Stability of Longitudinal Oscillations in a Uniform Magnetized Plasma with Anisotropic Velocity Distribution*

LAURENCE S. HALL AND WARREN HECKROTTE
Lawrence Radiation Laboratory, University of California, Livermore, California
(Received 13 January 1964)

We consider the longitudinal oscillations of a uniform magnetized plasma in which the particles are allowed to have anisotropic distributions in velocity space, corresponding to the two temperatures $T_\perp$ and $T_\parallel$ perpendicular and parallel to the magnetic field. It is shown that no instabilities can exist unless $T_\perp > T_\parallel$ for at least one of the species of plasma particles. An additional theorem, of use in surveying for possible unstable roots to the dispersion equation when $T_\perp > T_\parallel$, is also proved.

THE stability of longitudinal oscillations of a uniform plasma with anisotropic velocity distribution in the presence of a uniform magnetic field has been studied by a number of authors[1-3] in various approximations. We consider here a multicomponent nonrelativistic plasma in which the zeroth-order distribution function may be written as a bi-Maxwellian, viz.,

$$f_{0j}=(2\pi\kappa T_{\perp j}/m_j)^{-1}(2\pi\kappa T_{\parallel j}/m_j)^{-1/2}\times\exp\left[-\frac{m_jv_\perp^2}{2\kappa T_{\perp j}}-\frac{m_jv_\parallel^2}{2\kappa T_{\parallel j}}\right], \quad (1)$$

and show that no instability exists unless $T_{\perp j} > T_{\parallel j}$ for at least one species. (Here $\parallel$ and $\perp$, respectively, denote directions parallel and perpendicular to the magnetic field B_0; j denotes the species of particle; and m, T, and κ are, respectively, mass, temperature, and Boltzmann's constant.) An additional theorem of use in locating possible instabilities when $T_{\perp j} > T_{\parallel j}$ is also proved.

Assuming that all perturbations vary in space and time according to the form $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, where $\mathbf{k}$ is real and ω complex, and $\mathrm{Im}(\omega)>0$, one can follow step by step the prescription of Bernstein[4] in obtaining the dispersion relation for longitudinal oscillations. After a straightforward but laborious computation, one may write

$$1+\sum_j \omega_{pj}^2\Omega_j^{-2}F_j=0\,, \quad (2)$$

where $\omega_{pj}=(4\pi Z_j^2e^2n_j/m_j)^{1/2}=$ plasma frequency, $\Omega_j=|Z_je|B_0/m_jc=$ cyclotron frequency, and the sum is over particle species. If we write $\omega_j=\omega/\Omega_j$; $\gamma_j=k(2\kappa T_{\perp j}/m_j\Omega_j^2)^{1/2}=k\cdot$ (radius of gyration); $t_j^2=T_{\parallel j}/T_{\perp j}$; and take $n_\parallel$ and $n_\perp$, respectively, to be the direction cosines of the propagation vector parallel and perpendicular to the magnetic field, so that $n_\parallel^2+n_\perp^2=1$, we have

$$F_j=\int_0^\infty dx\,\exp\{i\omega_jx-\tfrac{1}{4}\gamma_j^2t_j^2n_\parallel^2x^2-\tfrac{1}{2}\gamma_j^2n_\perp^2(1-\cos x)\}\times[n_\parallel^2x+n_\perp^2\sin x]. \quad (3)$$

We now drop the subscript j as understood, and consider the integral in (3) noting that it may be rewritten after an integration by parts of the first term to read

$$F=2\gamma^{-2}t^{-2}+2\gamma^{-2}t^{-2}F_A+(1-t^{-2})n_\perp^2F_B\,, \quad (4)$$

where

$$F_A=i\omega\int_0^\infty dx\,\exp\{i\omega x-\mu x^2-\lambda(1-\cos x)\}\,,$$
$$F_B=\int_0^\infty dx\,\exp\{i\omega x-\mu x^2-\lambda(1-\cos x)\}\sin x\,, \quad (5)$$

and we have set $\mu=\frac{1}{4}\gamma^2t^2n_\parallel^2$ and $\lambda=\frac{1}{2}\gamma^2n_\perp^2$. All of the parameters are real except ω which is complex and which, for an instability, has positive imaginary part. We propose to show that, when $\mathrm{Im}(\omega)>0$,

$$\mathrm{Re}(\omega)\cdot\mathrm{Im}(F_A)\geq 0\,,\quad \mathrm{Re}(\omega)\cdot\mathrm{Im}(F_B)\geq 0\,. \quad (6)$$

