Self-Consistent Sets of Mesons in a Bootstrap Model

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(Received 2 December 1963)

Consideration is given to the possibility that the self-consistency requirements of some dispersion-theoretic bootstrap model may specify uniquely the set of strongly-interacting particles found in nature. The discussion is based on a particular, approximate model of the P (pseudoscalar) and V (vector) mesons. In this model the V exchange forces in the $P+P$ states produce the V mesons, and the V exchange forces in the $P+V$ states produce the P mesons. Attention is limited to systems in which the particles arising in a particular way are degenerate or nearly degenerate, and are represented by a small number of irreducible representations of a simple Lie group of first, second, or third rank. Three plausible self-consistency requirements are postulated. The smallest meson set that satisfies all three postulates corresponds to the group SU_3. The predicted particles in this scheme are a P-meson octet, a V octet, a V singlet, and a singlet particle of spin and parity 2^-. One of the self-consistency postulates is concerned with deviations from degeneracy, and leads to the Gell-Mann-Okubo sum rule for the SU_3 scheme. The double-septet scheme of the group G_2 does not satisfy any of the postulates.

I. INTRODUCTION

IN dispersion theoretic calculations a common approximation procedure is to consider only the relevant particles of lightest rest mass, neglecting strange particles whenever possible. The recent discovery of multiplets of particles of the same spins and parities, together with the success of unitary symmetry, makes this procedure seem very doubtful. It appears that one should include entire particle multiplets in the dynamic calculations. In one respect, this is unfortunate, since the inclusion of many types of particles leads to complicated equations. However, there is a compensating feature: A bootstrap model involving particle multiplets leads to the prediction of the ratios of interaction constants, and thus allows one to investigate the possible dynamical origin of interaction symmetries. The idea that the interaction symmetries may have a dynamical origin is not new,[1] but progress in investigating this idea can be made only when the many-particle aspects of the dispersion relations are considered.

In this paper, it is assumed that since the masses of baryon-antibaryon pairs are appreciably larger than those of the known P (pseudoscalar) and V (vector) mesons, it is a reasonable first approximation to neglect the baryons. It is hoped that eventually a dispersion theory of the bootstrap type will be developed that is consistent if applied to the existing set of mesons, and inconsistent if applied to any fictitious set, so that the existing set will be required dynamically. The term "fictitious set of mesons" is used to mean a set that differs from the existing set in one or more of the following properties—the total number of types of mesons, the spins, parities, internal quantum numbers (isotopic spin, etc.), masses, and interaction constants of the mesons.

The task of eliminating all fictitious meson sets theoretically is indeed formidable. Fortunately, if experimental evidence concerning the existing particles is used as a guide, it is not necessary to start at the beginning and work straight through to the end. A conceivable solution to the problem may be broken up into the following four steps, which may be investigated independently. First, it may be that consistent bootstrap equations are possible only if the mesons of lowest mass are pseudoscalar and vector mesons. Second, it may be that the meson multiplets and their interactions must correspond to representations of a simple Lie group. Third, only specific representations of one specific group may be allowed. Fourth, the mass differences and other symmetry breaking interactions may have a dynamic origin.[2] Some progress already has been in investigating each of these conjectures.[3–8]

The requirement that the existing set of particles leads to a solution to the model certainly is as important as the requirement that fictitious sets do not. It is assumed in this paper that the existing set consists of eight P mesons, (the π triplet, K and $\bar{K}$ doublets, and η singlet), and nine V mesons (the ρ triplet, the 885-MeV K^* and $\bar{K}^*$ doublets, and the ω and 1020-MeV φ singlets). The other existing mesons are assumed to arise in more complete developments of the model. The last two steps of the four-step procedure outlined above are investigated here. Only sets of P and V mesons corresponding to a small number of irreducible representations of a simple Lie group are considered. The term

[1] See, for example, G. F. Chew, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by Prentki (CERN, Geneva, 1962) pp. 525–528.

[2] It may be that in a complete bootstrap model there will be a solution involving degenerate multiplets, so that it will be necessary to postulate nondegeneracy. Even if this is so, it would be quite an achievement if such a postulate allowed one to calculate the physically observed mass splittings from self-consistency requirements.

[3] R. H. Capps, Phys. Rev. Letters **10**, 312 (1963).

[4] R. H. Capps, Nuovo Cimento **30**, 341 (1963).

[5] R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).

[6] S. L. Glashow, Phys. Rev. **130**, 2132 (1963).

[7] R. E. Cutkosky and Pekka Tarjanne, Phys. Rev. **132**, 1354 (1963).

[8] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963); E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 1831 (1963); P. Carruthers, Phys. Rev. Letters **10**, 538 and 540 (1963) and Phys. Rev. **133**, B497 (1964).

"scheme" will be used to designate a particular set of representations of a particular Lie group, together with the condition that the interactions of the mesons are invariant under the transformations of the group. The aims of the paper are to find a bootstrap model and a group-representation scheme leading to a solution involving the seventeen types of mesons listed above, and to investigate whether or not other schemes lead to solutions. Since there are an infinite number of possible schemes to be considered, we limit our investigation to sets of seventeen or fewer mesons, i.e., to sets no larger than the existing set. The bootstrap model considered is a generalization of that developed previously by the author.[3,4,9]

At the present stage in the development of dispersion theory, cutoffs, and other approximation procedures are needed in most calculations, so that it is difficult to test a particular assumption concerning the nature of the singularities that are important in a particular amplitude by comparing a calculated number with experiment. However, the question of which group-representation schemes lead to consistent solutions to a bootstrap model depends on the types of particle configurations and forces assumed to be important. Therefore, comparison of the predictions of group-theoretical bootstrap models with experiment may provide important evidence concerning the nature of the forces that are dominant physically.

In Sec. II the basic assumptions of the bootstrap model used here are outlined. Self-consistency requirements associated with the V-meson and P-meson poles are formulated in Secs. III and IV, and applied to the various group-representation schemes. The double-octet scheme of SU_3 is discussed in detail in Sec. IV B. In Sec. V the possibility of nondegenerate solutions is investigated; this section may be read before Secs. III and IV without loss of continuity. The double-septet scheme of G_2 is discussed in detail in Sec. VI.

II. BASIC ASSUMPTIONS

In a bootstrap model the P and V mesons must correspond to bound state or resonance poles in the appropriate scattering states of angular momentum and parity 0^- and 1^-. In this paper it is assumed that the most important 1^- states are of the type $P+P$, and the most important 0^- states are of the type $P+V$. It is further assumed that only these important configurations need be considered when testing whether or not a particular set of group representations leads to a self-consistent solution to the model.

In a complete bootstrap model the virtual particles that transmit the forces must all appear as poles in the appropriate amplitudes. However, this is not necessarily the case in an incomplete model. In general, we will assume that the V mesons of the model transmit the forces, but for certain considerations it will not be necessary to make a specific assumption concerning the forces.

The sets of mesons considered are self-conjugate, i.e., the set of antiparticles for either the P or V mesons is the set itself. The mesons are assumed to correspond to representations of a simple Lie group, and the interactions are assumed invariant to the group transformations.[10] The P and V representations (not necessarily irreducible) are denoted by D_P and D_V. The self-conjugate property of the meson sets implies that the identity representation must occur as a symmetric combination of the two particles in the reduction of the direct products $D_P\otimes D_P$ and $D_V\otimes D_V$. We use this fact in order to classify each irreducible representation as either single or double; the double representations are those that must occur in pairs if they occur at all. An irreducible representation that is not equivalent to its complex conjugate representation is double. For example, the two three-dimensional representations of SU_3 are double; $3\oplus 3^*$ may represent some of the P or V mesons, but neither 3 nor 3^* can occur alone. (Specific representations are denoted here by the numerical values of their dimensions; a star is used to distinguish one of two complex-conjugate representations.) Each irreducible representation α that is equivalent to its complex conjugate has the property that the identity representation 1 occurs exactly once in the reduction of $\alpha\otimes\alpha$. The representation α is single (double) if 1 occurs as a symmetric (antisymmetric) combination of the α. For example, of the irreducible representations of SU_2, those of odd dimension (integral I spin) are single, while those of even dimension (half odd-integral I spin) are double.

At present we consider only schemes in which D_P is a single, irreducible representation of a simple Lie group of rank one, two, or three. Double representations are considered in Sec. III C. The requirement that the V mesons occur in the P-wave, $P+P$ states implies that the only representations possible for the V mesons are those that occur antisymmetrically in the direct product $D_P\otimes D_P$. Investigation shows that there is one possible D_P, D_V scheme involving as few as seventeen particles for each of the second-rank groups SU_3, C_2 and G_2, eleven such schemes for the first-rank group SU_2, and only one scheme corresponding to a third-rank group. The third-rank group is B_3, the seven-dimensional rotation group. These schemes are listed in the first two columns of Table I. The symbol α,β refers to the scheme in which $D_P=\alpha$ and $D_V=\beta$.

