Scattering of Energy-Time Wave Packets from Many-Body Systems*

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The wave-packet nature of an experimental beam is discussed. The theory of wave-packet scattering is applied to wave packets with exponential time dependence scattered from resonances in both the isolated-resonance and overlapping-resonance regions. It is shown how a sequential description of scattering depends on some of the resonance parameters in the scattering amplitude. The meaning and usefulness of some experimentally possible wave-packet experiments is discussed.

1. INTRODUCTION

SCATTERING of a particle from a many-body system can be divided into two types with respect to a time interval τ and a corresponding energy interval $\delta(\sim\hbar/\tau)$.[1] If the scattering amplitude as a function of energy varies rapidly over the energy interval δ, the scattering is said to be compound scattering; if the amplitude varies slowly over this interval, we have direct scattering. τ is several orders of magnitude greater than the time it takes the incident particle to traverse a distance the size of the scatterer in free space.

In most actual scattering experiments time is not resolved. The incident beam has an energy spread. The cross section can be regarded as the weighted sum of cross sections for independent experiments, each with definite incident energy. This will be discussed in Sec. 2. It is still possible to define a time delay Δt according to the definition of Wigner,[2]

$$\Delta t = -i\hbar(d/dE)\ln S(E), \tag{1}$$

where $S(E)$ is the S-matrix element for the scattering problem. This time delay may be thought of as the time it takes for the phase of one incident wave to catch up with that of another whose energy differs from it by an infinitesimal amount, and whose phase shift in the scatterer is therefore infinitesimally different.

If the S-matrix element is divided according to some physical prescription into a rapidly and a slowly varying component, this definition is difficult to apply and it certainly does not tell us anything about each component separately. The rapidly varying component corresponds to a large Δt, that is, to particles that spend a long time on the average in the scatterer, and the slowly varying component corresponds to a small Δt, that is to particles that pass rapidly over the scatterer without significant time delay.

An experiment which measures the relative amount of direct and compound scattering has been suggested by Eisberg, Yennie, and Wilkinson.[3] The experiment also defines an energy interval which is the reciprocal of the time delay. The information is obtained from the energy spectrum of bremsstrahlung from elastic scattering of charged particles.

One would like to make a classical picture in which particles are described as being in the scatterer for varying times. To do this one must make a wave-packet argument. It will be shown in Sec. 2 that the physical situation corresponding to a meaningful wave-packet argument is necessarily a time-dependent scattering experiment.

Wave-packet arguments are made for example by Friedman and Weisskopf[4] for the case of shape-elastic and compound-elastic scattering. They show that for nonoverlapping levels (resonances) of the compound system the time delay is the average over the beam energy spread of the time delays for the individual levels. If time delay is defined according to Wigner's definition, this result is obtained at once. It has been done explicitly for example by Goldberger and Watson.[5]

If, however, time delay is defined as the average time delay in the emerging of a wave packet in a hypothetical time-dependent experiment complementary to the usual energy-dependent experiment, then the result is not so clear. The delayed wave packet would be expected to interfere with the immediately scattered wave packet from the shape elastic scattering. It is commonly stated that if the two wave packets are sufficiently short in time they will not interfere.[6] This

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[1] For example H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

[2] E. P. Wigner, Phys. Rev. **98**, 145 (1955).

[3] R. M. Eisberg, D. R. Yennie, and D. H. Wilkinson, Nucl. Phys. **18**, 338 (1960).

[4] F. L. Friedman and V. F. Weisskopf, *Niels Bohr and the Development of Physics* (Pergamon Press, Ltd., London, 1955).

[5] M. L. Goldberger and K. M. Watson, Phys. Rev. **127**, 2284 (1962).

