em} is the Hamiltonian of the radiation field; it is understood that Ω acts on the vacuum. This substitution is to be understood as a convenient shorthand; all the manipulations that will be made could be done by alternative methods without using operators as arguments of δ functions. With this substitution, the sum over infrared contributions in (2.3) may be carried out explicitly, with the result

$$\Omega|0\rangle=e^{\alpha B}(-2\pi i)\delta(\epsilon-H_{\text{em}})e^{C\dagger}\times\left[\gamma_0+\sum_{m=1}^{\infty}\frac{1}{m!}\int\cdots\int\gamma_{\text{m}}(\mathbf{k}_1\cdots\mathbf{k}_m)\times\prod_{i=1}^{m}a^\dagger(\mathbf{k}_i)\frac{d^3k_i}{(2\omega_i)^{1/2}}\right],\qquad(2.8)$$

where

$$C^\dagger=\int\sum_{\mu=1}^{2}T_\mu(\mathbf{k})a_\mu{}^\dagger(\mathbf{k})\frac{d^3k}{(2\omega)^{1/2}}.$$

The operator $C^\dagger$ creates a photon in the "bremsstrahlung mode" appropriate to the scattering of the electron. If the consideration is restricted to very small energy loss by the electron, which is the situation of primary concern here, then only the γ_0 term in (2.8) is of importance.[6] In this limit, the scattering state may be written as

$$\Omega|0\rangle=e^{\alpha B}(-2\pi i)\delta(\epsilon-H_{\text{em}})e^{C\dagger}\gamma_0(\mathbf{p}',\mathbf{p})|0\rangle.\qquad(2.9)$$

The factor $\gamma_0(\mathbf{p}',\mathbf{p})$ contains the contribution to the scattering amplitude due the basic potential scattering process as well as the noninfrared virtual photon contributions. Sometimes it is convenient to express energy conservation in a different form. One uses

$$\delta(\lambda)=(1/2\pi)\int_{-\infty}^{\infty}e^{iy\lambda}dy$$

and

$$H_{\text{em}}a^\dagger(\mathbf{k})=a^\dagger(\mathbf{k})(H_{\text{em}}+\omega)$$

to write

$$\Omega|0\rangle=-ie^{\alpha B}\int_{-\infty}^{\infty}dye^{iy\epsilon}\exp(\hat{C}^\dagger)\gamma_0|0\rangle,\qquad(2.10)$$

where

$$\hat{C}^\dagger=\int_0^{\epsilon}\sum_{\mu=1}^{2}T_\mu(\mathbf{k})a_\mu{}^\dagger(\mathbf{k})e^{-i\omega y}\frac{d^3k}{(2\omega)^{1/2}}.$$

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[4] The present discussion is similar to that of Sec. 2 of Ref. 3 above. The same notation is also employed. Familiarity with that section of Ref. 3 is assumed.

[5] It is more convenient for our purposes to treat the operator Ω than to treat the electron scattering cross section as is the case in Sec. 2 of Ref. 3.

[6] The discussion to follow may be still carried unchanged in essentials even if we do not leave out these terms. If the higher corrections $(m\geq 1)$ are retained, they lead to contributions of order ϵ^m since each photon's energy is bounded by ϵ.

Equation (2.10) may also be derived directly from (2.3) without the intermediate step (2.7).

The scattering probability may be written from (2.9):

$$\langle 1\rangle = \langle 0|\Omega^\dagger(\mathbf{p}',\mathbf{p})\Omega(\mathbf{p}',\mathbf{p})|0\rangle$$
$$= (2\pi)^2\langle 0|e^{\alpha B}e^{C}\gamma_0{}^*\delta(\epsilon-H_{\rm em})\delta(\epsilon-H_{\rm em})e^{\alpha B}e^{C\dagger}\gamma_0|0\rangle. \quad (2.11)$$

The product $\delta(\epsilon-H_{\rm em})\times\delta(\epsilon-H_{\rm em})$ may be replaced by the product $\delta(0)\times\delta(\epsilon-H_{\rm em})$. The factor $\delta(0)$ is the usual one appearing in transition probability calculations. This factor is disposed of by the usual methods in the calculation of the scattering cross section and of angular momentum expectation values. The factor $\delta(\epsilon-H_{\rm em})$ is then re-expressed as in (2.10) in terms of a y integration. Using the identity

$$\exp(C)\exp(\hat{C}^\dagger) = \exp(\hat{C}^\dagger)\exp(C)\exp\{[C,\hat{C}^\dagger]\}, \quad (2.12)$$

(2.11) may be reduced to the form given in Ref. 3

$$\langle 1\rangle = 2\pi\delta(0)\exp[2\alpha(B+\tilde{B})]|\gamma_0|^2\int_{-\infty}^{\infty} dy e^{iy\epsilon+D} \quad (2.13)$$

$\tilde{B}(\mathbf{p}',\mathbf{p})$ and $D(\mathbf{p}',\mathbf{p},y)$ are defined as follows:

$$\tilde{B}(\mathbf{p}',\mathbf{p}) = \frac{-1}{8\pi^2}\int_0^\epsilon\left(\frac{p_\mu'}{k\cdot p'}-\frac{p_\mu}{k\cdot p}\right)^2\frac{d^3k}{\omega}, \quad (2.14)$$

$$D(\mathbf{p}',\mathbf{p},y) = \frac{-\alpha}{4\pi^2}\int_0^\epsilon\left(\frac{p_\mu'}{k\cdot p'}-\frac{p_\mu}{k\cdot p}\right)^2(e^{-i\omega y}-1)\frac{d^3k}{\omega}. \quad (2.15)$$

The real photon infrared divergence is isolated in $\tilde{B}$, while D is finite. It is seen from (2.13) that the real photon infrared divergent contribution to the electron scattering cross section is also factorable as a multiplicative exponential factor $\{\exp[2\alpha\tilde{B}(\mathbf{p}',\mathbf{p})]\}$. Further, the infrared divergences of real and virtual photons have cancelled in the observable cross section. The functions B and $\tilde{B}$ are separately divergent, but their sum is finite as $\lambda\to 0$.

