of the infrared divergence is a mathematical, rather than a physical one.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor L. C. Biedenharn for asking the pertinent question which led to this investigation. Several discussions with Professor H. Suura helped clarify our ideas. We also thank Professor S. Gasiorowicz for helpful criticism of the manuscript.

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PHYSICAL REVIEW VOLUME 134, NUMBER 2B 27 APRIL 1964

# Asymptotic Conditions and Integral Equations for Nonrelativistic Invariant Functions*

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(Received 29 August 1963; revised manuscript received 27 December 1963)

The asymptotic behavior of the invariant functions introducted previously is investigated. It is shown that there exists a "direct" invariant function and that all scattering amplitudes for single-particle processes can be expressed in terms of this direct function and the asymptotic plane-wave part only of other invariant functions. Moreover, only the short-range part of the potentials enters into the expression for the scattering amplitude.

## INTRODUCTION

IN a previous paper[1] it was shown that

$$\begin{aligned}\langle N,\beta,\mathbf{K}'|\psi(\mathbf{x})|N+1,\alpha,\mathbf{K}\rangle &\equiv (N+1)^{1/2}\int \Psi_{\beta,\mathbf{K}'}{}^{*}(\mathbf{x}_1,\cdots,\mathbf{x}_N)\\ &\qquad\times\Psi_{\alpha,\mathbf{K}}(\mathbf{x},\mathbf{x}_1,\cdots,\mathbf{x}_N)d\mathbf{x}_1\cdots d\mathbf{x}_N\\ &=(2\pi)^{-3/2}e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{x}}\tilde{\psi}_{\beta,\alpha}\left(\frac{N}{N+1}\mathbf{K}-\mathbf{K}'\right),\end{aligned}\tag{1}$$

where $\beta$ and $\alpha$ label the internal states of the $N$ and $N+1$ particle systems and $\mathbf{K}'$ and $\mathbf{K}$ are their total momenta. The Fourier transforms of the functions $\tilde{\psi}_{\beta,\alpha}(\mathbf{k})$,

$$\psi_{\beta,\alpha}(\mathbf{x})=(2\pi)^{-3/2}\int\tilde{\psi}_{\beta,\alpha}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k},\tag{2}$$

were shown to satisfy the set of differential equations (in units $\hbar=2m=1$)

$$-\frac{N+1}{N}\nabla^2\psi_{\beta,\alpha}(\mathbf{x})+\mathbf{S}_\gamma v_{\beta,\gamma}(\mathbf{x})\psi_{\gamma,\alpha}(\mathbf{x})=(\mathcal{E}_\alpha-\mathcal{E}_\beta)\psi_{\beta,\alpha}(\mathbf{x}),\tag{3}$$

where $\mathcal{E}_\alpha$ is the internal energy of the state $\alpha$ and the potentials $v_{\beta,\gamma}(\mathbf{x})$ are defined by

$$v_{\beta,\gamma}(\mathbf{x})=\int V(\mathbf{x}-\mathbf{y})n_{\beta,\gamma}(\mathbf{y})d\mathbf{y},\tag{4}$$

$$\begin{aligned}\langle N,\beta,\mathbf{K}'|\psi^\dagger(\mathbf{x})\psi(\mathbf{x})|N,\gamma,\mathbf{K}\rangle &=(2\pi)^{-3/2}e^{i(\mathbf{K}-\mathbf{K}')\cdot\mathbf{x}}\tilde{n}_{\beta,\gamma}(\mathbf{K}-\mathbf{K}'),\end{aligned}\tag{5}$$

$$n_{\beta,\gamma}(\mathbf{y})=(2\pi)^{-3/2}\int\tilde{n}_{\beta,\gamma}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{y}}d\mathbf{k}.\tag{6}$$

In this paper, the asymptotic behavior of the solutions of (3) will be considered in detail. In particular, the difficulties arising from the fact that $v_{\beta,\gamma}(\mathbf{x})$ does not vanish asymptotically will be resolved. It will be shown that the asymptotic forms given in I go asymptotically to solutions of the asymptotic limits of Eq. (3). For the case of elastic scattering, the existence of direct and exchange wave functions will be demonstrated. Integral equations for the direct and exchange wave functions will be derived and used to produce useful expressions for the scattering amplitudes for both elastic and inelastic processes in which either the initial or final state consists of a single particle plus a bound group. All these results will be similar to those described in a recent paper,[2] but the discussion in the latter does not adequately cover the asymptotic properties of (3) and its solutions.