[6] For example, H. G. Preston, *Physics of the Nucleus* (Addison Wesley Publishing Co., Inc., Reading, Massachusetts, 1962). R. K. Adair, S. E. Darden, and R. E. Fields, Phys. Rev. **96**, 503 (1954).

situation is now extreme hypothetical. It requires very good time definition in the experiment. In an actual experimental situation, time is undefined and the wave packets interfere completely. It may also be asked whether the scattering amplitude, and hence, for example, the angular distribution will not be dependent on the width of the incident wave packet, since, for short wave packets (in time), we might not expect such complete interference as for long wave packets.[7]

The question of what actually happens to a scattered wave packet has been investigated by Sasakawa.[8] He shows that for the scattering of a wave packet from an isolated resonance, the variation of the cross section with energy is characterized not by the width of the resonance but, in addition, by the width of the wave packet. In Sec. 3 we will re-derive this result using a different mathematical technique and Lorentzian wave packets. The Lorentzian shape is one shape for the energy spectrum that can be realized experimentally.

In Sec. 4 we will consider the scattering of wave packets from a system in which the compound levels overlap. This gives new insight into the optical model, which is defined for this purpose as the model for which the time delay in the complementary time-dependent experiment would be small. Our treatment is more general than Sasakawa's treatment of the same situation in a special case.

The usual definition of the optical-model scattering amplitude[9] is that it is the average of the actual scattering amplitude over an energy interval δ. In the actual situation, amplitudes for different energies are independent and it is the cross sections that are experimentally averaged. The meaning of the energy averaging process will be discussed.

In Sec. 5 we will discuss the possibility of performing scattering experiments with time definition which are complementary to the usual ones in which energy but not time is resolved. One such experiment has been suggested by the present authors.[10]

2. DEPENDENCE OF THE SCATTERING CROSS SECTION ON THE INITIAL BEAM CHARACTERISTICS

A beam consists of particles whose energy is defined within certain limits. The time of arrival of particles at a specific point may also be defined within limits by modulating the amplitude of the beam. In general the limits on the time and energy resolution of the beam are wider than those imposed by the uncertainty principle. For example some particles may be moving faster than others because they were accelerated a little more in the accelerator. The fast and slow beams could in principle be separated by a magnetic spectrometer. This type of energy uncertainty will be called classical uncertainty. On the other hand, if the time of arrival of a particle at a point is defined within a time interval τ, we cannot in principle measure its energy with a definition better than $\delta=\hbar/\tau$. This type of uncertainty will be called quantal uncertainty.

A beam with both classical and quantal uncertainty is described by a collection of wave packets with different mean wave numbers $\mathbf{k}_j$ whose position at time t is measured with respect to different initial position vectors $\mathbf{r}_i$. The value of $\mathbf{r}_i$ specifies the starting time of the wave packet at the source. One such wave packet may be written

$$\xi_{ij}(\mathbf{r},t)=(2\pi)^{-\frac{3}{2}}\int\Delta(\mathbf{k}_j,\delta;\mathbf{k})\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_i)]d^3k. \quad (2)$$

Suppose that the number of packets of type j is given by n where

$$n=\Phi_j N. \quad (3)$$

N is the total number of wave packets.

We wish to know the cross section for a scattering experiment with an incident beam which can be described in this way. For simplicity we will consider only one spin state. We must be careful to distinguish two cases.

Case A. If the $\mathbf{r}_i$ are such that the wave packets do not overlap both before and after scattering, the scattering of each wave packet may be considered as an individual event. The total cross section is the sum of the cross sections for the scattering of each wave packet.

Case B. The $\mathbf{r}_i$ are such that the wave packets overlap. In this case different wave packets will interfere and the cross sections for individual wave packets do not add incoherently.

The mathematical description of these cases is as follows. For case A we will consider wave packets with the same value $\mathbf{r}_0$ of $\mathbf{r}_i$ and describe the beam by means of a density matrix. The density matrix at time t is

$$\rho(\mathbf{r},\mathbf{r}',t)=\textstyle\sum_j\Phi_j\xi_{0j}(\mathbf{r},t)\xi_{0j}^*(\mathbf{r}',t). \quad (4)$$

According to the standard theory of wave-packet scattering,[11] ξ_{0j} is given in the case where the space occupied by the wave packet is large compared to the size of the scatterer but small compared to the distance of the detector by

$$\xi_{0j}(\mathbf{r},t)=(2\pi)^{-\frac{3}{2}}\int\Delta(\mathbf{k}_j,\delta;\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r}_0-iEt/\hbar)\times[\exp(ikr)/r]f(k,\Omega)d^3k. \quad (5)$$

[7] I. E. McCarthy, *Proceedings of the International Symposium on Direct Interactions and Nuclear Reaction Mechanisms, Padua, 1962* (Gordon and Breach Publishers, Inc., New York, 1963).