The angular momentum operator for the electromagnetic field is

$$\mathbf{M}_{\rm em} = \int \mathbf{x}\times\mathbf{S}(\mathbf{x},t)d^3x = \int \mathbf{x}\times(\mathbf{E}\times\mathbf{H})d^3x, \quad (2.16)$$

where $\mathbf{x}$ is a radius vector, $\mathbf{S}(\mathbf{x},t)$ is Poynting's vector, and $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors, respectively. The expression (2.16) is expressible in terms of the creation and annihilation operators

$$\mathbf{M}_{\rm em} = -i\int[\sum_{\mu=1}^{2} a_\mu{}^\dagger(\mathbf{k})(\mathbf{k}\times\nabla_{\mathbf{k}})a_\mu(\mathbf{k})+\mathbf{a}^\dagger(\mathbf{k})\times\mathbf{a}(\mathbf{k})]d^3k, \quad (2.17)$$

$$\mathbf{a}(\mathbf{k}) = \sum_{\mu=1}^{2} a_\mu(\mathbf{k})\mathbf{e}_\mu(\mathbf{k}). \quad (2.18)$$

The first term of (2.17) may be spoken of as the orbital angular momentum and the second as the spin term.

The expectation value of $\mathbf{M}_{\rm em}$ in the scattering state may be calculated by using (2.9), (2.10), and (2.17). We may write

$$\langle\mathbf{M}_{\rm em}\rangle = \langle 0|\Omega^\dagger(\mathbf{p}',\mathbf{p})\mathbf{M}_{\rm em}\Omega(\mathbf{p}',\mathbf{p})|0\rangle$$
$$= (2\pi)^2 e^{2\alpha B}|\gamma_0|^2\langle 0|e^{C}\delta(\epsilon-H_{\rm em}) \times\mathbf{M}_{\rm em}\delta(\epsilon-H_{\rm em})\exp(C^\dagger)|0\rangle. \quad (2.19)$$

The operator $\mathbf{M}_{\rm em}$ commutes with $H_{\rm em}$ and can be moved to the right of $\delta(\epsilon-H_{\rm em})$ in (2.19). Then the product $\delta(\epsilon-H_{\rm em})\times\delta(\epsilon-H_{\rm em})$ is replaced by $\delta(0)\times\delta(\epsilon-H_{\rm em})$ as before. The state vector to the right is then written in the form (2.10). Thus, (2.19) is reduced to

$$\langle\mathbf{M}_{\rm em}\rangle = 2\pi\delta(0)e^{2\alpha B}|\gamma_0|^2\int_{-\infty}^{\infty}\langle 0|\exp(C)\mathbf{M}_{\rm em}\exp(\hat{C}^\dagger)|0\rangle e^{iy\epsilon}dy$$
$$= 2\pi\delta(0)e^{2\alpha B}|\gamma_0|^2\int_{-\infty}^{\infty} e^{iy\epsilon}\exp\{[C,\hat{C}^\dagger]\} \times[C,[\mathbf{M}_{\rm em},\hat{C}^\dagger]]dy. \quad (2.20)$$

The reduction of (2.19) is accomplished with the help of (2.12) and the following relations:

$$\mathbf{M}_{\rm em}\exp(\hat{C}^\dagger)|0\rangle = [\mathbf{M}_{\rm em},\exp(\hat{C}^\dagger)]|0\rangle = \exp(\hat{C}^\dagger)[\mathbf{M}_{\rm em},\hat{C}^\dagger]|0\rangle$$

and

$$\langle 0|e^{C}[\mathbf{M}_{\rm em},\hat{C}^\dagger]|0\rangle = \langle 0|[C,[\mathbf{M}_{\rm em},\hat{C}^\dagger]]|0\rangle.$$

The factor $[C,[\mathbf{M}_{\rm em},C^\dagger]]$ is a C number. Writing out (2.20) in detail, we find

$$\langle\mathbf{M}_{\rm em}\rangle = -i\int\frac{d^3k}{\omega}[\sum_{\mu=1}^{2} T_\mu(\mathbf{k})(\mathbf{k}\times\nabla_{\mathbf{k}})T_\mu(\mathbf{k}) \quad (2.21)$$
$$+\mathbf{T}(\mathbf{k})\times\mathbf{T}(\mathbf{k})]\langle 1\rangle|_{\epsilon\to\epsilon-\omega},$$

where

$$\mathbf{T}(\mathbf{k}) = \sum_{\mu=1}^{2} T_\mu(\mathbf{k})\mathbf{e}_\mu(\mathbf{k}). \quad (2.22)$$

It is immediate from (2.21) that the spin contribution is identically zero. Integrating the orbital angular momentum term by parts while recognizing that $(\mathbf{k}\times\nabla_{\mathbf{k}})\omega = 0$ and that the surface term is zero, one shows that the integral is equal to its negative; hence, it must be zero. Therefore, it follows that the expectation value of the angular momentum vector of the radiated photons in the scattering state is zero.

The expectation value of $\mathbf{M}_{\rm em}{}^2$ in the scattering state can be now calculated using the same techniques.

$$\langle\mathbf{M}_{\rm em}{}^2\rangle = \langle 0|\Omega^+(\mathbf{p}',\mathbf{p})\mathbf{M}_{\rm em}{}^2\Omega(\mathbf{p}',\mathbf{p})|0\rangle$$
$$= e^{2\alpha B}|\gamma_0|^2 2\pi\delta(0)\int_{-\infty}^{\infty} dy e^{iy\epsilon}\langle 0|e^{C}\mathbf{M}_{\rm em}{}^2\exp(\hat{C}^\dagger)|0\rangle. \quad (2.23)$$

The last factor of (2.23) may be reduced as follows:

$$\begin{aligned}\langle 0|e^C\mathbf{M}_{\text{em}}\cdot\mathbf{M}_{\text{em}}\exp(\hat{C}^\dagger)|0\rangle &=\langle 0|[C,\mathbf{M}_{\text{em}}]\cdot\exp\{[C,\hat{C}^\dagger]\}\\ &\qquad\times\exp(\hat{C}^\dagger)\exp(C)[\mathbf{M}_{\text{em}},\hat{C}^\dagger]|0\rangle\\ &=\langle 0|\{[[C,\mathbf{M}_{\text{em}}],\hat{C}^\dagger]+[C,\mathbf{M}_{\text{em}}]\}\\ &\quad\cdot\{[C,[\mathbf{M}_{\text{em}},\hat{C}^\dagger]]+[\mathbf{M}_{\text{em}},\hat{C}^\dagger]\}|0\rangle\exp\{[C,\hat{C}^\dagger]\}.\end{aligned}$$

