The hard core may be replaced with any other short-range force as long as the features (i)–(v) are maintained. It would also be interesting to find a short-range interaction for which (v) does not hold, because that would lead to a more complicated phase diagram, which could be treated by the same method.

APPENDIX II

First lemma. Let A be a real symmetrical matrix of the form

$$A_{ij}=a_i\delta_{ij}-b_{ij}\,,\quad b_{ij}=b_{ji}\geqslant 0\,.$$

This matrix is positive definite if

$$a_i>\sum_j b_{ij}\quad\text{for all } i\,. \tag{37}$$

Proof. Let λ be an eigenvalue and $\{x_i\}$ the corresponding eigenvector,

$$\lambda x_i=\sum_j A_{ij}x_j=a_ix_i-\sum_j b_{ij}x_j\,.$$

Let x_1 be the component of $\{x_i\}$ with the largest absolute value and let the arbitrary phase factor in $\{x_i\}$ be chosen such that $x_1>0$. Then, if (37) holds,

$$(a_1-\lambda)x_1=\sum_j b_{1j}x_j\leqslant x_1\sum_j b_{1j}<x_1a_1\,.$$

Hence, each eigenvalue λ is positive.

Second lemma. A is not positive definite if $a_i\leqslant\sum_j b_{ij}$ for all i.

Proof. It is easily seen that for the vector $\{y_i\}=\{1,1,1,\cdots\}$ one has $\sum_{ij} y_iA_{ij}y_j\leqslant 0$.

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Nonlinear Interaction of an Electromagnetic Wave with a Plasma Layer in the Presence of a Static Magnetic Field. III. Theory of Mixing*

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The Boltzmann transport equation coupled with Maxwell's equations has been solved under a small signal, plane-wave assumption for a uniform, weakly ionized plasma layer with a constant collision frequency in the presence of a static magnetic field. The effects of the nonlinear terms in the equations are included. The amplitudes of the sum and difference frequency waves produced within the plasma layer by two incident waves of different frequencies, which propagate in the extraordinary mode within the plasma, are derived as functions of the plasma and incident wave parameters. Resonances in the amplitudes of these waves occur for values of the static magnetic field in the neighborhood of plasma resonance for the waves at the fundamental and combination frequencies. The magnetic-field strength for which a given resonance occurs is a sensitive function of the electron density. This provides a mechanism through which the electron density in a plasma layer can be determined by a measurement of magnetic-field strength.

INTRODUCTION

A NUMBER of authors have discussed the problem of the mixing of two electromagnetic waves in a plasma. Ginzburg[1] has considered two different mechanisms for the generation of combination frequencies in a uniform plasma. In one case, a strong wave at frequency ω_1 traverses a homogeneous, isotropic plasma and produces a variable component of the electron-neutral particle collision frequency at frequency $2\omega_1$, due to the effect of the strong wave on the electron temperature, i.e., $\nu \propto \cos 2\omega_1 t$. A second and weaker wave of frequency ω_2 traversing the plasma results in waves of frequencies $\omega_2\pm 2\omega_1$. In the second case, in the presence of a dc magnetic field, the stronger wave at frequency ω_1 produces additional ionization, i.e., $n \propto \cos\omega_1 t$. The simultaneous incidence of a weaker wave at frequency ω_2 results in waves of frequency $\omega_2\pm\omega_1$. Vilenskii[2] discusses the case of an inhomogeneous plasma, in the absence of an external magnetic field, in which the applied electromagnetic field interacts with the electron density gradient to produce a first-order component of the electron density at frequency ω_1. This then interacts with a second, weaker wave at frequency ω_2 to produce waves at frequencies $\omega_2\pm\omega_1$. The case of an inhomogeneous plasma has been discussed in greater detail by Wetzel.[3,4] Taylor[5] has calculated the effect of the polarizing field in an inhomogeneous plasma upon the mixing of two waves, when the stronger wave is of much lower fre-

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† Portions of this work were performed while the authors were members of the former Palo Alto Laboratories, General Telephone and Electronics Laboratories.