[8] T. Sasakawa, Progr. Theoret. Phys. Suppl. **11**, 69 (1959).

[9] For example G. E. Brown, Rev. Mod. Phys. **31**, 893 (1959).

[10] L. R. Dodd and I. E. McCarthy, Phys. Rev. Letters **12**, 136 (1964).

[11] See, for example, E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961). A review of general wave-packet scattering with applications to nuclear reactions has been given by N. Austern in *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963).

$f(k,\Omega)$ is the scattering amplitude for scattering of particles into a detector whose angular position is Ω. $\Delta(\mathbf{k}_j,\delta;\mathbf{k})$ is a weight function which confines the wave packet to a volume in momentum space centered at $\mathbf{k}_j$ and of width δ. δ is the *quantal uncertainty*.

We will not be interested in the angular uncertainty in the wave packet, so we will write

$$\Delta(\mathbf{k}_j,\delta;\mathbf{k})=\frac{\delta(\theta_k-\theta_0)\delta(\phi_k-\phi_0)}{2\pi k^2\sin\theta_k}\Delta(k_j,\delta;k), \qquad (6)$$

where (θ_k,ϕ_k), (θ_0,ϕ_0) are, respectively, the angular coordinates of $\mathbf{k}$ and $\mathbf{k}_j$, the beam direction.

The probability $P_j(t)$ of detecting a particle at time t at the detector whose position is $\mathbf{r}_d$ is, for the wave packet j,

$$P_j(t)=N_j|\xi_{0j}(\mathbf{r}_d,t)|^2r_d^2d\Omega. \qquad (7)$$

This defines the normalization N_j of ξ_{0j}. The differential cross section is

$$(d\sigma/d\Omega)(\mathbf{k}_j,\delta)=v_0N_jr_d^2\int_{-\infty}^{\infty}|\xi_{0j}(\mathbf{r}_d,t)|^2dt, \qquad (8)$$

where v_0 is the group velocity of the wave packet.

We may write the differential cross section for the beam described by the density matrix (4) as

$$d\sigma/d\Omega=v_0N_jr_d^2\int_{-\infty}^{\infty}\rho(\mathbf{r}_d,\mathbf{r}_d,t)dt. \qquad (9)$$

Substituting from (4) and interchanging the order of summation and integration we have

$$d\sigma/d\Omega=\sum_j\Phi_jv_0N_jr_d^2\int_{-\infty}^{\infty}|\xi_{0j}(\mathbf{r}_d,t)|^2dt$$
$$=\sum_j\Phi_jd\sigma(\mathbf{k}_j,\delta)/d\Omega. \qquad (10)$$

This is the result stated above for case A. Since we are only interested in the differences of the peak energies E_j for different wave packets j, we may replace the sum in (10) by an energy integral.

$$\frac{d\sigma}{d\Omega}=\int\Phi(E')\frac{d\sigma}{d\Omega}(E',\delta)dE'. \qquad (11)$$

The cross section is the energy average of the cross sections for individual wave packets with different peak energies. In particular, for beams with negligible quantal uncertainty the cross section is the energy average of the cross sections for scattering events with plane wave and outgoing spherical wave boundary conditions. For this reason we will call ϕ, the width of the weight function $\Phi(E)$, the *classical uncertainty*.

We will now consider the more general case B and see under what circumstances the distinction between classical and quantal uncertainty can be maintained.

In terms of plane-wave states $\eta(\mathbf{k})$, the most general density matrix is

$$\rho(\mathbf{r},\mathbf{r}',t)=\int\int a(\mathbf{k},\mathbf{k}')\eta(\mathbf{k}')^*\eta(\mathbf{k})d^3kd^3k'. \qquad (12)$$

We wish to write (12) in diagonal form in terms of wave-packet states ξ_{ij}.

$$\rho(\mathbf{r},\mathbf{r}',t)=\sum_{ij}\Phi_j\xi_{ij}\xi_{ij}^*. \qquad (13)$$

We must know if this diagonalization is unique. If so, we can assign a definite classical weight factor Φ with uncertainty ϕ and a definite quantal weight factor Δ with uncertainty δ. If not, the distinction is meaningless.