However, $[C,[\mathbf{M}_{\text{em}},C^\dagger]]$ is zero, as was shown in calculating the expectation value of $\mathbf{M}_{\text{em}}$. Hence, this reduces to

$$\begin{aligned}\langle 0|[C,\mathbf{M}_{\text{em}}]\cdot[\mathbf{M}_{\text{em}},\hat{C}^\dagger]|0\rangle\exp\{[C,\hat{C}^\dagger]\}\\ =\exp\{[C,\hat{C}^\dagger]\}[[C,\mathbf{M}_{\text{em}}],\cdot[\mathbf{M}_{\text{em}},\hat{C}^\dagger]].\end{aligned}$$

In this reduction note that $[\mathbf{M}_{\text{em}},C^\dagger]$ is a creation operator while $[C,\mathbf{M}_{\text{em}}]$ is an annihilation operator. The expectation value of $\mathbf{M}_{\text{em}}{}^2$ then becomes

$$\langle\mathbf{M}_{\text{em}}{}^2\rangle=\int\frac{d^3k}{\omega}\sum_{\mu=1}^{2}\{[(\mathbf{k}\times\nabla_{\mathbf{k}}T_\mu(\mathbf{k})]^2+T_\mu{}^2(\mathbf{k})\}\langle 1\rangle_{\epsilon\to\epsilon-\omega}. \quad (2.24)$$

Equation (2.24) may be viewed as the sum of the expectation values of the square of the orbital and spin terms. The interference term between the spin and orbital angular momentum parts can be seen to be zero. The spin is, for a given $\mathbf{k}$, in the direction of $\mathbf{k}$ while the orbital angular momentum is perpendicular to $\mathbf{k}$. Consequently, the dot product is zero. The contribution of both the orbital and the spin angular momentum parts to (2.24) is infrared divergent as is easily seen by examining the expression of $T_\mu(\mathbf{k})$ for small k (or ω). Analogous analysis can be carried out for the case where scalar mesons of vanishing mass, instead of photons, are being emitted. The spin term is zero for this case; however, the expectation value of the square of the angular momentum for the emitted mesons still contains an infrared divergence; a fact which emphasizes that the divergence is not caused by the photon's spin.

It is worth noting, at this point, that the operator for emitting a photon $C^\dagger$ depends on $\mathbf{p}'$, the momentum of the scattered electron. Consequently it contributes to the scattered electron's angular momentum. Further, $\mathbf{M}_{\text{em}}$ is not a constant of the motion. Thus instead of calculating the expectation value $\langle\mathbf{M}_{\text{em}}\rangle$, let us calculate the expectation value of the total angular momentum and its square. In the representation being used, the total angular momentum operator is

$$\mathbf{M}_{\text{tot}}=\mathbf{M}_{\text{em}}+(-i\mathbf{p}'\times\nabla_{p'}). \quad (2.25)$$

The second term on the right-hand side of (2.25) is the orbital angular momentum operator for the scattered electron. The electron spin does not lead to infrared divergences and will not concern us here. The second term on the right-hand side of (2.25) will be denoted for brevity by $\mathbf{L}_{p'}$. The corresponding operator for the incident electron is denoted by $\mathbf{L}_p(=-i\mathbf{p}\times\nabla_{\mathbf{p}})$.

It will be helpful in performing the calculation of $\mathbf{M}_{\text{tot}}$ and $\mathbf{M}_{\text{tot}}{}^2$ to note, for a spherically symmetric potential, the identity

$$(\mathbf{M}_{\text{em}}+\mathbf{L}_{p'}+\mathbf{L}_p)\Omega|0\rangle=0. \quad (2.26)$$

We shall assume from here on that the scattering potential is spherically symmetric; this assumption is a simplifying rather than a crucial one. The identity (2.26) is a consequence of the rotational invariance of the S matrix and can be verified directly for our state (2.8). It may be viewed in terms of infinitesimal rotations to mean that an infinitesimal rotation of the incident particle, scattered particle, and the emitted radiation leaves the scattering amplitude unchanged.

We are now able to write

$$\langle 0|\Omega^\dagger\mathbf{M}_{\text{tot}}\Omega|0\rangle=-\langle 0|\Omega^\dagger\mathbf{L}_p\Omega|0\rangle, \quad (2.27)$$

which is in turn equal to

$$\begin{aligned}\langle\mathbf{M}_{\text{tot}}\rangle&=-\mathbf{L}_p\langle 0|\Omega^\dagger(\mathbf{p}',\mathbf{p}_1)\Omega(\mathbf{p}',\mathbf{p})|0\rangle|_{\mathbf{p}_1=\mathbf{p}}\\ &=-2\pi\delta(0)\mathbf{L}_p e^{\alpha[B(\mathbf{p}',\mathbf{p}_1)+B(\mathbf{p}',\mathbf{p})]}\gamma_0{}^*(\mathbf{p}',\mathbf{p}_1)\gamma_0(\mathbf{p}',\mathbf{p})\\ &\quad\times\int_{-\infty}^{\infty}dy\,e^{iy\epsilon}\exp\Big[\int\sum_{\mu=1}^{2}T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p}_1)\\ &\qquad\times T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p})e^{-i\omega y}\frac{d^3k}{2\omega}\Big]. \quad (2.28)\end{aligned}$$

The introduction of $\mathbf{p}_1$ is a trick so that the differentiation may be carried out after the expectation value is worked out. We now observe in (2.28) that

$$\begin{aligned}T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p}_1)T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p})&=\tfrac{1}{2}[T_\mu{}^2(\mathbf{k},\mathbf{p}',\mathbf{p}_1)+T_\mu{}^2(\mathbf{k},\mathbf{p}',\mathbf{p})]\\ &\quad-\tfrac{1}{2}[T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p}_1)-T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p})]^2, \quad (2.29\text{a})\end{aligned}$$

while

$$[T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p}_1)-T_\mu(\mathbf{k},\mathbf{p}',\mathbf{p})]^2=T_\mu{}^2(\mathbf{k},\mathbf{p}_1,\mathbf{p}). \quad (2.29\text{b})$$