The I-spin zero, $K+\bar{K}$ state is the only two-particle state of hypercharge and I-spin zero that can be formed from the π, K, and η. For this reason, the φ and ω

[9] R. H. Capps, Phys. Rev. **132**, 2749 (1963).

[10] A lucid discussion of some important properties of Lie groups is given by A. Salam, *Theoretical Physics* (lectures presented at a seminar, Trieste, summer of 1962) (International Atomic Energy Agency, Vienna, 1963), pp. 173–196. A detailed discussion of Lie groups of second rank is given by Behrends, Dreitlein, Fronsdal, and Lee, Rev. Mod. Phys. **34**, 1 (1962). Our notation for the representations is essentially that of this latter reference.

TABLE I. Results of self-consistency tests applied to simple schemes for which D_P is irreducible. The V-Bootstrap, P-Bootstrap, and Roots (a,b) tests are described in Secs. III, IV, and V. In some cases where one of the bootstrap tests is violated, the tests have not all been applied: A blank indicates such an omission.

Group	D_P, D_V	V Boot-strap	P Boot-strap	Roots (a,b)
SU_3	8, 8	Yes	Yes	(1/2, 1/4), (1/6, 1/36), $(-1/3, 1/9)^2$
C_2	5, 10	Yes	Yes	$(-1/4, 3/8)^2$
G_2	7, 7	No	No	$(-1/6, 1/36)^3$
SU_2	3, 3	Yes	No	(−1/2, 1/4)
SU_2	5, 3	Yes	Yes	(1/2, 7/20), (−2/3, 0)
SU_2	5, 7	Yes	Yes	(−1/14, 11/28), (−4/7, 3/70)
SU_2	5, $3\oplus7$	Yes	Yes	$(-1/4, 3/8)^2$
SU_2	7, 3	Yes	No	(3/4, 3/8), (1/6, 0), (−3/4, 0)
SU_2	7, 7	No	No	
SU_2	7, $3\oplus7$	No	No	
SU_2	9, 3	Yes	No	
SU_2	9, 7	No		
SU_2	11, 3		No	
SU_2	13, 3		No	
B_3	8, 7		No	

cannot both occur as $P+P$ resonances in the present model (the dispersion relations used are sufficiently simple that two resonance poles cannot occur in a one-channel problem). The double-octet scheme of SU_3 accounts for all the existing mesons except one of the isoscalar V mesons. Furthermore, as shown in Sec. IV B, the second isoscalar V meson does arise in this scheme when the bootstrap model is extended in a natural way. Thus, we assume the 8, 8 (SU_3) scheme corresponds to reality. Before discussing this scheme in detail we introduce the principal self-consistency tests of the model.

III. THE V MESON BOOTSTRAP CONDITION

A. The Self-Consistency Condition

We follow the model of Ref. 3 and assume that the forces that produce the V-meson resonances in the $P+P$ states result from the exchange of the same V mesons. Degenerate P and V multiplets are assumed. The dispersion theoretic method used is the matrix N/D method, with the numerator matrix (the force matrix) set equal to the Born-approximation amplitude.

Self-consistency requires that the forces be attractive in the resonating states and less attractive or repulsive in all nonresonating states. Since the elements of the N matrix are proportional, it is not necessary to calculate N in order to determine consistency. It is sufficient to determine the sign of the force and the crossing matrix for the $P+P\to P+P$ processes. Different conventions are used in defining crossing matrices, so we shall be explicit in our definitions. We first define the (PV) crossing matrices associated with the processes $P+V\to P+V$. We assume that the numbers of P and V mesons are n and q, respectively, so that there are nq different (PV) states, and $(nq)^2$ amplitudes connecting these states. The set of P mesons may consist both of self-conjugate particles P_α and particle-antiparticle pairs P_β, $\bar{P}_\beta$; i.e.,

$$\mathcal{C}P_\alpha=\Gamma_\alpha P_\alpha \quad \mathcal{C}P_\beta=\Gamma_\beta\bar{P}_\beta,$$

where $\mathcal{C}$ is the charge conjugation operator and the Γ_α and Γ_β are either 1 or -1. It is assumed that the phases of the P_β are so chosen that all the Γ_β and Γ_α are equal (a method for doing this is described later). We first consider a representation in which the (PV) states are simple products, i.e., $\psi_k=P_\alpha V_\beta$, where k ranges through the nq different (PV) states. In this representation the crossing matrix is an $(nq)^2$ by $(nq)^2$ matrix, defined by the equation,

$$T_{kl}{}^{cr}=\sum_{k'l'}C_{kl,k'l'}T_{k'l'},$$

where T_{kl} is the amplitude for the process $k\to l$, and $T^{cr}(P_\alpha V_\beta\to P_\gamma V_\delta)=T(\bar{P}_\gamma V_\beta\to P_\alpha V_\delta)$. ($\bar{P}_\alpha=P_\alpha$ for the self-conjugate P mesons.)

Since the interactions are assumed invariant to the group transformations, it is convenient to rewrite C in the representation (i,m), where i denotes an irreducible representation contained in the direct product $D_P\otimes D_V$ and m designates a state within the representation i. In the (i,m) representation the matrix amplitude T is diagonal, so we may limit our attention to the elements of C referring to elastic scattering.[11] These elements satisfy the equation,

$$T_{(i,m)}{}^{cr}=\sum_{(i',m')}C_{(i,m),(i',m')}T_{(i',m')}\,.$$

Invariance under the group transformations implies that $T_{(i',m')}$ is independent of m', so that a "reduced" crossing matrix $C_{ii'}$ may be defined by the equation

$$T_i{}^{cr}=\sum_{i'}C_{ii'}T_{i'}\,, \tag{1}$$

where $C_{ii'}=\sum_{m'}C_{(i,m),(i',m')}$. ($C_{ii'}$ is independent of m.)

One may carry out the transformation from the kl representation to the i representation by using the Clebsch-Gordan coefficients of the group in question, if the phases of the P_α are chosen in the manner discussed above. One method of choosing appropriate phases is to first assume any definite phase convention, form the identity representation $\psi(I)$ from the direct product $D_P\otimes D_P$, and then redefine the phases so that the signs of all terms in $\psi(I)$ are the same. For example, if D_P is the representation 3 of SU_2, and the phases are defined by the condition that the matrix elements of the isotopic lowering operator I_x-iI_y are not negative, the

[11] The statement that the amplitude is diagonal in the (i,m) representation is not completely general. If a particular irreducible representation α occurs twice in the reduction of $D_P\otimes D_V$, transitions between the two states characterized by α are possible. This situation does not occur in any of the cases considered in the present paper, however. In the SU_3 scheme, the representation 8 does occur twice in the decomposition of $8\otimes8$, but the two octets are of opposite symmetry, and since the PPV and VVP vertices are each of definite symmetry, there are no transitions between the two $P+V$ octets in our model.

isotopic singlet state of $3\otimes 3$ is

$$\psi(I)=3^{-\frac{1}{2}}(-\pi^+\pi^- - \pi^-\pi^+ + \pi^0\pi^0),$$

where π^i denotes the members of the P triplet. In this case, one should change the defined phase of the π^+ or π^- by 180°. The fact that the phases may always be chosen so that the terms in $\psi(I)$ have the same sign is obvious if D_P is a direct sum of a double representation and its conjugate representation, and follows from our definition of single representations if D_P is a single representation.

It is clear from the basic definition of T^{cr} that if the matrix amplitude T is a multiple of the unit matrix, T^{cr} must be the same multiple of the unit matrix. This condition implies that the $C_{ii'}$ of Eq. (1) satisfy the row sum rule,

$$\sum_{i'} C_{ii'}=1. \tag{2a}$$

Furthermore, it follows from the definition of T^{cr} that Trace $T=$Trace T^{cr}, for all possible choices of the T_{ij}. (Trace T is defined as the sum over all the nq elastic amplitudes.) If the T_i are chosen to be finite for only one of the irreducible representations of $D_P\otimes D_V$, the trace condition implies the column sum rule,

$$\sum_i w_i C_{ii'}=w_{i'}, \tag{2b}$$

where w_i denotes the dimension of the representation i.

We next consider the (PP) crossing matrices appropriate for P-wave amplitudes of the type $P+P\rightarrow P+P$. For such an amplitude the two crossed processes are both of the $P+P\rightarrow P+P$ type. The contributions of the two crossed processes cancel for the symmetric (unallowed) P-wave states, and are equal for the antisymmetric states. Furthermore, only antisymmetric states occur in the crossed processes, since these are also of the P-wave type. The dimension of the (PP) crossing matrix is the number of antisymmetric irreducible representations in the reduction of $D_P\otimes D_P$. One method of constructing a (PP) crossing matrix is to construct the (PV) crossing matrix corresponding to the case where D_P and D_V are the same representation, delete the rows and columns referring to the symmetric representations, and then double the remaining elements. The (PP) crossing matrices also satisfy the sum rules of Eqs. (2a) and (2b).