## ASYMPTOTICS

In general, the potential $v_{\beta,\gamma}(\mathbf{x})$ is of the form

$$v_{\beta,\gamma}(\mathbf{x})=\sum_{j=1}^{r_{\beta,\gamma}}w_{\beta,\gamma}{}^{j}e^{i\mathbf{K}_{\beta,\gamma}{}^{j}\cdot\mathbf{x}}+u_{\beta,\gamma}(\mathbf{x}),\tag{7}$$

where

$$u_{\beta,\gamma}(\mathbf{x})\to 0.$$

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* Work supported in part by the U. S. Atomic Energy Commission.

[1] M. Bolsterli, Phys. Rev. **129**, 2830 (1963), hereafter referred to as I.

[2] M. Bolsterli, Phys. Rev. **131**, 883 (1963), hereafter referred to as II.

Here and in the following the $\longrightarrow$ sign is to be associated with behavior as $|\mathbf{x}|\to\infty$. The plane-wave parts of $v$ arise when $\beta$ and $\gamma$ are continuum states. For example, if $\beta$ and $\gamma$ each consist of a single-particle plus a bound group, then as the distance between the particle and the bound group becomes large, $v$ has nonzero values in two regions of space, one near the single particle and one near the bound group. Both of these regions are at large values of $|\mathbf{x}|$.

In order to illustrate the derivation of the coefficients $w_{\beta,\gamma}{}^j$, consider the function $\tilde{n}_{\beta,\gamma}(\mathbf{k})$ for the case that $\beta$ is the state $N-n$, $b$; $n$, $c$; $\mathbf{p}$ and $\gamma$ is $N-n$, $g$; $n$, $h$; $\mathbf{q}$. Here $b$, $c$, $g$, and $h$ are bound states of the respective groups; $\mathbf{p}$ and $\mathbf{q}$ are relative momenta. Then

$$(2\pi)^{-3/2}\tilde{n}_{\beta,\gamma}(\mathbf{k})=\langle N,\beta,\mathbf{0}|\psi^\dagger(0)\psi(0)|N,\gamma,\mathbf{k}\rangle$$

$$=\left\langle N-n,\,b,\,-\mathbf{p};\,n,c,\mathbf{p}\,|\psi^\dagger(0)\psi(0)|\,N-n,\,g,\,\frac{N-n}{N}\mathbf{k}-\mathbf{q};\,n,\,h,\,\frac{n}{N}\mathbf{k}+\mathbf{q}\right\rangle. \quad (8)$$

The asymptotic part of $n_{\beta,\gamma}(\mathbf{x})$ comes from the delta-function part of $\tilde{n}_{\beta,\gamma}(\mathbf{k})$; the latter will be denoted $D(\tilde{n}_{\beta,\gamma}(\mathbf{k}))$:

$$(2\pi)^{-3/2}D(\tilde{n}_{\beta,\gamma}(\mathbf{k}))=\delta\left(\frac{N-n}{N}\mathbf{k}-\mathbf{q}+\mathbf{p}\right)\left\langle n,c,\mathbf{p}|\psi^\dagger(0)\psi(0)|\,n,\,h,\,\frac{n}{N}\mathbf{k}+\mathbf{q}\right\rangle\delta_{b,g}$$

$$+\delta\left(\frac{n}{N}\mathbf{k}+\mathbf{q}-\mathbf{p}\right)\left\langle N-n,\,b,\,-\mathbf{p}|\psi^\dagger(0)\psi(0)|\,N-n,\,g,\,\frac{N-n}{N}\mathbf{k}-\mathbf{q}\right\rangle\delta_{c,h}$$

$$=\left(\frac{N}{N-n}\right)^3\delta\left(\mathbf{k}-\frac{N(\mathbf{q}-\mathbf{p})}{N-n}\right)(2\pi)^{-3/2}\tilde{n}_{c,h}\left(\frac{n}{N}\mathbf{k}+\mathbf{q}-\mathbf{p}\right)\delta_{b,g}$$

$$+\left(\frac{N}{n}\right)^3\delta\left(\mathbf{k}-\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)(2\pi)^{-3/2}\tilde{n}_{b,g}\left(\frac{N-n}{N}\mathbf{k}-\mathbf{q}+\mathbf{p}\right)\delta_{c,h}, \quad (9)$$