Substituting these into (2.28) and recalling the definitions of $\tilde{B}(\mathbf{p}',\mathbf{p})$ and $D(\mathbf{p}',\mathbf{p},y)$, we find for (2.28)

$$\begin{aligned}\langle M_{\text{tot}}\rangle&=-2\pi\delta(0)\mathbf{L}_p\exp\{\alpha[B(\mathbf{p}',\mathbf{p}_1)+\tilde{B}(\mathbf{p}',\mathbf{p}_1)+B(\mathbf{p}',\mathbf{p})+\tilde{B}(\mathbf{p}',\mathbf{p})]\}\\ &\quad\times\gamma_0{}^*(\mathbf{p}',\mathbf{p}_1)\gamma_0(\mathbf{p}',\mathbf{p})\int_{-\infty}^{\infty}dy\,e^{iy\epsilon}e^{1/2D(\mathbf{p}',\mathbf{p}_1,y)+1/2D(\mathbf{p}',\mathbf{p},y)}e^{-1/2D(\mathbf{p},\mathbf{p}_1,y)}\exp[-\alpha\tilde{B}(\mathbf{p}_1,\mathbf{p})]|_{\mathbf{p}_1=\mathbf{p}}.\end{aligned}$$

The quantities $(B+\tilde{B})$ and D are finite so only the last exponential factor contains an infrared divergence. The contribution to angular momentum from this term is proportional to

$$\mathbf{L}_p \exp[-\alpha\tilde{B}(\mathbf{p},\mathbf{p}_1)]_{\mathbf{p}_1=\mathbf{p}} = -\alpha(\mathbf{L}_p\tilde{B}(\mathbf{p},\mathbf{p}_1)) \exp[-\alpha\tilde{B}(\mathbf{p},\mathbf{p}_1)]|_{\mathbf{p}_1=\mathbf{p}}.$$

Since $\tilde{B}(\mathbf{p}_1,\mathbf{p})$ is some function of $\mathbf{p}_1\cdot\mathbf{p}$, $\mathbf{L}_p\tilde{B}(\mathbf{p}_1,\mathbf{p})$ is proportional to $(\mathbf{p}_1\times\mathbf{p})$. Thus when $\mathbf{p}_1$ is set equal to $\mathbf{p}$, the result is zero. This means that (2.28) is free of infrared divergence, which in turn means that the expectation value of the total angular momentum in the scattering state is free of infrared divergence. Since $\mathbf{M}_{\text{em}}$ is zero, it follows that $\mathbf{M}_{\text{tot}}=\mathbf{L}_{p'}$ which means that the expectation value of the scattered electron's angular momentum in the scattering state is free of infrared divergence also.

The calculation of the expectation value of $\mathbf{M}_{\text{tot}}^2$ is performed in exactly the same way as that of $\mathbf{M}_{\text{tot}}$. The net result for $\mathbf{M}_{\text{tot}}^2$, which corresponds to (2.28) of $\mathbf{M}_{\text{tot}}$, differs from (2.28) by having $\mathbf{L}_p^2$ in front of the square brackets instead of $-\mathbf{L}_p$. That is,

$$\langle\mathbf{M}_{\text{tot}}^2\rangle = 2\pi\delta(0)\mathbf{L}_p^2 \exp\{\alpha[B+B+\tilde{B}+\tilde{B}]\}\gamma_0^*\gamma_0 \times\int_{-\infty}^{\infty} dy\, e^{iy\epsilon+1/2D+1/2D} \exp[-\alpha\tilde{B}(\mathbf{p}_1,\mathbf{p})]_{\mathbf{p}_1=\mathbf{p}}. \quad (2.30)$$

As in (2.28), any possible infrared divergence in (2.30) can only come from $\mathbf{L}_p^2$ acting on the last exponential factor; the noninfrared terms are indicated somewhat symbolically. Consider

$$\mathbf{L}_p^2 \exp[-\alpha\tilde{B}(\mathbf{p}_1,\mathbf{p})]|_{\mathbf{p}_1=\mathbf{p}}.$$

Since $\mathbf{p}_1$ is to be set equal to $\mathbf{p}$ after $\mathbf{L}_p^2$ acts on the exponential, it is sufficiently accurate to consider $\tilde{B}(\mathbf{p}_1,\mathbf{p})$ for small $(\mathbf{p}-\mathbf{p}_1)$. Let $\mathbf{p}-\mathbf{p}_1=\mathbf{q}$; then

$$\alpha\tilde{B}(\mathbf{p}_1,\mathbf{p})=(\alpha\mathbf{q}^2/3\pi m^2)\ln(\epsilon/\lambda)\equiv a^2\mathbf{q}^2, \quad (2.31)$$

where ϵ is the electron energy loss in the scattering process and λ is the photon mass. Inserting this into (2.30), we find

$$\mathbf{L}_p^2 \exp[-\alpha\tilde{B}(\mathbf{p}_1,\mathbf{p})]|_{\mathbf{p}_1=\mathbf{p}}=4a^2\mathbf{p}^2,$$

which is logarithmically divergent as $\lambda\to 0$. Thus, $\mathbf{M}_{\text{tot}}^2$ is infrared divergent. Similar calculations may be undertaken to show that $\mathbf{L}_{p'}^2$, the expectation value of the square of the angular momentum of the scattered electron, is also infrared divergent.

III. PHYSICAL BASIS OF THE DIVERGENCE

We saw in the preceding section, among other things, that the expectation value of the square of the total angular momentum in the scattering state is infrared divergent. Since that calculation involved a somewhat detailed knowledge of the infrared divergence problem, it is valuable to give a more physical discussion indicating the presence of the large angular momentum. This discussion will also indicate why a large angular momentum reaction on the electron is not observed experimentally.