In the model of this paper and Ref. 3, a positive crossing coefficient corresponds to an attractive V-meson exchange force in P-wave, $P+P$ states. Therefore, if the V mesons are represented by a single irreducible representation α, the self-consistency condition for the bootstrapping of the V resonances may be expressed simply in terms of elements of the (PP) crossing matrix K, i.e.,

$$K_{\alpha\alpha}>K_{i\alpha} \text{ for all } i\neq\alpha. \tag{3}$$

[Note that because of the sum rule of Eq. (2b) the above inequality implies $K_{\alpha\alpha}>0$, so that the force producing the resonance is attractive.] If D_V is a sum of m irreducible representations, the self-consistency conditions for degenerate V resonances are

$$\sum_{i=1}^{m} K_{\alpha i}=\sum_{i=1}^{m} K_{\beta i},$$

$$\sum_{i=1}^{m} K_{\alpha i}>\sum_{i=1}^{m} K_{\gamma i}, \tag{4}$$

where α, β, and i range through all the resonating representations, and γ ranges through all nonresonating representations.

B. Single P-Meson Representations

The (PP) crossing matrices for the 3, 3 (SU_2) and 5, 10 (C_2) schemes are one-dimensional unit matrices. Therefore, these schemes satisfy the V-bootstrap consistency condition. The crossing matrices for the P-meson quintet of SU_2 and the P-meson octet of SU_3 are two-by-two unit matrices, so the 5, 3, the 5, 7, and the 5, $3\oplus 7$ schemes of SU_2, and the 8, 8, the 8, 20, and the 8, $8\oplus 20$ schemes of SU_3 all satisfy the V-bootstrap condition. The 20-fold V-meson multiplet of SU_3 corresponds to the sum of the irreducible representations 10 and 10*. The schemes involving this multiplet are not included in Table I, because more than seventeen particles are involved.[12]

The (PP) crossing matrices are more complicated for the P-meson septet of G_2 and for the P-meson septet and nonet of SU_2. These three crossing matrices are listed below.

P Septet (G_2)

$$\begin{array}{c|cc} & 7 & 14 \\ \hline 7 & -1 & 2 \\ 14 & 1 & 0 \end{array}$$

P Septet (SU_2)

$$\begin{array}{c|ccc} & 3 & 7 & 11 \\ \hline 3 & 11 & 14 & -11 \\ 7 & 6 & -14 & 22 \\ 11 & -3 & 14 & 3 \end{array} \times\frac{1}{14}$$

P Nonet (SU_2)

$$\begin{array}{c|cccc} & 3 & 7 & 11 & 15 \\ \hline 3 & 627 & 1078 & 605 & -1320 \\ 7 & 462 & -382 & -770 & 1680 \\ 11 & 165 & -490 & 715 & 600 \\ 15 & -264 & 784 & 440 & 30 \end{array} \times\frac{1}{990}.$$

The numbers by the rows and columns are the dimensions of the corresponding representations. For these three P-meson multiplets, the only schemes involving

[12] The 8, $8\oplus 20$ (SU_3) solution to the V-meson bootstrap model is discussed in detail and compared with experiment by Donald E. Neville, Phys. Rev. **132**, 844 (1963).

TABLE II. The V-meson representations corresponding to some simple P-meson, double representations.

Group	D_P	V representations
SU_2	$2\oplus 2$	$1s$, $3s$, $2d$
SU_2	$4\oplus 4$	$1s$, $3s$, $5s$, $7s$, $2d$, $10d$
SU_3	$3\oplus 3^*$	$1s$, $8s$, $6d$
C_2	$4\oplus 4$	$1s$, $5s$, $10s$, $2d$, $10d$

as few as seventeen mesons that satisfy the self-consistency test of Eq. (3) or Eq. (4) are the 7, 3 and 9, 3 schemes of SU_2. Those schemes satisfying this test are indicated with a "yes" in the V-bootstrap column of Table I.

For any set of P mesons, one scheme satisfying this V-bootstrap test is obtained by assuming a V resonance in every P-wave, $P+P$ state.

C. Double P-Meson Representations

In this section we consider several fictitious schemes involving $2n$, P mesons associated with a double, n-dimensional irreducible representation. The reader not interested in these possibilities may skip to Sec. IV without loss of continuity.

Four double representations for the P mesons are considered; $2\oplus 2$ (SU_2), $4\oplus 4$ (SU_2), $3\oplus 3^*$ (SU_3), and $4\oplus 4$ (C_2). The K and $\bar{K}$ doublets are an example of the $2\oplus 2$ (SU_2) P-meson set. Particle-antiparticle pairs are split between the representations. The possible V-meson representations are of two types. The first type corresponds to P-meson pairs from the opposite P-meson representations; these V-meson irreducible representations occur singly and are denoted by their dimensions, followed by the letter s. The second type refers to pairs from the same P-meson representation; these V-meson representations appear in pairs (though representations of the type defined as single in Sec. II may be involved) and are denoted by twice the dimension of the irreducible representation, followed by the letter d. The possible V-meson representations corresponding to each of the four P-meson sets are listed in Table II.

The crossing matrices have been computed for each of these four cases; in each case only the scheme in which all possible V representations resonate satisfies the bootstrap equation, Eq. (3) or Eq. (4). The crossing matrix for the $3\oplus 3^*$ (SU_3) scheme is given below as an illustration.

$$\begin{array}{c|ccc} & 1s & 8s & 6d \\ \hline 1s & 1 & 8 & -6 \\ 8s & 1 & -1 & 3 \\ 6d & -1 & 4 & 0 \end{array} \times \tfrac{1}{3}.$$

There are many other possible sehemes involving multiple representations. The purpose of this section is only to show that it is more difficult to find self-consistent solutions for a certain type of complicated scheme than it is for the simple type of scheme discussed in Sec. IIIB.

IV. THE P-MESON BOOTSTRAP CONDITION

A. The Self-Consistency Condition

If a particular assumption is made concerning the nature of the force in the 0^- partial wave of the $P+V$ states, a further self-consistency condition is obtained. We assume that the force results from V meson exchange, and that the two interaction vertices are of the VVP type. We continued to assume degenerate multiplets. This model was introduced in Ref. 8 and will be extended here.

Since the pseudoscalar state of two V mesons is symmetric under exchange of the mesons, the VVP interactions may exist only if D_P is contained symmetrically in the reduction of $D_V\otimes D_V$. If this interaction does exist, the direct product $D_P\otimes D_V$ contains both D_V and D_P. The predicted magnitude of the VVP interaction constant cannot be determined unless detailed calculations are made, in which the coupling of the $P+V$ and $P+P$ states of partial wave 1^- is included. However, in order to determine the consistency of any scheme for which the VVP interaction exists, it is sufficient to know the sign of the force and the relevant (PV) crossing matrix. It is shown in Ref. 4 that a *negative* crossing coefficient corresponds to an attractive V-meson exchange force in the 0^- partial wave. Therefore, if D_V and D_P are irreducible, the self-consistency conditions for the bootstrapping of the P mesons are,

$$\begin{aligned} &C_{PV}<C_{iV}, \quad \text{all} \quad i\neq P, \\ &C_{PV}<0, \end{aligned} \tag{5}$$

where the $C_{ii'}$ are the elements of the (PV) crossing matrix defined by Eq. (1).

B. The Extended SU_3 Scheme

The VVP interaction does exist in the 8, 8 (SU_3) scheme, since the reduction of $8\otimes 8$ contains a symmetric 8 as well as an antisymmetric 8. In this scheme the elements of that column of the (PV) crossing matrix corresponding to the exchange of the V octet are

$$\begin{aligned} &C_{8a}=-\tfrac{1}{2}, \quad C_{10}=C_{10^*}=\tfrac{2}{5}, \\ &C_{8a}=-\tfrac{3}{10}, \quad C_1=1, \quad C_{27}=\tfrac{1}{5}, \end{aligned} \tag{6}$$

where the column index is suppressed and the $8s$ and $8a$ denote the symmetric and antisymmetric octets contained in $8\otimes 8$. In this SU_3 scheme the PPV interaction is antisymmetric to the exchange of corresponding V and P mesons, as well as to exchange of the two P mesons. Therefore, the representation $8a$ represents the P mesons and the bootstrap condition of Eq. (5) is satisfied.