[1] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. **35**, 1573 (1958) [English transl.: Soviet Phys.—JETP **35** (8), 1100 (1959)].

[2] I. M. Vilenskii, Zh. Eksperim. i Teor. Fiz. **22**, 544 (1952); **26**, 42 (1954).

[3] L. Wetzel, J. Appl. Phys. **32**, 327 (1961).

[4] L. Wetzel, Phys. Rev. **123**, 722 (1961).

[5] L. S. Taylor, J. Appl. Phys. **33**, 2913 (1962).

quency than that of the weaker wave. His results indicate that one can obtain combination frequencies $\omega_2 \pm m\omega_1$, where m is an integer.

In the above investigations, it was assumed that one of the two incident waves is much more intense than the other and that the stronger wave modified the existing plasma. The interaction of the second, weaker wave with the modified ionized gas was then examined. The present work discusses frequency mixing in magnetoplasmas when the incident waves are assumed to be of low intensity and of comparable amplitude. The electron-neutral particle collision frequency remains constant, and the electron density remains constant to first order. The nonlinear equations describing the plasma are solved, yielding the amplitudes of the combination waves as functions of the plasma and incident wave parameters. A detailed comparison of these results with experiment is presented in the following paper.[6]

MIXING

In parts I and II[7,8] (hereafter referred to as I and II, respectively) the theory of harmonic generation by an electromagnetic wave propagating through a uniform, weakly ionized layer in the presence of a static magnetic field was derived. In the present work these calculations are extended to include the case of mixing of two electromagnetic waves in a plasma. The theoretical model to be discussed is as follows. Two linearly polarized plane waves at frequencies ω_1 and ω_2 are incident normally on a plasma layer. The plasma layer is assumed to be of thickness d in the direction of propagation of the incident signals and infinite in all other directions. A uniform dc magnetic field is impressed upon the plasma layer in a direction normal to the direction of propagation. The electric field of each incident wave is perpendicular to the direction of the dc magnetic field. The plasma is assumed to be electrically neutral and of uniform electron density in the absence of the electromagnetic forces. The motion of the ions, as well as any thermal effects is neglected. Plane-wave solutions for the fields are examined under the assumption of a constant electron-neutral particle collision frequency. Due to the nonlinear nature of the Boltzmann transport equation, waves at the harmonic frequencies and at the sum and difference frequencies of the fundamental waves are assumed to exist within the plasma layer.

The equations to be solved are Maxwell's equations, coupled with the first two velocity moments of the Boltzmann equation. A detailed discussion of the assumptions and of the method for solving the nonlinear equations is presented in I. It is assumed that each of the variables can be expanded in a Fourier series of the form

$$f(x,t)=\sum_{\substack{m=-\infty\\ n=-\infty}}^{+\infty} f_{mn}(x)e^{-i(m\omega_1+n\omega_2)t}, \tag{1}$$

where f_{00} is a constant and f_{mn} decreases in magnitude as $|m|+|n|$ increases. With this assumption, the resulting equation to be solved is given by

$$\begin{aligned}[\nu-i(m\omega_1+n\omega_2)-\boldsymbol{\omega}_c\times]&\\ \times[\nabla\times\nabla\times\mathbf{E}_{mn}-[(m\omega_1+n\omega_2)/c]^2\mathbf{E}_{mn}]&\\ -i(m\omega_1+n\omega_2)(\omega_p/c)^2\mathbf{E}_{mn}&\cong\mathbf{G}_{mn}\,,\end{aligned} \tag{2}$$

where $\mathbf{E}_{mn}$ is the electric-field intensity at frequency $m\omega_1+n\omega_2$; ν is the electron-neutral particle collision frequency; c is the velocity of light in free space; and

$$\boldsymbol{\omega}_c=-(e/m_e)\mathbf{B}_{00}\,,\quad \omega_p^2=n_{00}e^2/m_e\epsilon_0\,, \tag{3}$$