It will be necessary to review some of the results of the analysis of the infrared problem.[3] The functions B and $\tilde{B}$ are both infrared divergent, and we will concentrate our attention here on the infrared dependence. Let the angular integral occurring in (2.14) be called $\frac{1}{2}A$, so that

$$\partial\tilde{B}(\mathbf{p}',\mathbf{p},\epsilon)/\partial\epsilon=A/2\epsilon. \quad (3.1)$$

This same factor A is associated with all the infrared divergences. For example,

$$\tilde{B}(\mathbf{p}',\mathbf{p},\epsilon)=\tfrac{1}{2}A\ln(\epsilon/\lambda)+\cdots, \quad (3.2a)$$

$$B(\mathbf{p}',\mathbf{p})=\tfrac{1}{2}A\ln\lambda+\cdots, \quad (3.2b)$$

where the dots indicate terms which remain finite as λ or ϵ tend toward zero. The function D may also be expressed in terms of A:

$$D=\alpha A\int_0^{\epsilon}\frac{dk}{k}(e^{-iyk}-1). \quad (3.3)$$

The y integration in (2.12) yields[3]

$$(2\pi\alpha A/\epsilon)F(\alpha A), \quad (3.4)$$

where F is very close to unity except for ultrahigh energies. The $(\alpha A/\epsilon)$ dependence of (3.4) is characteristic of single photon emission. The effect of multiple emission is contained in

$$\exp(2\alpha\tilde{B})=(\epsilon/\lambda)^{\alpha A}\times(\cdots).$$

It is seen that this factor makes the differential cross section for energy loss slightly less singular than ϵ^{-1} as $\epsilon\to 0$. In fact, one may now define a cross section with energy resolution

$$\sigma(\Delta E)=\int_0^{\Delta E}\frac{d\sigma}{d\epsilon}d\epsilon =\sigma_0\exp\{2\alpha[B(\mathbf{p}',\mathbf{p})+\tilde{B}(\mathbf{p}',\mathbf{p},\Delta E)]\}F(\alpha A), \quad (3.5)$$

where σ_0 is the elastic scattering cross section, including the effects of virtual noninfrared photons. Now let us split this cross section up into various parts according to the number of photons which have been emitted. This is not an observable decomposition since the photons are not actually detected. Since $\tilde{B}$ refers to real photons, the desired cross section is

$$\sigma_n(\Delta E)=(1/n!)(2\alpha\tilde{B})^n e^{2\alpha B}F(\alpha A)\sigma_0 =(1/n!)(2\alpha\tilde{B})^n\exp(-2\alpha\tilde{B})\sigma(\Delta E). \quad (3.6)$$

Thus the real photons are emitted according to a Poisson distribution; notice that this is true for detection with energy resolution rather than for detection with definite energy loss. In the case of nonrelativistic incident particles, this becomes

$$\sigma_n(\Delta E)=(1/n!)(2a^2\mathbf{q}^2)^n e^{-2a^2\mathbf{q}^2}\sigma(\Delta E)\,, \qquad (3.7)$$

where a^2 is defined in (2.31) and $\mathbf{q}^2$ is now given by $2\mathbf{p}^2(1-\cos\theta)$, where θ is the scattering angle. Equation (3.7) is plotted as a function of θ in solid lines (Fig. 1) for $n=0$, 5, 10, and 20. A convenient photon mass (as a lower cutoff) has been used. The dashed line shows the cross section for electron scattering when the contributions from all photon channels ($n=0, 1, 2, \cdots$) are added. This cross section is independent of the photon mass, due to the cancellation of infrared divergent contributions from virtual and real photons, as was pointed out earlier. However, the solid lines of various photon channels become infinitely narrow as the photon mass is made to go to zero. Simultaneously, the height of the curves is suppressed down to zero (except $n=0$). Thus the infinite narrowing of these curves suggests the presence of large angular momentum for each "number of photons" scattering channel. When the contributions to the square of angular momentum from all photon channels, properly weighted, are added, the result is infrared divergent.

Since the incident plane-wave state used in the calculations contains arbitrarily large angular momentum components, there is no violation of angular momentum conservation. In fact, one can show explicitly with the help of the optical theorem that the elastically scattered wave ($n=0$ in Fig. 1) interferes with the incident wave in such a way as to account for over-all angular momentum conservation. On the other hand, it may be argued that since the calculations of Sec. II are carried out for a fairly general potential, then for a potential of finite range a finite section of the plane wave passes through the scattering region. This finite section contains only finite angular momentum. The question thus arises: Why is there infinite angular momentum, independent of the potential, in the scattering state? The point to observe is that the wave function of the incident electron represents its center-of-mass motion. The physical electron, with a core (bare electron) and a photon cloud, has internal structure of infinite extent (due to the zero rest mass of the photons). The fluctuations in the photon cloud cause fluctuations in the core due to recoil. Hence an incident electron whose center of mass does not pass through the scattering potential can still interact through its extended core. This can produce scattering at arbitrarily large impact parameter of the center of mass, which leads to a large angular momentum in the scattering state.

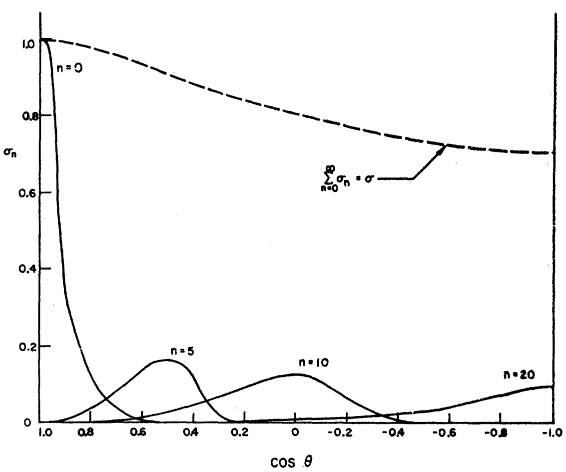

FIG. 1. The partial cross sections for electron scattering at angle θ with emission of exactly n photons ($n=0, 1, \cdots$) of any energy and momentum (subject to total energy conservation) are plotted as the solid curves for representative values of n. A convenient photon mass has been used. The cross section for scattering with emission of an arbitrary number of photons ($\Sigma_n\sigma_n$) is shown as a dashed line. For reasonable energy resolution, it is very close to the cross section in which all radiative corrections are ignored.