In order to check the consistency of the model more thoroughly, we must examine partial waves other than the $1^-(P+P)$ and $0^-(P+V)$ waves, to see if any other resonances are predicted. The $P+P$ states of angular momenta 0 and 2 are being investigated currently; the results will be reported in a later publication. The dispersion integrals associated with the various $P+V$ partial waves are so highly divergent that it is difficult to make a sensible comparison of two arbitrary partial waves.[4] However, if two partial waves correspond to the same orbital angular momentum, a comparison may be made of the values of the Born-approximation amplitudes at the threshold energy. Thus, we may compare the 0^-, $P+V$ amplitudes with the two other P-wave amplitudes of total angular momenta and parities 1^- and 2^-. The symbol B is used to denote the Born-approximation value of the partial-wave amplitude T at the threshold energy, where the elements of T are defined in terms of the corresponding elements of the unitary S matrix by the formula

$$T_{ij}=\frac{3\pi(S_{ij}-\delta_{ij})W\mu(2m-\mu)}{2iq^3}.$$

The quantities W and q are the total energy and particle momentum in the center-of-mass system, and μ and m are the masses of the P and V mesons. The constant factors are included in the definition in order that the expressions for the B be simple. A positive value of B corresponds to an attractive force.

We make the assumption that a P-wave, $P+V$ amplitude corresponding to the partial-wave j and representation α resonates if and only if $B_\alpha(j)\geqq B_{8a}(0^-)$, where $8a$ is the representation of the resonating P mesons. As in Ref. 4, the VVP vertex function is taken to be proportional to $\epsilon_{\mu\nu\lambda\sigma}p_\mu q_\nu$, where p and q denote the four-momenta or the two V mesons, and λ and σ are the polarization indices for the two V mesons. The computation of the B is complicated, but straightforward, and leads to the results

$$\begin{aligned}B_\alpha(0^-)&=-C_\alpha f^2,\\ B_\alpha(1^-)&=(\tfrac{1}{2}+\mu^2/m^2)C_\alpha f^2,\\ B_\alpha(2^-)&=\tfrac{1}{2}C_\alpha f^2,\end{aligned}\tag{7}$$

where C_α is the appropriate element of the crossing matrix, and f^2 is the VVP interaction constant, defined to be equal to the γ^2 of Ref. 4. The calculated value of μ^2/m^2 in the approximate model of Ref. 3 is $\sim\frac{1}{6}$; we assume here only that $0<\mu^2/m^2<\frac{1}{2}$. It is seen from a comparison of Eq. (6) and Eq. (7) that in the 8, 8 (SU_3) scheme, 1^- and 2^- resonances in the $P+V$ states are expected in the singlet representation, and not in any other representations.

Since the corresponding members of the P and V octets are of opposite G parity, the predicted unitary singlets are of odd G parity. The singlet V meson and the isoscalar member of the V octet may be identified with the physical ω and φ mesons, although it is not clear what linear combinations of the theoretical particles should correspond to the experimental particles. To the author's knowledge, there is no evidence for a unitary singlet meson of spin and parity 2^-. We conclude that the model predicts the existence of the seventeen particles listed in Sec. I, plus one other (probably nonexistent) particle.

In a consistent bootstrap model, all resonances must be included in the input. Therefore, we must consider the effects of the V singlet on the fundamental scattering states and forces. The spin-2 meson is neglected, since no attempt has been made to be complete in the sense of including all significant partial waves. There is no PPV interaction involving the V singlet, since the identity representation does not occur antisymmetrically in the direct product $8\otimes 8$. Therefore, the presence of the V singlet does not affect the argument of Sec. III, concerning the bootstrapping of the V octet from $P+P$ states.

The only interaction involving the singlet V meson in the model is the $V_1V_8P_8$ interaction, where the subscripts denote the dimensions of the representations involved. This interaction does affect the argument concerning the bootstrapping of the P octet, V singlet, and 2^- singlet from the $V+P$ states. We consider first the force in the P_8+V_8 state resulting from the exchange of the V singlet. The elements of the singlet column of the (PV) crossing matrix are

$$\begin{aligned}C_{8s,1}&=C_{1,1}=C_{27,1}=\tfrac{1}{8},\\ C_{8a,1}&=C_{10,1}=C_{10^*,1}=-\tfrac{1}{8}.\end{aligned}\tag{8}$$

Since an attractive force corresponds to a negative crossing coefficient for the 0^- partial wave, and to a positive crossing coefficient for the 1^- partial wave, the V-singlet exchange force is attractive for both the 0^- antisymmetric octet and the 1^- singlet. The V-meson exchange force is not more attractive in any other 0^- or 1^- states, so its presence does not alter the previous conclusions concerning which representations resonate.

The threshold values of the Born-approximation amplitudes in the resonating P octet and V singlet states are

$$\begin{aligned}B_{8a}(0^-)&=\tfrac{1}{2}f^2+g^2,\\ B_1(1^-)&=(\tfrac{1}{2}+\mu^2/m^2)(f^2+g^2),\end{aligned}\tag{9}$$

where g^2 is the $V_1V_8P_8$ interaction constant. [The crossing factor $\frac{1}{8}$ of Eq. (8) has been absorbed in the definition of g^2.] If g^2/f^2 is very small, then $B_1(1^-)>B_{8a}(0^-)$; i.e., the force is strongest in the V-singlet state. However, if g^2/f^2 is sufficiently great, the attractive force is greatest in the $8a(0^-)$ state. Since the P octet actually is lighter than the V singlet, the presence of the V-singlet exchange force improves the agreement of the model with experiment. The threshold amplitude for the partial-wave 2^- singlet is $B_1(2^-)=\frac{1}{2}(f^2+g^2)$, so the predicted mass of this particle is greater than those of the P octet and V singlet for any nonzero value of g^2/f^2.

It is necessary also to consider the effect of the P_8+V_1 states. These states are not coupled to the resonating singlet and antisymmetric octet combinations of the P_8+V_8, but only to the symmetric octet states.[13] Thus, the Born-approximation amplitudes (forces) in the symmetric octet, $P+V$ states are represented by two-by-two matrices denoted by $\mathbf{B}_{8s}$. In such a situation the energy of the lowest predicted resonance is inversely proportional to the largest positive eigenvalue of $\mathbf{B}_{8s}$. We consider first the 0^- partial wave. For self-consistency, it is necessary that there be no positive eigenvalue of $\mathbf{B}_{8s}(0^-)$ larger than $B_{8a}(0^-)=\frac{1}{2}f^2+g^2$. It can be shown that $\mathbf{B}_{8s}(0^-)$ is given by

$$\mathbf{B}_{8s}(0^-)=\begin{pmatrix} \frac{3}{10}f^2-g^2 & fg \\ fg & -g^2 \end{pmatrix},$$

where the first row and column refer to the P_8+V_8 states, and the second row and column refer to the P_8+V_1 states. It is easy to verify that for any value of g^2/f^2, the largest eigenvalue of the matrix $\mathbf{B}_{8s}(0^-)$ is smaller than $\frac{1}{2}f^2+g^2$. Hence, the assumption of no pseudoscalar resonance in the symmetric octet is justified. The ratio of the largest eigenvalue of $\mathbf{B}_{8s}(0^-)$ to $\frac{1}{2}f^2+g^2$ is smaller when $\frac{1}{2}f^2$ and g^2 are comparable then when g^2 is zero. Therefore, the assumption of an appreciable $V_1V_8P_8$ interaction improves the model by leading to a larger difference between the forces in the resonating and nonresonating octet states.

We now consider the partial-wave 1^-, symmetric octet state. This is the state in which the V-octet resonance develops. In order to treat the model completely, it would be necessary to consider the coupling of the V-octet resonance to the P_8+V_8 (symmetric octet) and the P_8+V_1 state, as well as to the P_8+P_8 state. Such a treatment is not attempted here.

The experimental measurement of the small partial width for the decay $\varphi\to\rho+\pi$ indicates that the interaction constant $\gamma_{\varphi\rho\pi}$ is small.[14] A small $\gamma_{\varphi\rho\pi}$ does not imply that the $V_1V_8P_8$ interaction constant is small, since the experimental φ may be a mixture of the unitary singlet and isoscalar member of the V octet.[15]

[13] The allowed couplings of the P_8, V_8 and V_1 may be understood simply from the facts that the G parity of the V singlet is odd, and the G parities of corresponding members of the P and V octets are opposite. Thus, the antisymmetric octet combination of P_8+V_8 has the G parity of the P octet and is not coupled to P_8+V_1.

[14] P. L. Connolly, E. L. Hart, K. W. Lai, G. London, G. C. Moneti, *et al.*, Phys. Rev. Letters **10**, 371 (1963).

[15] See the discussion of S. L. Glashow, Phys. Rev. Letters **11**, 48 (1963).