In mathematical terms the question is as follows. Can the Hermitian matrix $\rho(\mathbf{r},\mathbf{r}',t)$ in (12) be diagonalized by more than one transformation of the type (2) of the basis vectors? A necessary condition for the diagonalization to be unique[12] is that the states ξ_{ij} must be orthogonal like the states η.

In fact we have

$$\int\xi_{ij}\xi_{i'j'}d^3r=(2\pi)^{-3}\int\Delta(\mathbf{k}_j,\delta;\mathbf{k})\Delta(\mathbf{k}_{j'},\delta;\mathbf{k})^*$$
$$\times\exp[-i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_{i'})]d^3k. \qquad (14)$$

The overlap integral in (14) is approximately zero if the wave packets have sufficiently different starting times given by $\mathbf{r}_i$ and $\mathbf{r}_{i'}$ because of the rapid oscillations of the integrand.

In case A the wave packets have a definite physical significance. The quantal uncertainty leads to alterations in the usual cross sections, as will be shown in the succeeding sections. In case B the wave packets may be regarded as mathematical aids to the visualization of the scattering process. For example we can decompose a plane wave into an infinite number of overlapping wave packets and then follow the propagation of one of these packets. Nevertheless, this would be without physical significance because the scattering amplitudes for all the packets must be recombined to obtain the cross section.

The vital point is that the wave packets both before and after the scattering must be quite distinct in space and time for the quantal uncertainty to have any physical significance.

3. SCATTERING OF A WAVE PACKET FROM A RESONANCE

In the usual theory of scattering it is assumed that $f(k,\Omega)$ in Eq. (5) varies much more slowly with energy than the wave packet amplitude factor $\Delta(\mathbf{k}_j,\delta;\mathbf{k})$; that is, that the quantal uncertainty δ is negligible in com-

[12] U. Fano, Rev. Mod. Phys. **29**, 74 (1957).

parison with the width of the energy fluctuations of the scattering amplitude.

We will consider the case where δ is comparable with Γ_s, the width of the compound state $|s\rangle$ of the system comprising the incident particle and the scatterer whose energy ϵ_s is near the central energy E_0 of the wave packet. In the present section we will suppose that only one state $|s\rangle$ contributes to the scattering amplitude at the energies covered by the wave packet.[12a] We will be interested in only one wave packet $j=0$.

Since we are interested in the energy rather than the momentum of the wave packet, it will be convenient to transform the integral in (5) to an energy integral. The transformation is trivial when k is proportional to E as it is for photons ($E=\hbar ck$).

For our discussion of hypothetical time-dependent experiments with finite-mass particles complementary to the usual energy-dependent experiments, we can make the assumption, following Friedman and Weisskopf,[4] that the source and detector are sufficiently close to the scatterer for wave packet spreading to be negligible. We will neglect the final term in the following expansion of E:

$$E=\frac{\hbar^2}{2m}[2\mathbf{k}\cdot\mathbf{k}_0-k_0^2+|\mathbf{k}-\mathbf{k}_0|^2]. \tag{15}$$

Photon wave packets do not spread for practical purposes.

We must now consider a particular form for the energy amplitude factor $\Delta(E_0,\delta;E)$ corresponding to $\Delta(\mathbf{k}_0,\delta;\mathbf{k})$. Wave packets made by electronic means (see Sec. 5) would have a rise and decay function in time that is something like an exponential. We will therefore consider a wave packet that has an exponential rise and decay, with time constant $\hbar/\delta$. We will assume that the beam is switched off as soon as it has attained full strength, so that there is no time for which the beam intensity is constant. The experimental difficulty is in fast switching on and off. Since we want as short a packet in time as possible, this is then the most realistic form. The energy amplitude factor for this form is

$$\Delta(E_0,\delta;E)=\frac{\delta/2\pi}{(E-E_0)^2+\delta^2/4}. \tag{16}$$

One shape of wave packet is already available in nature. This is the photon wave packet from the decay of a level $|s\rangle$ of a many-body system. It has a half-exponential time spectrum. If the moment of excitation of the level is taken as zero time, then the probability of emission of a photon decays exponentially with a time constant $\hbar/\Gamma_s$. In this case we have

$$\Delta(E_0,\Gamma_s;E)=\frac{i/2\pi}{E-E_0+i\Gamma_s/2}. \tag{17}$$

We will use the form (16) for most of our discussion.