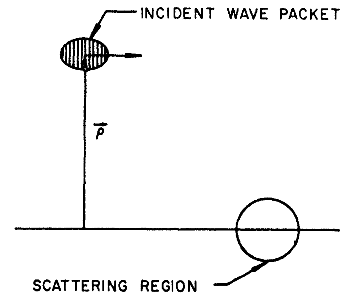

FIG. 2. A schematic diagram showing an incident wave packet on a scattering region. The impact parameter is chosen so that the wave-packet configuration does not overlap the scattering region.

To illustrate the above remarks further, let us consider the scattering state with a wave-packet amplitude used to describe the incident electron. This state now takes the form

$$-i\int_{-\infty}^{\infty}dy\int d^3p_1\phi(\mathbf{p}_1)e^{\alpha B(\mathbf{p}',\mathbf{p}_1)}e^{iy\epsilon_1}\times\exp[\hat{C}^\dagger(\mathbf{p}',\mathbf{p}_1)]\gamma_0(\mathbf{p}',\mathbf{p}_1)|0\rangle\,. \qquad (3.8)$$

$\phi(\mathbf{p}_1)$ is the incident wave packet amplitude in the momentum representation. The scattering probability may be obtained by taking the inner product of (3.8) with itself. Using the same techniques employed in Sec. II, one finds

$$2\pi\int_{-\infty}^{\infty}e^{iy\epsilon}dy\int\cdots\int\delta(E_1-E_2)\times\gamma_0^*(\mathbf{p}',\mathbf{p}_2)\phi^*(\mathbf{p}_2)\gamma_0(\mathbf{p}',\mathbf{p}_1)\phi(\mathbf{p}_1)e^{Q(\mathbf{p}',\mathbf{p}_1)+Q(\mathbf{p}',\mathbf{p}_2)}\times\exp[-\tfrac{1}{2}\tilde{B}(\mathbf{p}_1,\mathbf{p}_2)]e^{-1/2D(\mathbf{p}_1,\mathbf{p}_2)}d^3p_1d^3p_2\,. \qquad (3.9)$$

The function $Q(\mathbf{p}',\mathbf{p})$ is defined by

$$Q(\mathbf{p}',\mathbf{p})=\alpha[B(\mathbf{p}',\mathbf{p})+\tilde{B}(\mathbf{p}',\mathbf{p})]+\tfrac{1}{2}D(\mathbf{p}',\mathbf{p},y)\,. \qquad (3.10)$$

$Q(\mathbf{p}',\mathbf{p})$ is free of infrared divergence. Let us choose the incident wave packet to have an impact parameter ρ relative to the center of the scattering region (cf. Fig. 2). This wave function ϕ may be written in terms of a wave-packet amplitude ϕ_0 whose impact parameter is zero. Thus,

$$\phi(\mathbf{p})=e^{i\mathbf{p}\cdot\rho}\phi_0(\mathbf{p}). \tag{3.11}$$

Let $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2$. Making the approximation of (2.31) for $\tilde{B}(\mathbf{p}_1,\mathbf{p}_2)$, one then writes (3.9)

$$2\pi\int_{-\infty}^{\infty}e^{iy\epsilon}dy\int\cdots\int\delta(E_1-E_2)\gamma_0^*(\mathbf{p}',\mathbf{p}_2)\gamma_0(\mathbf{p}',\mathbf{p}_1)\times\phi_0^*(\mathbf{p}_1+\mathbf{q})\phi_0(\mathbf{p}_1)e^{Q(\mathbf{p}',\mathbf{p}_2)+Q(\mathbf{p}',\mathbf{p}_1)-1/2D(\mathbf{p}_1,\mathbf{p}_2,y)}\times\exp(i\boldsymbol{\rho}\cdot\mathbf{q})e^{-a^2\mathbf{q}^2}d^3p_1d^3q. \tag{3.12}$$

We assume that since a is very large for small λ, one can always choose the wave-packet spread Δx to be small relative to a. Then the Gaussian factor $\exp(-a^2\mathbf{q}^2)$ limits the values of q in such a way that the $\mathbf{q}$ dependence of $\phi_0^*(\mathbf{p}_1+\mathbf{q})$, $\gamma_0^*(\mathbf{p}',\mathbf{p}_2)$ and $Q(\mathbf{p}',\mathbf{p}_2)$ may be neglected. With this approximation, (3.12) becomes

$$2\pi\int_{-\infty}^{\infty}dye^{iy\epsilon}\int\cdots\int|\phi_0(\mathbf{p}_1)|^2e^{2Q(\mathbf{p}',\mathbf{p}_1)}\times\exp[-(\boldsymbol{\rho}^2/4a^2)]|\gamma_0(\mathbf{p}',\mathbf{p}_1)|^2\exp[-(a\mathbf{q}-i\boldsymbol{\rho}/2a)^2]\times2m\delta(2\mathbf{p}_1\cdot\mathbf{q}-\mathbf{q}^2)d^3p_1d^3q, \tag{3.13}$$

where we have used

$$E_1-E_2=(\mathbf{p}_1^2-\mathbf{p}_2^2)/2m=(2\mathbf{p}_1\cdot\mathbf{q}-\mathbf{q}^2)/2m.$$

Because of the δ function the q integration is really two dimensional, this integration is easily carried out and we find

$$\frac{\pi^2}{2a^2}\exp[-(\rho^2/4a^2)]\int_{-\infty}^{\infty}dye^{iy\epsilon}\int\frac{m}{p_1}\exp2Q(\mathbf{p}',\mathbf{p}_1)\times|\gamma_0(\mathbf{p}',\mathbf{p}_1)|^2|\phi_0(\mathbf{p}_1)|^2d^3p_1. \tag{3.14}$$