C. Groups Other than SU_3

An interaction of the type VVP does not exist for the double-representation schemes discussed in Sec. IIIC, and for many of the schemes listed in Table I. It turns out that for those listed schemes that allow the existence of the VVP interaction, the P-meson bootstrap condition of Eq. (5) is satisfied. These schemes are indicated with a "yes" in the P-bootstrap column of Table I.

The scheme involving the fewest particles that satisfies both the V-bootstrap and P-bootstrap tests is the 5, 3 (SU_2) scheme. The elements of that column of the (PV) crossing matrix corresponding to the exchange of a V triplet in this scheme are

$$C_3=\tfrac{1}{10},\quad C_5=-\tfrac{3}{10},\quad C_7=\tfrac{3}{5},\tag{10}$$

where the column index is suppressed. We follow the procedure of Sec. IVB, using Eq. (7) to see if additional resonances are predicted in the 1^- or 2^- partial waves. The approximate value $\mu^2/m^2\sim\frac{1}{6}$ calculated in Ref. 3 for the 8, 8 (SU_3) scheme applies also to the 5, 3 (SU_2) scheme [and to all other schemes in which the appropriate element of the (PP) crossing matrix is unity]. It is seen that resonances are predicted in the 1^- and 2^- septet states. The consistency of this system when the V septet is included in the input has not been investigated. However, even if the extended 5, 3 scheme is consistent, it now involves 22 particles and is more complicated than the SU_3 scheme.

On the other hand, if similar considerations are applied to the 5, 7 (SU_2) and 5, 10 (C_2) schemes, no additional 1^- or 2^- resonances are predicted.

V. SELF-CONSISTENCY FOR NONDEGENERATE MULTIPLETS

In this section we study the possibility of solutions involving nondegenerate multiplets. It is hoped that no such solutions exist for many of the group-representation schemes, so that these schemes will be eliminated theoretically, if an argument for the necessity of deviations from degeneracy can be found.

The masses of the P and V mesons are denoted by μ and m, respectively; μ_0 and m_0 are the values corresponding to the solution in which the P and V multiplets are each degenerate (the "degeneracy solution"). The symbols δ_i and Δ_j denote the fractional deviations in the squares of the masses of the P-meson i and the V-meson j, i.e.

$$\delta_i=(\mu_i^2-\mu_0^2)/\mu_0^2 \quad\text{and}\quad \Delta_j=(m_j^2-m_0^2)/m_0^2.$$

We consider first the amplitudes in which the P-meson poles occur. The wave function associated with the pole of P_i is of the form $\psi(P_i)=\sum_{jk}(P_jV_k)\beta_{ijk}$, where the β_{ijk} are real coefficients. The mass deviations δ_i depend on the mass deviations of the particles in the wave-function $\psi(P_i)$ and of the virtual particles that transmit the forces. It is assumed that the deviations are sufficiently small that an expansion in powers of δ_i and Δ_i may be made, and terms of order greater than the second neglected. The equation for δ_i, determinable from the dispersion relations for the bound-state P_i, is of the form

$$\delta_i=\sum_j X_{ij}\delta_j+\sum_j Y_{ij}\Delta_j+O_2\,, \tag{11}$$

where the X_{ij} and Y_{ij} are real numbers and O_2 represents symbolically the terms of second order. A similar equation for the Δ_i may be determined from the dispersion relations involving the V poles. Since $\psi(V_i)$ is of the form $P+P$, we omit terms involving the Δ_j in the right side of the equation, i.e.,

$$\Delta_i=\sum_j Z_{ij}\delta_j+O_2\,. \tag{12}$$

Combination of Eqs. (11) and (12) leads to self-consistency equations for the δ_i and Δ_i. A solution other than the degeneracy solution, $\delta_i=\Delta_i=0$, is desired.

Accurate calculations of the X_{ij}, Y_{ij}, Z_{ij} and O_2 cannot be made, because of the incomplete nature of the bootstrap model and the inexact state of present-day dispersion theory. Furthermore, the equations resulting from detailed calculations would be rather complicated, while our aim is to find a simple criterion that may be applied easily to any group-representation scheme. Therefore, we will attempt to find a simple, but plausible, approximation to Eqs. (11) and (12).

The type of approximation to be used is based on the results of Ref. 9. In this reference the values of the Δ_i resulting from assumed values of the δ_j are calculated for the 8, 8 (SU_3) scheme from the bootstrap equations for the partial-wave 1^- amplitudes. A rough, approximate rule for the result is that the factor Z_{ij} of Eq. (12) of the present paper may be replaced by the probability of the P-meson P_j in the degeneracy-solution wave function $\psi(V_i)$. The basic reason for the approximate validity of this rule is that in the self-consistent dispersion relations for the V poles, the partial derivative of $\omega(V_i)$ with respect to μ_j^2 is proportional to this probability, where $\omega(V_i)$ is the energy of the zero in the resonance denominator corresponding to V_i.

The above considerations suggest that the X, Y, and Z coefficients of Eqs. (11) and (12) be replaced as follows,

$$X_{ij}=\alpha_{PP}\theta(P_iP_j)\,,\quad Y_{ij}=\alpha_{PV}\theta(P_iV_j)\,,\quad Z_{ij}=\alpha_{VP}\theta(V_iP_j)\,, \tag{13}$$

where $\theta(\alpha_i\beta_j)$ denotes the probability of the meson β_j in the degeneracy-solution expression for $\psi(\alpha_i)$, and α_{PP}, α_{PV}, and α_{VP} are positive constants. The effect of mass splitting in the virtual multiplets that transmit the forces is neglected in this approximation. Since absolute masses cannot be determined from the dispersion relations, the assumption that all the δ_i and Δ_i are equal must lead to a solution of both Eqs. (11) and (12). It follows from this condition that if the probabilities are normalized according to the convention $\sum_j\theta(\alpha_i\beta_j)=1$, the coefficients α of Eq. (13) must satisfy the equations

$$\alpha_{VP}=1\,,\quad \alpha_{PP}+\alpha_{PV}=1\,. \tag{14}$$

The basic idea of expanding in powers of δ_i may seem unrelated to reality, since the actual deviation of the P and V mesons from degeneracy are not small. However, in the present calculation the small deviation requirement is that the ratios of the interaction constants should not differ greatly from their values in the degeneracy solution, in order that the use of degeneracy-solution wave functions be justified. In the calculation of Ref. 9, it is shown that the relative deviations of the coupling-constant ratios are much smaller than the relative P-mass deviations. Furthermore, the success of many of the experimental predictions of unitary symmetry supports the point of view that the symmetry-breaking may be regarded as a perturbation.

It is convenient to regard the θ functions of Eq. (13) as rectangular matrices and the δ_i, Δ_i, and O_2 as column matrices. Boldface symbols will be used to represent the matrices. The matrix elements of the $\boldsymbol{\theta}$ are $\boldsymbol{\theta}_{VP,ij}=\theta(V_iP_j)$, etc. If Eqs. (12), (13), and (14) are substituted into Eq. (11), the resulting equation, in matrix notation is

$$\boldsymbol{\delta}=[(1-\alpha_{PV})\boldsymbol{\theta}_{PP}+\alpha_{PV}\boldsymbol{\theta}_{PV}\boldsymbol{\theta}_{VP}]\boldsymbol{\delta}+\mathbf{O}_2\,. \tag{15}$$

We assume that the mass of a particle is equal to that of its antiparticle, so the dimension of $\boldsymbol{\delta}$ and of the square matrices $\boldsymbol{\theta}_{PP}$ and $\boldsymbol{\theta}_{PV}\boldsymbol{\theta}_{VP}$ is the sum of the number of self-conjugate particles and the number of particle-antiparticle pairs in the set of P mesons.