For the scattering amplitude we will use the expansion of Siegert, Humblet, and Rosenfeld,[13]

$$f[k(E),\theta]=\frac{1}{2i}\sum_l(2l+1)P_l(\cos\theta)\times\left[C_l(E)+\sum_n\frac{R_{ln}}{E-\epsilon_{ln}+\frac{1}{2}i\Gamma_{ln}}\right]. \tag{18}$$

The nonresonant term $C_l(E)$ varies slowly with energy. The splitting off of this term is not unique and can be made according to some physical prescription. Equation (5) now becomes

$$\xi(\mathbf{r},t)=\frac{K}{2\pi}\sum_l(2l+1)P_l(\cos\theta)\int_0^\infty\frac{\delta}{(E-E_\delta)(E-\bar{E}_\delta)}\times\exp(iEX)\left[C_l(E)+\sum_n\frac{R_{ln}}{E-\epsilon_{ln}+i\Gamma_{ln}/2}\right]dE, \tag{19}$$

where

$$X=[(r_0+r)/v_0-t]/\hbar,\quad E_\delta=E_0-i\delta/2,\quad K\bar{K}=1/8\pi r^2. \tag{20}$$

We have dropped the subscripts on $\xi(\mathbf{r},t)$ because we are now interested only in one particular wave packet.

The integral in (19) is now in a convenient form for contour integration if we make the approximation of extending the lower limit to $-\infty$, thus neglecting the contributions of bound states.

We will first consider the scattering of the wave packet according to the nonresonant scattering amplitude $C_l(E)$ whose variation with energy can be neglected over the energies of the wave packet. Performing the integration in Eq. (19) we find

$$\begin{aligned}\xi(\mathbf{r},t)&=K\sum_l(2l+1)P_l(\cos\theta)C_l(E)\exp(i\bar{E}_\delta X),\quad X>0\\&=K\sum_l(2l+1)P_l(\cos\theta)C_l(E)\times\exp(iE_\delta X),\quad X<0.\end{aligned} \tag{21}$$

The wave packet is centered at $X=0$, i.e., at

$$t=(r_0+r)/v_0. \tag{22}$$

Hence, the nonresonant packet is propagated without time delay. The time spectrum of $|\xi(\mathbf{r},t)|^2$ is an exponential rise and fall, $\exp(-\delta t/\hbar)$.

The differential cross section is, according to Eq. (8),

$$d\sigma/d\Omega=\tfrac{1}{4}|\sum_l(2l+1)P_l(\cos\theta)C_l(E)|^2. \tag{23}$$

Thus, the differential cross section for nonresonant (potential) scattering is independent of the quantal uncertainty δ and identical with that for a normal beam in which δ is negligible.

[12a] *Note added in proof.* The decay of a single resonance has been investigated by R. G. Newton, Ann. Phys. (N. Y.) **14**, 333 (1961).

[13] A. Siegert, Phys. Rev. **56**, 750 (1939). J. Humblet and L. Rosenfeld, Nucl. Phys. **26**, 529 (1961).

We will now consider the scattering from an isolated resonance state $|s\rangle$. Taking one term of the sum in (18) we have for the integral in (18)

$$I=\int_{-\infty}^{\infty}\frac{\delta/2\pi}{(E-E_\delta)(E-\bar{E}_\delta)}\frac{R_s}{E-E_s}\exp(iEX)dE, \quad (24)$$

where

$$E_s=\epsilon_s-i\Gamma_s/2. \quad (25)$$

Integration round an infinite semicircle in the upper half plane gives

$$I=\frac{R_s}{\bar{E}_\delta-E_s}\exp(i\bar{E}_\delta X), \quad X>0. \quad (26)$$