Since both $\mathrm{Im}(F_A)$ and $\mathrm{Im}(F_B)$ are antisymmetric in $\mathrm{Re}(\omega)$, it is only necessary to prove (6) for ω in the first quadrant of its complex plane. We then use the fact that if $f(\omega)$ is an analytic function of ω in the region of consideration, the map of any closed contour C in the ω plane onto the f plane encircles all values of f attained by letting ω assume values within C. We choose C as in Fig. 1, and note that F_A and F_B both vanish on the infinite quarter circle. Further, for ω on the imaginary axis both $\mathrm{Im}(F_A)$ and $\mathrm{Im}(F_B)$ vanish. Thus, we need only show that $\mathrm{Im}(F_A)$ and $\mathrm{Im}(F_B)$ are sign-definite for ω real and positive in order to show they are sign-definite for ω anywhere in the first quadrant.

First, consider $\mathrm{Im}(F_A)$ and use the identity

$$e^{\lambda\cos x}=\sum_{l=-\infty}^{\infty} I_l(\lambda)e^{ilx}\,, \quad (7)$$

where the $I_l(\lambda)$ are Bessel functions of the first kind

* This work performed under the auspices of the U. S. Atomic Energy Commission.

[1] E. G. Harris, J. Nucl. Energy **C2**, 138 (1961), and Oak Ridge National Laboratory Report ORNL-2728, 1959 (unpublished).
[2] Y. Ozawa, I. Kaji, and M. Kito, J. Nucl. Energy **C4**, 271 (1962), and references cited therein.
[3] T. Kammash and W. Heckrotte, Phys. Rev. **131**, 2129 (1963); **132**, (1963).
[4] I. Bernstein, Phys. Rev. **109**, 10 (1958), Appendix II.

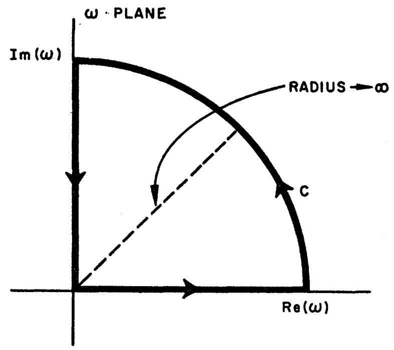

FIG. 1. Contour for the examination of various integrals.

and imaginary argument, obtaining for $\omega>0$

$$\mathrm{Im}(F_A)=\omega(\pi/4\mu)^{1/2}\sum_{l=-\infty}^{\infty}e^{-\lambda}I_l(\lambda)\times\exp\{-(\omega+l)^2/4\mu\}>0. \quad (8)$$

Similarly, with the use of (7) together with the fundamental relation $I_{l-1}(\lambda)-I_{l+1}(\lambda)=2lI_l(\lambda)/\lambda$, one can also write that for $\omega>0$

$$\mathrm{Im}(F_B)=\lambda^{-1}(\pi/\mu)^{1/2}\sum_{l=1}^{\infty}e^{-\lambda}I_l(\lambda)\times\exp\{-(\omega^2+l^2)/4\mu\}l\sinh(l\omega/2\mu)>0. \quad (9)$$

According to our previous discussion, (8) and (9) establish (6). Now consider the dispersion relation (2) which can be satisfied only if either (a) $\mathrm{Im}(F)$ vanishes for all species or (b) $\mathrm{Re}(\omega)\cdot\mathrm{Im}(F)<0$ for at least one species.[5] Condition (a) can be satisfied in general only when $\mathrm{Re}(\omega)=0$, which, if unstable, corresponds to purely growing waves. However, as pointed out by Kammash and Heckrotte,[3] $\mathrm{Re}(F)$ is positive definite in this limit and no purely growing modes can exist. Hence, an instability can only occur if $\mathrm{Re}(\omega)\cdot\mathrm{Im}(F)<0$ for at least one species. From (4) and (6), then, we have the result that no instability exists unless $t^2=T_\parallel/T_\perp<1$ for at least one species in the plasma.