The elements of the $\boldsymbol{\theta}$ may be determined by using standard methods of group theory.[10] As an illustration, we write the wave-functions $\psi(P_i)$ corresponding to the 8, 8 (SU_3) scheme.[16]

$$\begin{aligned}
\psi(\eta)=&-(\tfrac{1}{4})^{1/2}(K^+\bar{M}^-)-(\tfrac{1}{4})^{1/2}(K^0\bar{M}^0)\\
&+(\tfrac{1}{4})^{1/2}(\bar{K}^0M^0)+(\tfrac{1}{4})^{1/2}(\bar{K}^-M^+)\,,\\
\psi(\pi^0)=&-(\tfrac{1}{3})^{1/2}(\pi^+\rho^-)+(\tfrac{1}{3})^{1/2}(\pi^-\rho^+)-(\tfrac{1}{12})^{1/2}(K^+\bar{M}^-)\\
&+(\tfrac{1}{12})^{1/2}(K^0\bar{M}^0)-(\tfrac{1}{12})^{1/2}(\bar{K}^0M^0)\\
&+(\tfrac{1}{12})^{1/2}(\bar{K}^-M^+)\,,\\
\psi(\pi^+)=&(\tfrac{1}{3})^{1/2}(\pi^+\rho^0)-(\tfrac{1}{3})^{1/2}(\pi^0\rho^+)-(\tfrac{1}{6})^{1/2}(K^+\bar{M}^0)\\
&+(\tfrac{1}{6})^{1/2}(\bar{K}^0M^+),\\
\psi(K^+)=&(\tfrac{1}{4})^{1/2}(K^+\varphi)-(\tfrac{1}{4})^{1/2}(\eta M^+)+(\tfrac{1}{12})^{1/2}(K^+\rho^0)\\
&+(\tfrac{1}{6})^{1/2}(K^0\rho^+)-(\tfrac{1}{12})^{1/2}(\pi^0M^+)\\
&-(\tfrac{1}{6})^{1/2}(\pi^+M^0)\,,\\
\psi(K^0)=&(\tfrac{1}{4})^{1/2}(K^0\varphi)-(\tfrac{1}{4})^{1/2}(\eta M^0)-(\tfrac{1}{12})^{1/2}(K^0\rho^0)\\
&+(\tfrac{1}{6})^{1/2}(K^+\rho^-)+(\tfrac{1}{12})^{1/2}(\pi^0M^0)\\
&-(\tfrac{1}{6})^{1/2}(\pi^-M^+)\,.
\end{aligned} \tag{16}$$

[16] The construction of the wave functions corresponding to the various representations in the direct product $8\otimes 8$ of SU_3 is discussed by S. L. Glashow and J. J. Sakurai, Nuovo Cimento **25**, 337 (1962).

The wave functions for the π^-, K^-, and $\bar{K}^0$ may be determined by applying the charge-conjugation operator to $\psi(\pi^+)$, $\psi(K^+)$ and $\psi(K^0)$. The matrix $\boldsymbol{\theta}_{PP}$, determined from the above equations, is

$$\boldsymbol{\theta}_{PP}=\begin{array}{c} \\ \eta \\ \pi^0 \\ \pi^\pm \\ K^\pm \\ K^0\bar{K}^0 \end{array}\begin{array}{c}\begin{array}{ccccc}\eta & \pi^0 & \pi^\pm & K^\pm & K^0\bar{K}^0\end{array}\\ \begin{bmatrix} 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 8 & 2 & 2 \\ 0 & 4 & 4 & 2 & 2 \\ 3 & 1 & 2 & 4 & 2 \\ 3 & 1 & 2 & 2 & 4 \end{bmatrix}\end{array}\times\frac{1}{12}.$$

In the 8, 8 (SU_3) scheme, the PPV interaction is antisymmetric to the exchange of corresponding V and P mesons, as well as to the exchange of P mesons. Because of this fact, the product-matrix $\boldsymbol{\theta}_{PV}\boldsymbol{\theta}_{VP}$ is equal to the square of $\boldsymbol{\theta}_{PP}$ in this scheme.

For each group-representation scheme, a complete set of eigenvectors and eigenvalues of $\boldsymbol{\theta}_{PP}$ is defined by the equation $\boldsymbol{\theta}_{PP}\boldsymbol{\delta}_i=a_i\boldsymbol{\delta}_i$. It can be shown that the $\boldsymbol{\delta}_i$ are eigenvectors of the product matrix $\boldsymbol{\theta}_{PV}\boldsymbol{\theta}_{VP}$ also, i.e., $\boldsymbol{\theta}_{PV}\boldsymbol{\theta}_{VP}\boldsymbol{\delta}_i=b_i\boldsymbol{\delta}_i$. Equation (15) may be written as a set of equations in the $\boldsymbol{\delta}_i$, i.e.,

$$c_i\boldsymbol{\delta}_i=\mathbf{O}_{2,i}\,, \tag{17}$$

$$c_i=1-(1-\alpha_{PV})a_i-\alpha_{PV}b_i. \tag{18}$$

For every scheme the trivial set of eigenvalues $a=1$, $b=1$ exists, and corresponds to equality of all the components of $\boldsymbol{\delta}_i$, i.e., to no breaking of the degeneracy. The nontrivial eigenvalue sets have been computed for many schemes and are listed in the "Roots (a,b)" column of Table I. The symbol $(i,j)^n$ means that $a=i$, $b=j$ is an n-fold root. Since no assumption is made in this section with regard to the virtual multiplet that transmits the forces (except that deviations from degeneracy in the multiplet may be neglected) the present considerations apply to a more general model than that considered in Secs. III and IV. Therefore, the roots (a,b) are computed for many schemes that violate one or both of the bootstrap tests.

A mass-deviation $\boldsymbol{\delta}$ that satisfies the second-order expansion of the dispersion relations is most likely to approximate an actual solution to the unexpanded equations if $|\boldsymbol{\delta}|$ is small. The second-order terms $\mathbf{O}_{2,i}$ cannot be determined without detailed calculations. However, the magnitude of a $\boldsymbol{\delta}$ satisfying Eq. (17) is smallest if the $\boldsymbol{\delta}_i$ that participate appreciably in the deviation all correspond to small $|c_i|$. It is seen from Eq. (18) that the condition $0<\alpha_{PV}<1$, together with the fact that the value of all the nontrivial a_i and b_i are less than one, implies that $c_i>0$. Therefore, c_i is smallest when both a_i and b_i are appreciable and positive. It is seen from Table I that all the eigenvalues b_i are in the range $0\leqq b_i<\frac{1}{2}$, while the a_i are of both signs. We assume that α_{PV} is either about $\frac{1}{2}$ or smaller; i.e., that P-mass deviations are of comparable or greater importance than V-mass deviations in the equations for the P-meson poles. It is conjectured that a mass-deviation eigenvector $\boldsymbol{\delta}_i$ cannot participate appreciably in an actual solution if a_i is negative. This conjecture leads to the following self-consistency requirement for the existence of a nondegenerate solution corresponding to a particular group-representation scheme:

$$a_i>0 \quad \text{for at least one } i. \tag{19}$$

The physical meaning of this condition is that large masses of the P mesons in the wave-function $\psi(P_j)=P+V$ should lead to a large μ_j^2.

It is seen from Table I that for many of the schemes considered, there are no positive, nontrivial values of a_i. In fact, the only schemes satisfying Eq. (19), as well as the two bootstrap conditions of Secs. III and IV, are the 5, 3 (SU_2) and 8, 8 (SU_3) schemes. It is shown in Sec. IVC that the 5, 3 scheme is not simpler than the 8, 8 scheme.

For any set of P mesons, the V-bootstrap condition of Section III is satisfied if all antisymmetric P-wave states resonate. We now show that this general scheme does not satisfy the nondegeneracy consistency requirement, Eq. (19). Let n_1 denote the number of self-conjugate P mesons, and n_2 the number of particle-antiparticle pairs, so that there are n_1+n_2 independent P-meson masses. In the wave-function $\psi(P_i)$, the antiparticle $\bar{P}_i$ is absent ($\bar{P}_i$ is P_i if i refers to a self-conjugate particle, of course) and all other P_j occur with equal probability. If we now use the indices i and j to refer only to self-conjugate particles, and k and l to refer only to particle-antiparticle pairs, the matrix elements of $\boldsymbol{\theta}_{PP}$ have the simple form

$$\begin{aligned}&\boldsymbol{\theta}_{PP,ii}=0\,,\quad \boldsymbol{\theta}_{PP,ik}=2y\,,\\ &\boldsymbol{\theta}_{PP,ij}=y \quad \text{for}\quad i\neq j\,,\\ &\boldsymbol{\theta}_{PP,kk}=y\,,\quad \boldsymbol{\theta}_{PP,ki}=y\,,\\ &\boldsymbol{\theta}_{PP,kl}=2y \quad \text{for}\quad k\neq l\,,\\ &\quad y=(n_1+2n_2-1)^{-1}.\end{aligned}$$

It is seen that all rows of the matrix $\boldsymbol{\theta}_{PP}+y\mathbf{1}$ are identical. This is the condition that $a_i=-y$ is an (n_1+n_2-1) fold eigenvalue of $\boldsymbol{\theta}_{PP}$. However, there are only n_1+n_2-1 nontrivial roots, so no positive, nontrivial a_i exists. [The $m+n-1$ fold eigenvalue of $\boldsymbol{\theta}_{PV}\boldsymbol{\theta}_{VP}$ in this scheme is $b=\frac{1}{2}(1-y)$.]