This means that the leading edge of the wave packet, that is, the part for times greater than $(r_0+r)/v_0$, is propagated with the same shape as it originally had, but, of course, with a different magnitude. The propagation of the trailing edge (the tail) of the wave packet is given by integration round a contour in the lower half plane

$$I=\frac{R_s}{E_\delta-E_s}\exp(iE_\delta X)-i\delta\frac{R_s}{(E_s-E_\delta)(E_s-\bar{E}_\delta)}\exp(iE_sX), \quad X<0. \quad (27)$$

The time spectrum given by $|I|^2$ is a decaying oscillatory function. In particular, if the time width of the wave packet is much less than the decay constant of the state $|s\rangle$, the time spectrum of the tail has the shape $\exp(-\Gamma_s t/\hbar)$. This corresponds to exciting the resonance suddenly and watching it decay with its natural time constant.

The differential cross section is

$$\frac{d\sigma}{d\Omega}=\frac{R_s\bar{R}_s[(E-\epsilon_s)^2\Gamma_s/2+(\Gamma_s+\delta)^2(\Gamma_s/2+\delta)/4]}{2\Gamma_s[(E-\epsilon_s)^2+(\Gamma_s+\delta)^2/4]^2}. \quad (28)$$

This reduces to the Breit-Wigner form for very small quantal uncertainty δ. The most significant thing is that the width of the energy spectrum is $\Gamma_s+\delta$. Note also that the differential cross section for scattering from a single resonance is reduced in magnitude for large quantal uncertainty. For large δ, $d\sigma/d\Omega$ is of order $1/\delta$.

In view of the arguments often made about the noninterference of potential and resonance scattering for large δ, it is interesting to consider the interference term. Denote the additional cross section due to the interference of potential and resonant scattering by $d\sigma_I/d\Omega$. It arises from the last term in the splitting of the scattering into potential and resonant parts

$$|\xi(\mathbf{r},t)|^2=|\xi_P(\mathbf{r},t)|^2+|\xi_R(\mathbf{r},t)|^2+2\,\mathrm{Re}\xi_P(\mathbf{r},t)\xi_R(\mathbf{r},t). \quad (29)$$

Taking only the s state for simplicity of notation,

$$\frac{d\sigma_I}{d\Omega}=\tfrac{1}{2}\left\{R_s\bar{C}_0(E)\frac{(E-\epsilon_s)+i(\Gamma_s/2+\delta)}{[(E-\epsilon_s)+i(\Gamma_s+\delta)^2/4]^2}\right\}, \quad (30)$$

$$=\tfrac{1}{2}\,\mathrm{Re}[R_s\bar{C}_0(E)]\frac{(E-\epsilon_s)(E-\epsilon_s)^2+(\Gamma_s+\delta)(\Gamma_s+3\delta)/4}{[(E-\epsilon_s)^2+(\Gamma_s+\delta)^2/4]^2}+\tfrac{1}{2}\,\mathrm{Im}[R_s\bar{C}_0(E)]\frac{(E-\epsilon_s)^2\Gamma_s/2+(\Gamma_s+\delta)^2(\Gamma_s+2\delta)/8}{(E-\epsilon_s)^2+(\Gamma_s+\delta)^2/4}. \quad (31)$$

The coefficient of $\mathrm{Im}[R_s\bar{C}_0(E)]$ is of order $1/\delta$ for large δ. Hence the interference term in the cross section does become smaller for larger δ, that is for better defined wave packets in time, but so also does the resonant scattering term, both being of order $1/\delta$. The potential scattering term is not affected by δ.

4. SCATTERING FROM MANY LEVELS AND THE OPTICAL MODEL

We will now consider the energy region where, for a given channel, the resonances in the scattering amplitude are such that the average width is greater than the average spacing, so that many levels contribute at each energy over the energy spread of the wave packet.

The levels in this region are defined by some plausible model such as has been discussed by Brown[9] for nuclei. Their widths are of the order of 1 eV, so that the corresponding lifetime, 10^{-15} sec, is too short for a practical wave-packet experiment. However, it is interesting to consider a hypothetical wave-packet experiment in order to give a proper quantum mechanical sequential description of the passage of a particle through the scatterer, when we know the S-matrix element for the scattering.