so that

$$D(\tilde{n}_{\beta,\gamma}(\mathbf{k}))=\left(\frac{N}{N-n}\right)^3\delta\left(\mathbf{k}-\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\tilde{n}_{c,h}\left(\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\delta_{b,g}+\left(\frac{N}{n}\right)^3\delta\left(\mathbf{k}-\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\tilde{n}_{b,g}\left(\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\delta_{c,h}. \quad (10)$$

$$D(\tilde{v}_{\beta,\gamma}(\mathbf{k}))=(2\pi)^{3/2}\tilde{V}(\mathbf{k})D(\tilde{n}_{\beta,\gamma}(\mathbf{k}))$$

$$=(2\pi)^{3/2}\left(\frac{N}{N-n}\right)^3\delta\left(\mathbf{k}-\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\tilde{V}\left(\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\tilde{n}_{c,h}\left(\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\delta_{b,g}$$

$$+(2\pi)^{3/2}\left(\frac{N}{n}\right)^3\delta\left(\mathbf{k}-\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\tilde{V}\left(\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\tilde{n}_{b,g}\left(\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\delta_{c,h}$$

$$=\left(\frac{N}{N-n}\right)^3\delta\left(\mathbf{k}-\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\tilde{v}_{c,h}\left(\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\delta_{b,g}+\left(\frac{N}{n}\right)^3\delta\left(\mathbf{k}-\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\tilde{v}_{b,g}\left(\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\delta_{c,h}. \quad (11)$$

Let $A(v_{\beta,\gamma}(\mathbf{x}))$ be the asymptotic plane-wave part of $v_{\beta,\gamma}(\mathbf{x})$. Then

$$A(v_{\beta,\gamma}(\mathbf{x}))=\left(\frac{N}{N-n}\right)^3 e^{i(N/N-n)(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}\tilde{v}_{c,h}\left(\frac{N}{N-n}(\mathbf{q}-\mathbf{p})\right)\delta_{b,g}+\left(\frac{N}{n}\right)^3 e^{i(N/n)(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\tilde{v}_{b,g}\left(\frac{N}{n}(\mathbf{p}-\mathbf{q})\right)\delta_{c,h}, \quad (12)$$

and (7) is verified. Note that if $\beta$ or $\gamma$ consists of a single bound group, then $w_{\beta,\gamma}=0$, since the density $n_{\beta,\gamma}(\mathbf{x})$ falls off exponentially with $|\mathbf{x}|$ in this case.

It follows that the asymptotic form of (3) is

$$-\frac{N+1}{N}\nabla^2\chi_{\beta,\alpha}(\mathbf{x})+\mathbf{S}_\gamma\sum_{j=1}^{r_{\beta,\gamma}}w_{\beta,\gamma}{}^j e^{i\mathbf{K}_{\beta,\gamma}{}^j\cdot\mathbf{x}}\chi_{\gamma,\alpha}(\mathbf{x})=(\mathcal{E}_\alpha-\mathcal{E}_\beta)\chi_{\beta,\alpha}(\mathbf{x}). \quad (13)$$

Consider first the asymptotic behavior of $\psi_{\beta,\gamma}(\mathbf{x})$ for the case that $\alpha$ is the state $b\mathbf{p}^{\text{in}}$ consisting at $t\to-\infty$ of a single particle with relative momentum $\mathbf{p}$ incident on the bound group $b$. As shown in I and II,

$$\psi_{\beta,\alpha}(\mathbf{x})\to\varphi_{\beta,\alpha}{}^D(\mathbf{x})+\varphi_{\beta,\alpha}{}^{\text{Ex}}(\mathbf{x}), \quad (14)$$

$$\varphi_{\beta,\alpha}{}^D(\mathbf{x})=(2\pi)^{-3/2}\delta_{\beta,b}e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (15)$$

$$\varphi_{\beta,\alpha}{}^{\mathrm{Ex}}(\mathbf{x})=\mp N^3(2\pi)^{-3/2}e^{iN(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}\tilde{\psi}_{\delta^{\mathrm{in}},b}\left(N\mathbf{q}-\frac{N^2-1}{N}\mathbf{p}\right),\quad \text{if } \beta=\delta^{\mathrm{in}}\mathbf{q}^{\mathrm{in}},$$
$$=0,\quad \text{otherwise},\tag{16}$$

where $\delta^{\mathrm{in}}$ is an arbitrary "in" state of $N-1$ particles, and $\delta^{\mathrm{in}}\mathbf{q}^{\mathrm{in}}$ is the state consisting of a single-particle incident with relative momentum $\mathbf{q}$ on the group $\delta^{\mathrm{in}}$. The asymptotic direct function satisfies (13) because the only $w$ functions that occur are zero. The exchange asymptotic function $\varphi_{b\mathbf{p}}{}^{\mathrm{Ex}}$ does *not* satisfy (13). However, the $w$ term in (13) can be divided into two parts such that ($\alpha=b\mathbf{q}^{\mathrm{in}}$)