The largest values of a_i and b_i for the double-octet scheme are $a=\frac{1}{2}$, $b=\frac{1}{4}$. The components of the eigenvector $\boldsymbol{\delta}_i$ corresponding to this root are

$$\begin{aligned}&\delta_{K^\pm}=\delta_{K^0}\,,\\ &\delta_\eta=2\delta_{K^0}\,,\\ &\delta_{\pi^\pm}=\delta_{\pi^0}=-2\delta_{K^0}.\end{aligned} \tag{20}$$

These equations correspond to the conservation of isotopic spin[17] and to the famous Gell-Mann-Okubo sum rule.[18] There is one other deviation from degeneracy involving the conservation of isotopic spin, corresponding to the unfavorable root ($a=-\frac{1}{3}$, $b=\frac{1}{9}$). This deviation, $\delta_\eta=-3\delta_K$, $\delta_\pi=-\frac{1}{3}\delta_K$, is associated with the hypercharge zero, I-spin zero operator of the 27-fold representation.[19] The conclusion concerning the favorability of the Okubo type deviation is similar to that obtained for the V-meson model of Cutkosky and Tarjanne,[7] because the VVV interaction in the V-meson model and the PPV interaction of the present model are both totally antisymmetric in the octet scheme of SU_3. [The PPV interaction is totally antisymmetric for the 7, 7 (G_2) and 3, 3 (SU_2) schemes also.]

Similar self-consistency criteria for nondegenerate multiplets may be applied to other models. In order to illustrate this, we modify the present model by assuming that the dominant states in the wave functions for the P mesons are of the type $V+V$ rather than $P+V$. In the SU_3 scheme, the appropriate wave-function $\psi'(P_i)$ are

$$\psi'(\eta)=(\tfrac{1}{5})^{1/2}(\varphi\varphi)-(\tfrac{2}{5})^{1/2}(\rho^+\rho^-)-(\tfrac{1}{5})^{1/2}(\rho^0\rho^0)+(\tfrac{1}{10})^{1/2}(M^+\bar{M}^-)+(\tfrac{1}{10})^{1/2}(M^0\bar{M}^0),$$
$$\psi'(\pi^+)=-(\tfrac{2}{5})^{1/2}(\rho^+\varphi)-(\tfrac{3}{5})^{1/2}(M^+\bar{M}^0),$$
$$\psi'(K^+)=(\tfrac{1}{10})^{1/2}(\varphi M^+)-(\tfrac{3}{10})^{1/2}(\rho^0 M^+)-(\tfrac{3}{5})^{1/2}(\rho^+ M^0).$$

The other $\psi'(P_i)$ may be determined by applying the isotopic lowering operator and charge conjunction operator to these equations. The ij component of the probability matrix $\boldsymbol{\theta}_{PV}'$ is defined to be the probability of V_j in the wave-function $\psi'(P_i)$. The significant roots a_i for this model are defined by the eigenvalue equation $\boldsymbol{\theta}_{PV}'\boldsymbol{\theta}_{VP}\boldsymbol{\delta}_i=a_i'\boldsymbol{\delta}_i$, where $\boldsymbol{\theta}_{VP}$ is the matrix defined below Eq. (13). A procedure similar to that used before leads to Eq. (17) with the quantity c_i given by

$$c_i=(1-a_i').$$

A positive a_i' is desired. It is easy to verify that for the 8, 8 (SU_3) scheme all four nontrivial roots a_i' are negative, so no type of nondegenerate solution appears likely.

This example does not show that a nondegenerate bootstrap model in which the $V+V$ states are dominant in the 0^- partial waves is impossible, only that in such a model the 8, 8 (SU_3) scheme would not likely be a solution. The example also illustrates that the favorability of a particular group-representation scheme and a particular type of mass splitting may depend critically on the types of configurations and interactions assumed important. We believe that realistic sets of particles must be considered if we are to find out why a particular group and type of mass splitting are realized in nature.

The magnitude and sign of a deviation from degeneracy cannot be determined unless terms of order greater than the first in the δ_i are calculated. At this point we make some extremely speculative remarks concerning the possible effects of higher order terms on a deviation from degeneracy of the Okubo type for the SU_3 scheme. The remarks are based on a comparison of the calculations of Ref. 9 (a first-order calculation of the dependence of the Δ_i on the δ_i in the bootstrap model of the V octet) with that of Ref. 20 (a similar calculation of the mass of the M meson only, but including all orders in the deviations from P-meson degeneracy). The two configurations $(\pi+K)$ and $(\eta+K)$ contribute to the M-meson amplitude. The most significant difference between the results of Ref. 9 and Ref. 20 is that the calculated M mass is higher in the first-order calculation for assumed physical values of the P masses. This effect results primarily from the fact that when the masses of the $(\pi+K)$ and $(\eta+K)$ states are taken to be very different, the probability of the state with the lighter mass in the M-wave function is appreciably greater than its value in the degeneracy solution, so that the second effect of the large splitting is to decrease the M mass.

For the sake of argument we assume that the most important type of second-order effect in the present bootstrap model is the type discussed above. We imagine a series of calculations in which progressively larger deviations of the Okubo type are assumed for the P-meson masses, the V masses are then calculated from the equations for the 1^- amplitudes, and then the P masses are calculated from the 0^- amplitudes. The subtraction energy (or cutoff parameter) is adjusted so that the average of the squares of the calculated P masses is equal to the average originally assumed. Therefore, consistency is obtained if the calculated π and η masses agree with those originally assumed. The degeneracy-solution wave functions for the π and η are

$$\psi(\eta)=(\tfrac{1}{2})^{1/2}(K\bar{M})_0+(\tfrac{1}{2})^{1/2}(\bar{K}M)_0,$$
$$\psi(\pi)=(\tfrac{2}{3})^{1/2}(\pi\rho)_1+(\tfrac{1}{6})^{1/2}(K\bar{M})_1-(\tfrac{1}{6})^{1/2}(\bar{K}M)_1,$$

where the subscript denotes the isotopic spin. Clearly, there are no states of different mass in $\psi(\eta)$. However, since the signs of $m_\rho^2-m_M^2$ and $\mu_\pi^2-\mu_K^2$ are the same in the present model, the second-order effect may be large for the π pole. For a very small deviation, the inequality $c_i>0$ [where c_i is defined in Eq. (18)] implies that the calculated δ_π is smaller in absolute magnitude than that originally assumed. The second-order effect discussed above implies a negative second

[17] It is well known that SU_2 can be contained as a subgroup of SU_3 in three different ways, corresponding to the choice of the $\pi^\pm$, or the $K^\pm$, or the K^0 and $\bar{K}^0$ as "outer" members of the SU_2 triplet. [See C. A. Levinson, H. J. Lipkin, and S. Meshkov, Nuovo Cimento **23**, 236 (1962).] The fact that the $\pi^\pm$ are singled out as members of the isotopic triplet in Eq. (20) results from our identification of the π^0 and η as distinct particles. Had we chosen as distinct particles either of the combinations, $\pi'=\frac{1}{2}[(3)^{\frac{1}{2}}\eta\pm\pi]$, $\eta'=\frac{1}{2}[(3)^{\frac{1}{2}}\pi\mp\eta]$, different SU_2 multiplets would have resulted.

[18] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[19] This alternate mass formula is discussed by J. J. deSwart, CERN Report No. 6488/TH. 345, 1963 (unpublished).

[20] R. H. Capps, Phys. Rev. **131**, 1307 (1963).

derivative of the calculated μ_π^2 with respect to the assumed $|\delta_i|$. Therefore, as larger $\pi-K$ mass differences are assumed, consistency for μ_π^2 is not likely to be obtained if $\mu_\pi^2>\mu_K^2$. If consistency with respect to μ_π^2 is obtained, one can attempt to obtain consistency with respect to μ_η^2 also by varying the ratio $R=(\mu_\eta^2-\mu_K^2)/(\mu_K^2-\mu_\pi^2)$ from the value given by the Okubo formula.[18] This plausibility argument suggests two conclusions: (1) It is not at all clear that a nondegenerate solution exists. (2) If one does exist, it is likely that $\mu_\pi^2-\mu_K^2<0$ and that the value of R is smaller than that given by the Okubo formula. Clearly, this speculation should be checked by detailed calculations based on a simplified form of the bootstrap model. If the question of the existence of a nondegenerate solution is to be investigated, *it will not be sufficient to include in the calculations only terms of first and second order in the* δ_i.

VI. THE DOUBLE-SEPTET SCHEME OF G_2

Recently, Behrends and Landovitch have pointed out that present experimental data does not rule out the hypothesis that the mesons and baryons are associated with representations of the group G_2.[21] Under this hypothesis the π, K, ρ, and M mesons may be associated with the 7, 7 (G_2) scheme. It is seen from Table I that this scheme violates both the V-meson and P-meson consistency tests, as well as the consistency test for mass splitting of Eq. (19).[22] The 7, 14 scheme of G_2 also violates the P-bootstrap condition, as may be seen from the (PP) crossing matrix of G_2 given in Sec. IIIB.