This gives us another way of looking at the optical and direct interaction models. We will restrict ourselves to elastic scattering for simplicity. The optical model has been considered in two ways, each with a different

starting point. The first way starts with a Schrödinger equation. The optical-model Hamiltonian is shown to be an approximation to the many-body Hamiltonian. The second way is independent of any Hamiltonian assumption and starts with the scattering amplitude. The average over some energy interval I of the scattering amplitude S is shown to be capable of being calculated from an optical-model Hamiltonian. The average cross section is split into two terms.

$$\langle\sigma\rangle=|\langle S\rangle|^2+\langle|S-\langle S\rangle|^2\rangle. \tag{32}$$

The first term is the optical-model or shape elastic cross section. The second term is the fluctuation or compound elastic cross section.

A third, classical, definition of the optical model is sometimes given. The optical-model cross section is the cross section for propagation without time delay while the compound elastic cross section is the remainder.

It is not clear that the last two definitions are equivalent. The splitting in Eq. (32) requires a definition of the averaging procedure. Some authors, for example Brown,[9] have gone to much trouble to discuss the averaging procedure. Brown's argument is essentially a wave-packet argument since it uses an average of the scattering amplitude over an energy interval I. We have seen in Sec. 2 that for a beam without significant time resolution it is the cross sections, not the amplitudes, that are averaged.

Another condition for the validity of this definition of the optical model is required. The average over amplitudes must be nearly equivalent to the average over cross sections. This means that the compound elastic cross section must be very small compared with the optical model cross section.

A quantal statement of the third definition of the optical model must be given by a detailed wave-packet description. It is quite conceivable that the scattering amplitudes for certain problems are such that a large proportion of the cross section is due to time-delayed wave packets. This question will be discussed in detail by one of us (LRD) in a subsequent publication.

At present we will just consider the propagation of the wave packet in general and show how it leads to a large proportion of immediate propagation in the case $\Gamma\gg D$ where Γ is the average level width and D is the average spacing.

Considering only the S wave for simplicity and omitting the potential scattering term, Eq. (19) gives for the time dependence of the trailing edge of the wave packet

$$\xi(\mathbf{r},t)=K\exp(iEX)\sum_n\frac{R_n}{E-\epsilon_n+i(\Gamma_n-\delta)/2}\Bigg\{\exp(\delta X/2)-\frac{i\delta\exp(\Gamma_nX/2)\exp i(\epsilon_n-E)X}{E-\epsilon_n+i(\Gamma_n+\delta)/2}\Bigg\}. \tag{33}$$

The second term in the bracket is the one containing the properties of the scattering amplitude rather than the wave packet. Each level n contributes an exponential tail to the amplitude with a time-delay constant $\hbar/\Gamma_n$. However, this contribution is multiplied by a phase factor $\exp[i(\epsilon_n-E)X]$ which gives a partial cancellation of the tails when the packets from different levels are superposed.

If the level widths Γ_n are much greater than the average spacing D, the factor ϵ_n-E can be large so that the phase factor oscillates rapidly. In this case the phases of the contributions from different levels tend to be random so the tails cancel out giving a large proportion of propagation without time delay.

Thus the sequential description of scattering gives the same result as the usual energy description. The optical model is valid when $\Gamma\gg D$.

The dependence of (33) on the magnitude of the quantal uncertainty δ is also interesting. As δ is increased the magnitude of the contribution from each level to the scattered packet decreases. However more levels contribute significantly to the sum. If the phases of the residues R_n are random the magnitude of the scattered packet will decrease rapidly with increasing δ. If they are correlated, the magnitude will decrease less rapidly. The range of the correlations between levels can be determined in principle by varying δ. This will be discussed in detail in a subsequent publication.

5. POSSIBLE WAVE-PACKET EXPERIMENTS

The condition for a time-dependent scattering experiment is that the experimental definition of time must be accurate in comparison with the characteristic time of the scattering amplitude. That is, we must have δ not much less than Γ.

Experimental definition of time is at present possible for times as short as about 10^{-10} sec. Typical nuclear values for $\hbar/\Gamma$ are 10^{-15} sec, so wave-packet experiments cannot be performed with nuclei except in special cases. These cases are metastable states which can have lifetimes as long as 10^{-7} sec.