$$-\frac{N+1}{N}\nabla^2\varphi_{\beta,\alpha}{}^{Ex}(\mathbf{x})+\mathbf{S}_\gamma\sum_{j=1}^{t_{\beta,\gamma}}w_{\beta,\gamma}{}^je^{i\mathbf{K}_{\beta,\gamma}{}^j\cdot\mathbf{x}}\varphi_{\gamma,\alpha}{}^{\mathrm{Ex}}(\mathbf{x})=(\mathcal{E}_\alpha-\mathcal{E}_\beta)\varphi_{\beta,\alpha}{}^{\mathrm{Ex}}(\mathbf{x}),\tag{17}$$

$$\mathbf{S}_\gamma\sum_{j=t_{\beta,\gamma}+1}^{r_{\beta,\gamma}}w_{\beta,\gamma}{}^je^{i\mathbf{K}_{\beta,\gamma}{}^j\cdot\mathbf{x}}\varphi_{\gamma,\alpha}{}^{\mathrm{Ex}}(\mathbf{x})\to 0.\tag{18}$$

It can be shown that those parts of $w$ in which a single particle is contracted must be included in (17), the rest of $w$ in (18); that is, if $c$ and $h$ are single-particle groups, then the second term in (12) is to be used in (17), while the first term is to be used in (18), and similarly for states $\beta$ and $\gamma$ other than those used in (12). It follows from (17) and (18) that $\varphi^{\mathrm{Ex}}$ is asymptotically a solution of (13); i.e., there is a solution $\chi^{\mathrm{Ex}}$ of (13) such that

$$\chi_{b\mathbf{p}}{}^{\mathrm{Ex}}(\mathbf{x})\to\varphi_{b\mathbf{p}}{}^{\mathrm{Ex}}(\mathbf{x}).\tag{19}$$

Since $\varphi^D$ and $\varphi^{\mathrm{Ex}}$ are not connected asymptotic forms, it follows that $\psi^D$ and $\psi^{\mathrm{Ex}}$ exist, where each of $\psi^D$ and $\psi^{\mathrm{Ex}}$ satisfies (3) and

$$\psi^D\to\varphi^D+\text{o.s.w.},\quad \psi^{\mathrm{Ex}}\to\varphi^{\mathrm{Ex}}+\text{o.s.w.},\tag{20}$$

with o.s.w. standing for outgoing spherical waves.

Similarly, if $\alpha$ is the state $N-1, b; 2, c; \mathbf{p}^{\mathrm{in}}$ consisting at $t\to-\infty$ of two-particle bound-state $c$ incident with relative momentum $\mathbf{p}$ on the $N-1$ particle bound-state $b$ (see I), the asymptotic functions

$$\varphi_{\beta,\alpha}{}^S(\mathbf{x})=(2\pi)^{-3/2}\left(\frac{N}{N-1}\right)^3e^{i[N/(N-1)](\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\times\tilde{\psi}_{1,c}\left(\frac{N+1}{2(N+1)}\mathbf{p}-\frac{N}{N-1}\mathbf{q}\right),\quad \text{if } \beta=b,\mathbf{q}^{\mathrm{in}},$$
$$=0,\quad\text{otherwise},\tag{21}$$

$$\varphi_{\beta,\alpha}{}^{HPS}(\mathbf{x})=(2\pi)^{-3/2}\left(\frac{N}{2}\right)^3e^{i(N/2)(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}}\times\tilde{\psi}_{\delta^{\mathrm{in}},b}\left(\frac{N}{2}\mathbf{q}-\frac{(N-2)(N+1)}{2(N-1)}\mathbf{p}\right),\quad \text{if } \beta=(N-2\,\delta^{\mathrm{in}},2c)\mathbf{q}^{\mathrm{in}};$$
$$=0,\quad\text{otherwise},\tag{22}$$

satisfy equations like (17) and (18), where the part of $w$ to be taken in (17) depends on $\alpha$ and on which of the functions $\varphi^S$ or $\varphi^{HPS}$ is involved. Again $\psi^S$ and $\psi^{HPS}$ can be defined as the solution of (3) that approach $\varphi^S$ and $\varphi^{HPS}$ asymptotically.