It is also of interest to be able to determine whether or not roots to the dispersion relation occur when $t^2<1$, and if they exist, where they lie. In this connection, the labor involved in exploring the ω plane can often be reduced by utilization of the theorem, which we refer to as (10):

If $Re(\omega)\cdot Im[F(\omega)]$ is non-negative for ω real and lying in the interval $l\leq\omega\leq l+1$, where l is an integer, it is also non-negative in the entire strip $l\leq Re(\omega)\leq l+1$, $Im(\omega)\geq 0$. (10)

Hence, if one can show that $\mathrm{Re}(\omega)\cdot\mathrm{Im}(F)\geq 0$ for some integral range of ω on the real axis, one also can eliminate the possibility of unstable roots appearing anywhere in the strip above that range.

In order to prove (10), we rewrite (3) in the form[6]

$$F=n_\parallel^2F_C+n_\perp^2F_B, \quad (11)$$

where

$$F_C=\int_0^\infty xdx\exp\{i\omega x-\mu x^2-\lambda(1-\cos x)\}, \quad (12)$$

and investigate F along the contours $\mathrm{Re}(\omega)=l$. We have already proved $\mathrm{Re}(\omega)\cdot\mathrm{Im}(F_B)>0$, and so we need only prove $l\cdot\mathrm{Im}(F_C)>0$ when $\mathrm{Re}(\omega)=l$ to establish (10). Setting $\omega=l+i\sigma$ and again using the expansion (7), we can rewrite (12) after a little manipulation to read

$$\mathrm{Im}(F_C)=\sum_{k=1}^{\infty}e^{-\lambda}[I_{k-l}(\lambda)-I_{k+l}(\lambda)]\times\int_0^\infty xdxe^{-\sigma x-\mu x^2}\sin kx. \quad (13)$$

Consider the integral $K(\omega')=\int_0^\infty xdx\exp\{i\omega'x-\mu x^2\}$, $\mathrm{Im}(\omega')>0$, whose imaginary part is antisymmetric in $\mathrm{Re}(\omega')$. Evaluating K along the contour C in Fig. 1, again we find $\mathrm{Im}(K)=0$ along the imaginary axis and the infinite quarter circle. Along the real axis $k\cdot\mathrm{Im}[K(k)]=\frac{1}{4}\pi^{1/2}k^2\mu^{-3/2}\exp(-k^2/4\mu)\geq 0$. Hence k times the integral in (13), which is just $\mathrm{Im}[K(k+i\sigma)]$, is non-negative. Moreover, the Bessel function $I_k(\lambda)$ is a monotonically decreasing function of the absolute value of its order, so that $[I_{k-l}(\lambda)-I_{k+l}(\lambda)]\geq 0$. Hence $\mathrm{Re}(\omega)\cdot\mathrm{Im}(F_C)>0$ and (10) is proved.

In conclusion, we have shown that in a uniform magnetized plasma whose particles are distributed anisotropically in velocity space in zero order according to the bi-Maxwellian form (1), no unstable longitudinal oscillations exist unless $T_\perp>T_\parallel$ for at least one species. In addition, instabilities can exist when $T_\perp>T_\parallel$ only for complex frequencies lying in the strip $\mathrm{Im}(\omega)>0$, $l<\mathrm{Re}(\omega)<l+1$, where l is an integer, and where $\omega\,\mathrm{Im}(F)<0$ for some real ω lying in the range $l<\omega<l+1$.

[5] We consider that the plasma has at least two species of particles. If one species only is principally responsible for the motion, we always consider here that the small but vanishing contribution from the neglected particles is in the stabilizing sense, as will almost always correspond to the physical situation. This means that any instability arising as a limiting case of (b) must be such that $\mathrm{Im}(F)\to 0^-$ for the active species.

[6] We may note in passing that since F may be written in either of the forms (4) and (11), the inequalities (6) show that there can be no instabilities whenever either $n_\parallel=0$ or $n_\perp=0$. This is an extension of the result noted by Harris (Ref. 1) for the special case $T_\parallel=0$.