It is instructive to consider a modified form of the model for the double-septet scheme, a form in which the dominant force in the P-wave, $P+P$ state is assumed particularly attractive in the resonating septet state. The particle multiplet transmitting this force, and the nature of the partial-wave 0^- amplitudes, are unspecified. If the Z_{ij} of Eq. (12) may be replaced by $\theta(V_iP_j)$, it is easy to show that a positive K-π mass difference leads to a negative M-ρ mass difference, in contradiction with experiment. We conclude that the G_2 scheme is difficult to reconcile with a simple bootstrap model, self-consistent, and consistent with experiment.

Sakurai has pointed out that if degenerate P- and V-meson octets of the isotopic spins and hypercharges appropriate to SU_3 are assumed, the requirements that the second-order effects of the PPV interactions leave both multiplets degenerate lead to the same interaction-constant ratios as does the simple V-meson bootstrap model (the ratios corresponding to SU_3).[23] The bootstrap requirements are more general than those of Sakurai, since degeneracy assumptions are not necessary in the bootstrap model.[9] Nevertheless, it is interesting to ask whether or not Sakurai's criteria are equivalent to the bootstrap criteria if consideration is limited to the coupling-constant ratios predicted when the multiplets are degenerate. We investigate this question by applying Sakurai's criteria to the 7, 7 (G_2) scheme.[24] The three interaction constants $\gamma_{\rho\pi\pi}^2$, $\gamma_{\rho KK}^2$, and $\gamma_{M\pi K}^2$ are defined in the same manner as in Ref. 3; i.e., γ_{ijk}^2 is proportional to the partial width for decay of the V-meson i into the $P+P$ state $j+k$. The requirement that V-meson degeneracy is maintained to second order is

$$\gamma_{\rho\pi\pi}^2+\gamma_{\rho KK}^2=\gamma_{M\pi K}^2,$$

and the requirement that P-meson degeneracy is maintained leads to the equation

$$2\gamma_{\rho\pi\pi}^2+\tfrac{4}{3}\gamma_{M\pi K}^2=\tfrac{3}{2}\gamma_{\rho KK}^2+\gamma_{M\pi K}^2.$$

These equations predict $\gamma_{\rho\pi\pi}^2:\gamma_{\rho KK}^2:\gamma_{M\pi K}^2=1:2:3$, which are the ratios corresponding to the G_2 group. However, as discussed above, this scheme does *not* satisfy the V-meson bootstrap condition, so this condition clearly is not equivalent to those of Sakurai.

VII. CONCLUSIONS

There are three main results of this paper. The first is that a second isoscalar, hypercharge-zero, vector meson arises naturally in an extension of the double-octet bootstrap model of Ref. 4, and the inclusion of this V meson improves the correspondence of the model with experiment.

The second result concerns the possibility that the self-consistency requirements of bootstrap-dispersion relations may specify uniquely the set of strongly interacting particles found in nature. In this paper a particular bootstrap model of the pseudoscalar and vector mesons is considered. Two self-consistency conditions are obtained from the requirements that the forces are more attractive in the resonating states than in the other states of angular momenta and parity 1^- and 0^-. These conditions are formulated for the case of degenerate multiplets; it is clear, however, that their implications are not changed essentially if small deviations from degeneracy are assumed. A third self-consistency condition is conjectured, applicable only if nondegenerate multiplets are assumed. If all three conditions are required, and it is required that mesons of the same spin arising from the same type of configuration correspond to an irreducible representation of a simple Lie group of first, second, or third rank, there

[21] R. E. Behrends and L. F. Landovitz, Phys. Rev. Letters **11**, 296 (1963).

[22] This conclusion concerning the V-bootstrap test is not new. It is contained in Ref. 4, where all subsets of the P and V octets are examined. Furthermore, since the PPV interaction is totally antisymmetric for the 7, 7 scheme, this scheme has many of the algebraic properties of the V-meson schemes considered by Cutkosky in Ref. 5. The theorem of Ref. 5 shows that this scheme violates the bootstrap condition, since 7 is not the regular representation of G_2.

[23] J. J. Sakurai, Phys. Rev. Letters **10**, 446 (1963).

[24] Making this comparison was suggested to the author by Professor L. M. Brown.

is no solution involving fewer particles than the double-octet solution of unitary symmetry. The 5, 3 (SU_2) scheme, discussed in Sec. IVC, may lead to a solution of simplicity comparable to that of the double-octet scheme.

The third result is that deviations from degeneracy of the Okubo type are favored in the double-octet scheme.

The model is incomplete in several aspects. The various assumptions concerning the partial waves and configurations that are important have not been checked with detailed, dispersion-theoretic calculations. The criterion used for nondegenerate solutions is crude; its chief virtue is its simple applicability. Furthermore, no reason has been given why the particle multiplets should be nondegenerate. There is no compelling reason, other than simplicity, for the neglect of the baryon-antibaryon states. In fact, it is hoped that in more accurate bootstrap models the baryons will be necessary.

Thus, even if our basic assumption is right, i.e., that nature chooses the simplest self-consistent set of particles, the true consistency criteria may be quite different from those assumed here. The primary purpose of this paper is to demonstrate the falseness of the common assumption that if simple representations of one Lie group satisfy a particular bootstrap model, simple representations of any other Lie group must satisfy a similar model. The consistency criteria of Secs. III, IV, and V are examples of plausible criteria that distinguish between different group-representation schemes.

PHYSICAL REVIEW VOLUME 134, NUMBER 2B 27 APRIL 1964

Translational Inertial Spin Effect with Moving Particles

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(Received 8 July 1963; revised manuscript received 22 November 1963)

Although the final conclusion of a preceding paper was incorrect, as we shall explain, the main point remains, and should entail the existence of *sui generis* recoil effects associated with nonzero values of curl $\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$ is the spin density). These should be observed by testing, not with solids as was previously proposed, but with the probability fluids associated with moving particles; this more refined type of experiment should be able to select, among the set of integrally equivalent energy-momentum tensors, the one describing locally the true or physical energy-momentum flux. In this paper it is shown, by an explicit calculation, that cylindrical type solutions of the extreme relativistic Dirac equation exist with no z dependence of the wave function (and thus no k_z component of the momentum) but still with a z component of the Dirac probability current; as this conclusion is reached with a t dependence of the wave function of strictly the form $\exp(-iWt/\hbar)$, there is no question of having to perform a Foldy-Wouthuysen transformation to extract the positive energy contribution (or equivalently, to use the Newton-Wigner position operator). The "transverse inertial spin effect" we predict is locally described by the flux of the Dirac current per time dt and surface $d\mathbf{s}$, and corresponds to the local transition probabilities between the dynamical state of the beam and a pointlike localization of the incident particles.

I. INTRODUCTION

IN a preceding paper[1] it has been argued that the true, physical, energy-momentum tensor associated with a spin-$\frac{1}{2}$ wave is Tetrode's asymmetrical tensor

$$T^{ij}=-\tfrac{1}{2}c\hbar\bar{\psi}[\partial^i]\gamma^j\psi+ieA^i\bar{\psi}\gamma^j\psi \tag{1}$$

so that, according to the well-known[2] formula

$$\Theta^{ij}=T^{ij}-T^{ji}=-\partial_k\sigma^{ijk}=ic\epsilon^{ijkl}(\partial_l\sigma_k-\partial_k\sigma_l)\,, \tag{2}$$

where σ denotes Dirac's spin density

$$\sigma^{ijk}=ic\epsilon^{ijkl}\sigma_l=\tfrac{1}{2}c\hbar\bar{\psi}\gamma^{ijk}\psi\,, \tag{3}$$

the kinematical current lines and the energy-momentum lines may, under appropriate circumstances, be noncollinear. (Latin indexes run from 1 to 4; $x^4=ict$; $h=2\pi\hbar$, denotes Plank's constant, ϵ^{ijkl} Levi-Civita's indicator, γ^i the von Neumann matrices, $\bar{\psi}=\psi^\dagger\gamma^4$, $[\partial^i]$ the Gordon current operator, e the electron charge, A^i the electromagnetic potential; $\gamma^{ij\cdots}=\gamma^i\gamma^j\cdots$ if all indexes are different, 0 otherwise.)

The final conclusion of this preceding paper[1] was incorrect, as we shall explain later. However, the main point, which the above paragraph recalls, remains true; the present paper intends to show that by using as a test material the probability fluid associated with moving spin-$\frac{1}{2}$ particles rather than a solid, the recoil effect corresponding to the "transverse momentum"[1] should appear.

The test material, which is a beam of spin-$\frac{1}{2}$ particles, has the three following fundamental properties: (a) a

[1] O. Costa de Beauregard, Phys. Rev. **129**, 466 (1963); all the notations of this paper are retained here, except for Θ^{ij} which is taken in a different sense.

[2] H. Tetrode, Z. Physik **48**, 52 (1928).