In this way, it is seen that part of the long-range part of the potential (7) is already included in the asymptotic form of the wave-function $\psi_{\beta,\alpha}$, while part is not. However, owing to (18), the part of $w$ that remains does not contribute plane waves at $\infty$, but only outgoing spherical waves.

## INTEGRAL EQUATIONS

As in II, let $G_{\beta,\gamma}{}^{\mathrm{in}}(\mathbf{x}-\mathbf{x}';E)$ be defined by

$$\left(E+i0-\mathcal{E}_\beta+\frac{N+1}{N}\nabla^2\right)G_{\beta,\gamma}{}^{\mathrm{in}}(\mathbf{x}-\mathbf{x}';E)=\delta(\mathbf{x}-\mathbf{x}')\mathbf{1}_{\beta,\gamma}{}^N,\tag{23}$$

where $\beta$ and $\gamma$ are $N$-particle internal states and $\mathbf{1}_{\beta,\gamma}{}^N$ is the generalization of the unit matrix in the subspace of $N$-particle internal states

$$\mathbf{S}_\gamma\mathbf{1}_{\beta,\gamma}{}^Nf_\gamma=f_\beta.\tag{24}$$

$G_{\beta,\gamma}$ is zero if $\beta$ and $\gamma$ are different types of states, that is, unless $\beta$ and $\gamma$ contain the same sets of bound groups and differ only in values of relative momenta.

In terms of $G$, Eqs. (17) and (18) can be written ($\alpha=b\mathbf{p}^{\mathrm{in}}$)

$$\mathbf{S}_{\gamma,\delta}\int G_{\beta,\gamma}(\mathbf{x}-\mathbf{x}',\mathcal{E}_\alpha)v_{\gamma,\delta}(\mathbf{x}')\psi_{\delta,\alpha}{}^{\mathrm{Ex}}(\mathbf{x}')d\mathbf{x}'\to\varphi_{\beta,\alpha}{}^{\mathrm{Ex}}(\mathbf{x})+\text{o.s.w.},\tag{25}$$

while clearly

$$\mathbf{S}_{\gamma,\delta}\int G_{\beta,\gamma}(\mathbf{x}-\mathbf{x}',\mathcal{E}_\alpha)v_{\gamma,\delta}(\mathbf{x}')\psi_{\delta,\alpha}{}^D(\mathbf{x}')d\mathbf{x}'\to\text{o.s.w.}\tag{26}$$

Similarly, for $\alpha=N-1, b; 2c; \mathbf{p}^{\mathrm{in}}$

$$\mathbf{S}_{\gamma,\delta}\int G_{\beta,\gamma}(\mathbf{x}-\mathbf{x}',\mathcal{E}_\alpha)v_{\gamma,\delta}(\mathbf{x}')\psi_{\delta,\alpha}{}^{S,HPS}(\mathbf{x}')d\mathbf{x}'\to\varphi_{\beta,\alpha}{}^{S,HPS}(\mathbf{x})+\text{o.s.w.}\tag{27}$$

Therefore, the integral equations for $\psi^D$, $\psi^{\text{Ex}}$, and $\psi^S$ are (in symbolic notation)

$$\psi^D=\varphi^D+Gv\psi^D\,, \tag{28}$$

$$\psi^{\text{Ex}}=Gv\psi^{\text{Ex}}\,, \tag{29}$$

$$\psi^S=Gv\psi^S\,. \tag{30}$$

Equation (28) can be iterated to give

$$\psi^D=(1+Gv+GvGv+\cdots)\varphi^D\,. \tag{31}$$

In order to obtain iterative equations for (29) and (30) it is necessary to write

$$\begin{aligned} v&=w^{\text{Ex}}+v-w^{\text{Ex}}\\ &=w^S+v-w^S\,, \end{aligned} \tag{32}$$

where $w^{\text{Ex}}$ is the part of $w$ that occurs in (17), and $w^S$ is the part of $w$ that occurs in the corresponding equation for $\varphi^S$. Then Eq. (17) can be written

$$Gw^{\text{Ex}}\varphi^{\text{Ex}}=\varphi^{\text{Ex}}\,, \tag{33}$$

and (29) becomes

$$\begin{aligned} \psi^{\text{Ex}}-\varphi^{\text{Ex}}&=Gv(\psi^{\text{Ex}}-\varphi^{\text{Ex}})+Gv\varphi^{\text{Ex}}-\varphi^{\text{Ex}}\\ &=G(v-w^{\text{Ex}})\varphi^{\text{Ex}}+Gv(\psi^{\text{Ex}}-\varphi^{\text{Ex}})\\ &=(1+Gv+GvGv+\cdots)G(v-w^{\text{Ex}})\varphi^{\text{Ex}}. \end{aligned} \tag{34}$$