One experiment has actually been done using the Mössbauer effect with the 14-keV γ ray from Fe^{57}, which has a decay constant of 10^{-7} sec, by Holland, Lynch, Perlow, and Hanna.[14] The time spectrum of the incident wave packet was defined by using as zero time the time of formation of the 14-keV state, which was defined by the time of emission of the 128-keV γ ray (a fast decay) from the next highest state in the γ-ray cascade from the decay of Co^{57}. The wave packet has an exponential time spectrum with $\delta=\hbar/10^{-7}$ sec. This wave packet was scattered resonantly from an Fe^{57} target. The time spectrum of the final state and the increased width of the absorption line were both

[14] R. E. Holland, F. J. Lynch, G. J. Perlow, and S. S. Hanna, Phys. Rev. Letters **4**, 181 (1960); F. J. Lynch, R. E. Holland, and M. Hamermesh, Phys. Rev. **120**, 513 (1960).

observed. In this case $\delta=\Gamma_s$. The width of the wave packet in this experiment is of course fixed. By looking at the scattering at times less than 10^{-7} sec, Holland *et al.* were able to observe greater widths, but for these cases the shape of the incident packet was not defined. The spectrum of total elapsed time gives only an upper limit to the time width of the incident wave packet because it is not known if the delay occurred in the source or the scatterer.

The Mössbauer effect is, of course, a wave-packet scattering experiment in which the absorption cross section is measured. If we do not define time but merely observe the cross section we lose the wave-packet property, that is the quantal uncertainty. The cross section is obtained using Eq. (17) to define the energy amplitude factor of the wave packet.

$$\frac{d\sigma}{d\Omega}=\frac{R_s^2}{2\Gamma_s}\frac{1}{(E_0-E_s)^2+\Gamma_s^2}. \tag{34}$$

Thus, we have the well-known result that the linewidth in the Mössbauer effect is twice the width of the level, assuming all the nuclei in the target are capable of absorbing resonantly.

Defining the starting time of the excitation of the metastable state puts a quantal uncertainty into the beam equal to Γ_s. The method of Holland *et al.* defines the starting time with a minimum tolerance equal to the lifetime of the next highest state in the γ-ray cascade. This is shorter than the experimental time resolution, which is itself much shorter than $\hbar/\Gamma_s$.

One way of varying the time width of the wave packet would be to vary the resolution with which the starting time is measured. It is possible by this method to decrease δ, but not to increase it.

A better experimental way of varying the wave-packet width in the Mössbauer effect has been suggested by the present authors. A third resonant absorber is introduced between the source and the target. This absorber is accelerated in a very short time (10^{-9} sec) to a speed sufficient to shift the resonance so that the incident beam is no longer absorbed and can hit the target. It is then slowed down again quickly so that the time duration of the pulse is of the order of 10^{-9} sec. This method would produce approximately the exponential wave packet [Eq. (16)] that we have used in the calculations of Secs. 3 and 4. It is experimentally just possible to achieve the requisite acceleration by using a very thin foil of Fe^{57} as one plate of a parallel plate condenser in a vacuum which is charged first with one sign and then with the opposite sign by an rf pulse. However, the acceleration may be achieved more easily using a piezo-electric crystal. Another possibility is to use the Stark effect to shift the resonance. This requires a tightly bound dielectric crystal containing nuclei with a metastable state.

Wave-packet experiments in the atomic energy region may be interesting. An absorber of laser material placed in a laser beam and moved for a short time as suggested above would produce wave packets of laser intensity. The quantal uncertainty would be much greater than that of a beam from a pulsed laser, whose quantal uncertainty can be no greater than the width of the laser state.

All such experiments observe only the scattering of a wave packet from a single resonance with a trivial angular distribution. Overlapping resonances, even in atoms, would probably have widths of the order of a few tenths of an electron volt. Wave-packet experiments in this region with $\delta\sim\Gamma$ would require time lengths of 10^{-15} sec which seems impossible at present. The possibility of doing time-dependent electron-scattering experiments from atoms is not experimentally remote and would be interesting.

Using laser wave packets, it may be possible to observe the changes in angular distribution due to the interference of potential and resonant scattering as the quantal uncertainty is changed. The potential scattering could be obtained by diffraction of laser light round a small crystal of laser material.

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