Now let us take experimental conditions such that the incident beam is reasonably well specified with a mean momentum $\mathbf{p}_1^0$ (compatible with the spatial extension of the packet being small compared with a). Then (3.14) may be approximated by

$$\frac{\pi^2m}{2a^2p_1^0}\exp[-(\boldsymbol{\rho}^2/4a^2)]\int_{-\infty}^{\infty}dye^{iy\epsilon}e^{2Q(\mathbf{p}',\mathbf{p}_1^0)}|\gamma_0(\mathbf{p}',\mathbf{p}_1^0)|^2=\frac{1}{4\pi a^2p_1^0m}\exp[-(\boldsymbol{\rho}^2/4a^2)]\frac{d\sigma}{d\epsilon}, \tag{3.15}$$

with

$$\epsilon=E_1^0-E'.$$

It is seen from (3.15) that the scattering probability of a wave packet of a given parameter decreases like $\exp[-(\rho^2/4a^2)]$ as ρ increases. For calculating the scattering cross section, one has to consider a beam of infinite extent of uniformly distributed wave packets incident on the scattering region. That is, one has to integrate (3.15) over wave packets of all impact parameters; the scattering cross section is then

$$\int_0^{\infty}\frac{1}{4\pi a^2}\exp[-(\rho^2/4a^2)]\frac{d\sigma}{d\epsilon}\cdot2\pi\rho d\rho=\frac{d\sigma}{d\epsilon}. \tag{3.16}$$

$2\pi\rho d\rho$ is the density measure of the incident wave packets in a ring of radius ρ and width $d\rho$, in the plane perpendicular to the direction of the incident beam. The factor p_1^0m introduced in going from (3.15) to (3.16) is essentially a Jacobian; Eq. (3.15) gives the probability of scattering into an element (d^3p') of momentum space and (3.16) gives the probability for energy range $d\epsilon$. We have recovered the same result for the observable cross section which was derived originally by considering an incident plane wave. It is free of infrared divergence even though the scattering probability for given impact parameter does depend on λ.

The expectation value of the square of angular momentum in the scattering state may be obtained, with the help of the above remarks, straightforwardly. The square of the angular momentum of a wave packet $\phi(p_1)$ of impact parameter ρ may be approximated by $\rho^2p_1^{02}$. Thus the expectation value of the square of the angular momentum in the scattering state for given impact parameter may be obtained by multiplying $\rho^2p_1^{02}$ by the scattering probability. Summing this over all impact parameters, we find

$$\int_0^{\infty}\frac{1}{4\pi a^2}\exp[-(\rho^2/4a^2)]\frac{d\sigma}{d\epsilon}(\rho p_1^0)^22\pi\rho d\rho=4(ap_1^0)^2\frac{d\sigma}{d\epsilon}, \tag{3.17}$$

which confirms the result obtained in the previous section.

IV. SUMMARY AND RESULTS

We have investigated the angular momentum associated with bremsstrahlung and found it to be infinite due to an infrared divergence, as had been conjectured by Biedenharn. This result applies either to an incident plane wave or an incident beam of wave packets which are uniformly distributed in the transverse plane. As Biedenharn points out, it seems paradoxical that a scattering electron can transfer an infinite angular momentum to the electromagnetic field and not experience a drastic modification in the angular dependence of its scattering cross section. The paradox is resolved by noting that if one detects only the electron, the cross section is really a sum over an infinite number of incoherent channels corresponding to emission of different numbers of photons. Even though each channel has a rapid angular dependence, as indicated in Fig. 1, the complete cross section has an angular dependence which, for reasonable energy resolution, does not differ qualitatively from the cross section in which all radiative corrections are ignored.

One may raise the question whether it is possible, in principle, to devise an experiment in which this angular momentum would be observable. No method of doing this has occurred to the authors. It would be impossible, for example, to devise an experiment in which one could say with certainty that exactly n photons have been emitted by the scattering electron. The trouble is that any experimental arrangement will always admit the possibility of emission of an arbitrary number of very soft photons which will escape detection. The infrared divergence (logarithmic dependence on the photon mass λ) will then always cancel out leaving a dependence on the resolution of the various detectors.

In spite of these remarks, there is a sense in which the angular momentum manifests itself. With sufficiently good energy resolution (small ΔE), the radiative corrections do modify the angular distribution. If the fine structure constant were of order one or greater, we would be quite conscious of this effect and its connection with angular momentum. However, with reasonable experimental values for ΔE and the actual value of the fine structure constant, these corrections are small and their connection with angular momentum has previously been overlooked.

As we have seen, the infinite angular momentum is really a manifestitation of something more basic, the internal structure of the physical particle. We have seen how it is possible for the particle to scatter even though its wave packet entirely misses the potential. The scattering probability for a given impact parameter ρ is proportional to $a^{-2}\exp(-\rho^2/4a^2)$, where a, given by (2.31), goes to infinity as λ tends to zero. This means that a collimated beam of particles, where the width of the beam is small compared to a, would not be scattered from a potential. However, there is no slit system which could provide such a collimated beam. For, if a wave packet were headed toward the opening of a slit, it would have only a very small chance of getting through because of the long-range interaction. On the other hand, a wave packet directed into the material away from the slit opening would have a small chance of getting through corresponding to the bare particle part of the physical particle being at the slit.

Since the photon mass is actually zero, the remarks of the preceding paragraph seem to introduce hopeless complications into the realistic description of a scattering process, including the questions of beam formation, detection, etc. As we will now try to argue, this is a consequence of an over-idealization of the scattering process. We usually try to conceive of the scattering process as being from a physical one-particle state to a state consisting of a superposition of physical particle states. Apparently this point of view cannot be maintained when one of the interacting fields has zero mass, as in quantum electrodynamics.[7]

Let us turn to the semiclassical description of the bremsstrahlung process. This is usually thought of as a shaking off of the proper field of the incident particle together with a radiation field to compensate for the fact that at the instant after scattering the particle cannot have developed its final proper field. We want to fix our attention on the contribution to the amplitude associated with the final motion of the particle. At the first instant after scattering, the radiation field is the negative of the proper field, canceling it out at large distances. As the electron moves along the radiation field moving with velocity c begins to disengage from the proper field so that after an infinite time the two are completely separate. However, at any finite time, the outer regions of the proper and radiation field must still cancel since the information about the scattering can propagate out only with the speed of light. From the quantum-mechanical point of view, this means that the physical particle states cannot be attained in a finite period of time. The lowest Fourier components of the radiation field are not disengaged. On the other hand, if the photon had a mass, the physical electron could become completely developed within a finite time after scattering, since the information that scattering has occurred would need to propagate only a finite distance. In considering a realistic scattering situation, we have to recognize that the incident electron has scattered at some finite time in the past, even if the time is considered very large from the microscopic point of view. Thus we never deal with completely free-incident particles. We may look on the incident particle as "semidressed," or more precisely as represented by a superposition of states whose spectral decomposition is concentrated very close to, but not quite on, the particle's mass shell.[7]