$$\psi^{\text{Ex}}=\varphi^{\text{Ex}}+(1+Gv+GvGv+\cdots)G(v-w^{\text{Ex}})\varphi^{\text{Ex}}, \tag{35}$$

and similarly

$$\psi^S=\varphi^S+(1+Gv+GvGv+\cdots)G(v-w^S)\varphi^S. \tag{36}$$

## *T*-MATRIX EXPRESSIONS

As noted in II, the direct elastic $T$-matrix element is

$$\langle b\mathbf{q}|T|b\mathbf{p}\rangle_D = (2\pi)^{-3/2}\int e^{-i\mathbf{q}\cdot\mathbf{x}}\mathbf{S}_\gamma v_{b,\gamma}(\mathbf{x})\psi_{\gamma,b\mathbf{p}}{}^{\text{in}D}(\mathbf{x})d\mathbf{x}\,, \tag{37}$$

while the exchange elastic amplitude is

$$\begin{aligned} \langle b\mathbf{q}|T|n\mathbf{p}\rangle^{\text{Ex}} &= (2\pi)^{-3/2}\int e^{-i\mathbf{q}\cdot\mathbf{x}}\mathbf{S}_\gamma v_{b,\gamma}(\mathbf{x})\psi_{\gamma,b\mathbf{p}}{}^{\text{in Ex}}(\mathbf{x})d\mathbf{x}\\ &=\int \varphi_{b\mathbf{q}}{}^{D*}v\psi_{b\mathbf{p}}{}^{\text{Ex}}\\ &=\int \varphi_{b\mathbf{q}}{}^{D*}v[1+G^{\text{in}}(v-w^{\text{Ex}})+G^{\text{in}}vG^{\text{in}}(v-w^{\text{Ex}})\\ &\qquad +G^{\text{in}}vG^{\text{in}}vG^{\text{in}}(v-w^{\text{Ex}})+\cdots]\varphi_{b\mathbf{p}}{}^{\text{Ex}}\\ &=\int \varphi_{b\mathbf{q}}{}^{D*}v\varphi_{b\mathbf{p}}{}^{\text{Ex}}+\int [(G^{\text{out}}v+G^{\text{out}}vG^{\text{out}}v+\cdots)\\ &\qquad \times\varphi_{b\mathbf{q}}{}^{D}]^*(v-w^{\text{Ex}})\varphi_{b\mathbf{p}}{}^{\text{Ex}}\\ &=\int \varphi_{b\mathbf{q}}{}^{D*}v\varphi_{b\mathbf{p}}{}^{\text{Ex}}+\int (\psi_{b\mathbf{q}}{}^{\text{out}D*}-\varphi_{b\mathbf{q}}{}^{D*})\\ &\qquad \times(v-w^{\text{Ex}})\varphi_{b\mathbf{p}}{}^{\text{Ex}}\\ &=\int \psi_{b\mathbf{q}}{}^{\text{out}D*}(v-w^{\text{Ex}})\varphi_{b\mathbf{p}}{}^{\text{Ex}}\,, \end{aligned} \tag{38}$$

where use has been made of

$$\varphi_{b\mathbf{q}}{}^{D*}w=0\,, \tag{39}$$

since $w$ is zero for bound state and $\varphi_{b\mathbf{q}}{}^{D}$ has only a bound-state component. Similarly,

$$\langle b\mathbf{q}|T|N-1,b';2c;\mathbf{p}\rangle_{S,HPS} =\int \psi_{b\mathbf{q}}{}^{\text{out}D*}(v-w^{S,HPS})\varphi_{b',c\mathbf{p}}{}^{S,HPS}. \tag{40}$$

As in II, Eqs. (39) and (40) show that it is not necessary to solve (3) for $\psi^{\text{Ex}}$, $\psi^S$, $\psi^{HPS}$, etc. Only $\psi_{b\mathbf{q}}{}^{D}$ is required for $T$-matrix elements for single-particle processes.


## ACKNOWLEDGMENTS

One of the authors (M.B.) thanks the Aspen Institute for Humanistic Studies for its hospitality during most of this work. Several discussions with Professor N. Austern, who was also at the Aspen Institute, were quite helpful.