Fortunately, we need not deal with these details every time we discuss a scattering process. The observable cross sections are not extremely sensitive to the past history of the incident particle. In effect, the previous scatterings and detection provide an infrared cutoff in the scattering of interest. However, it is known that the observable results of any experiment are insensitive to the details of the infrared cutoff provided that cutoff is at a low enough energy. The idea that a realistic description of scattering would include a physical infrared cutoff has also been stressed by Ascoli.[8] Now let us estimate the cutoff as being given by the dimensions of the laboratory, say 10^4 cm. Then with any reasonable energy resolution, the parameter a turns out to be very much smaller than the electron's Compton wavelength, which in turn is smaller than any target dimensions. We believe, then, that the problem

[7] B. Schroer, Fortschr. Physik **11**, 1 (1963).
[8] R. Ascoli, Nuovo Cimento **12**, 192 (1959).

of the infrared divergence is a mathematical, rather than a physical one.

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Asymptotic Conditions and Integral Equations for Nonrelativistic Invariant Functions*

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The asymptotic behavior of the invariant functions introducted previously is investigated. It is shown that there exists a "direct" invariant function and that all scattering amplitudes for single-particle processes can be expressed in terms of this direct function and the asymptotic plane-wave part only of other invariant functions. Moreover, only the short-range part of the potentials enters into the expression for the scattering amplitude.

INTRODUCTION

IN a previous paper[1] it was shown that

$$\begin{aligned}\langle N,\beta,\mathbf{K}'|\psi(\mathbf{x})|N+1,\alpha,\mathbf{K}\rangle &\equiv (N+1)^{1/2}\int \Psi_{\beta,\mathbf{K}'}^{*}(\mathbf{x}_1,\cdots,\mathbf{x}_N)\\ &\qquad\times\Psi_{\alpha,\mathbf{K}}(\mathbf{x},\mathbf{x}_1,\cdots,\mathbf{x}_N)d\mathbf{x}_1\cdots d\mathbf{x}_N\\ &=(2\pi)^{-3/2}e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{x}}\tilde{\psi}_{\beta,\alpha}\left(\frac{N}{N+1}\mathbf{K}-\mathbf{K}'\right),\end{aligned}\qquad(1)$$

where β and α label the internal states of the N and $N+1$ particle systems and $\mathbf{K}'$ and $\mathbf{K}$ are their total momenta. The Fourier transforms of the functions $\tilde{\psi}_{\beta,\alpha}(\mathbf{k})$,

$$\psi_{\beta,\alpha}(\mathbf{x})=(2\pi)^{-3/2}\int\tilde{\psi}_{\beta,\alpha}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}\,,\qquad(2)$$

were shown to satisfy the set of differential equations (in units $\hbar=2m=1$)

$$-\frac{N+1}{N}\nabla^2\psi_{\beta,\alpha}(\mathbf{x})+\mathbf{S}_\gamma v_{\beta,\gamma}(\mathbf{x})\psi_{\gamma,\alpha}(\mathbf{x})=(\mathcal{E}_\alpha-\mathcal{E}_\beta)\psi_{\beta,\alpha}(\mathbf{x})\,,\qquad(3)$$

where $\mathcal{E}_\alpha$ is the internal energy of the state α and the potentials $v_{\beta,\gamma}(\mathbf{x})$ are defined by

$$v_{\beta,\gamma}(\mathbf{x})=\int V(\mathbf{x}-\mathbf{y})n_{\beta,\gamma}(\mathbf{y})d\mathbf{y}\,,\qquad(4)$$

$$\langle N,\beta,\mathbf{K}'|\psi^\dagger(\mathbf{x})\psi(\mathbf{x})|N,\gamma,\mathbf{K}\rangle=(2\pi)^{-3/2}e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{x}}\tilde{n}_{\beta,\gamma}(\mathbf{K}-\mathbf{K}')\,,\qquad(5)$$

$$n_{\beta,\gamma}(\mathbf{y})=(2\pi)^{-3/2}\int\tilde{n}_{\beta,\gamma}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{k}\,.\qquad(6)$$

In this paper, the asymptotic behavior of the solutions of (3) will be considered in detail. In particular, the difficulties arising from the fact that $v_{\beta,\gamma}(\mathbf{x})$ does not vanish asymptotically will be resolved. It will be shown that the asymptotic forms given in I go asymptotically to solutions of the asymptotic limits of Eq. (3). For the case of elastic scattering, the existence of direct and exchange wave functions will be demonstrated. Integral equations for the direct and exchange wave functions will be derived and used to produce useful expressions for the scattering amplitudes for both elastic and inelastic processes in which either the initial or final state consists of a single particle plus a bound group. All these results will be similar to those described in a recent paper,[2] but the discussion in the latter does not adequately cover the asymptotic properties of (3) and its solutions.

ASYMPTOTICS

In general, the potential $v_{\beta,\gamma}(\mathbf{x})$ is of the form

$$v_{\beta,\gamma}(\mathbf{x})=\sum_{j=1}^{r_{\beta,\gamma}}w_{\beta,\gamma}{}^{j}e^{i\mathbf{K}_{\beta,\gamma}{}^{j}\cdot\mathbf{x}}+u_{\beta,\gamma}(\mathbf{x})\,,\qquad(7)$$

where

$$u_{\beta,\gamma}(\mathbf{x})\rightarrow 0\,.$$

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[1] M. Bolsterli, Phys. Rev. **129**, 2830 (1963), hereafter referred to as I.

[2] M. Bolsterli, Phys. Rev. **131**, 883 (1963), hereafter referred to as II.