where ϵ_0 is the dielectric constant of free space; e is the charge of an electron and is negative; m_e is the mass of an electron; $\mathbf{B}_{00}$ is the dc magnetic induction; and n_{00} is the steady-state electron density. Mks units are used throughout this discussion. Hereafter, $\mathbf{B}_{00}$ and n_{00} will be represented by $\mathbf{B}_0$ and n_0, respectively. $\mathbf{G}_{mn}$ is given by

$$\begin{aligned}\mathbf{G}_{mn}=[i(m\omega_1+n\omega_2)\omega_p^2(1-\delta_{mn})/2c^2]\sum_{s,r}\{&\mathbf{v}_{m-s,n-r}\times\mathbf{B}_{s,r}\\ &-(m_e/e)(\mathbf{v}_{m-s,n-r}\cdot\nabla)\mathbf{v}_{s,r}\\ &+(m_e/en_0)[\nu-i(m\omega_1+n\omega_2)-\boldsymbol{\omega}_c\times]\\ &\times n_{m-s,n-r}\mathbf{v}_{s,r}\}\,,\end{aligned} \tag{4}$$

where

$$\begin{aligned}&1\leqslant|m-s|\leqslant|m|-1\,,\\ &1\leqslant|s|\leqslant|m|-1\,,\\ &1\leqslant|n-r|\leqslant|n|-1\,,\\ &1\leqslant|r|\leqslant|n|-1\,.\end{aligned}$$

$\mathbf{v}_{mn}$ is the average electron velocity, n_{mn} is the average electron density, and $\mathbf{B}_{mn}$ is the magnetic induction vector, at frequency $m\omega_1+n\omega_2$, and

$$\delta_{mn}\begin{cases}=0\,, & \text{for } m \text{ and } n\neq 0\,,\\ =1\,, & \text{for } m \text{ or } n=0\,.\end{cases} \tag{5}$$

The effects of the higher order terms in Eq. (4) have been neglected (see I).

Equation (2) subject to the appropriate boundary conditions at the surfaces of the layer is now to be solved for the waves at frequencies $m\omega+n\omega_2$. The procedure for solving Eq. (2) is that presented in I, and only the unique features of this particular problem will be outlined here. The propagation constants k_{m0} and k_{0n}, of the plane-wave solutions, are easily determined because the right-hand side of Eq. (2) is zero and the propagation characteristics are identical to those found

[6] S. J. Tetenbaum, R. F. Whitmer, and E. B. Barrett, following paper, Phys. Rev. **135**, A374 (1964).

[7] R. F. Whitmer and E. B. Barrett, Phys. Rev. **121**, 661 (1961).

[8] R. F. Whitmer and E. B. Barrett, Phys. Rev. **125**, 1478 (1962).

in the linear case. Using these quantities, the first-order approximation to the quantity $\mathbf{G}_{mn}$ is calculated directly. The resulting equation is of the form

$$\mathbf{G}_{mn}=\sum_{j=1}^{4} \mathbf{g}_{mn}{}^{(j)} \exp[\pm i(mk_{10}+nk_{01})x] \times\exp[-i(m\omega_1+n\omega_2)t], \quad (6)$$

where $m=\pm1$ and $n=\pm1$. In Eq. (6) a particular m and n are first chosen, and j indicates a particular combination of the $\pm$ signs in the exponent. The sum is then taken over all combinations of the $\pm$ signs. $\mathbf{g}_{mn}$ is proportional to $\mathbf{E}_{m0}{}^{\pm}\mathbf{E}_{0n}{}^{\pm}$, where the $\pm$ signs indicate the direction of propagation of the waves. Equation (6) indicates several differences between the properties of the second harmonic waves of I and the properties of the waves at the sum and difference frequencies. First, the forward and backward waves are coupled, whereas these terms canceled to first order in the calculation of the wave at the second harmonic frequency. Second, the power at the sum and difference frequencies is proportional to the product of the input powers of the incident waves. Finally, for ν approaching zero, resonances in the combination waves should occur near the plasma resonances given by

$$\omega_c=(\omega_1{}^2-\omega_p{}^2)^{1/2}$$

and (7)

$$\omega_c=(\omega_2{}^2-\omega_p{}^2)^{1/2},$$

since k_{10} and k_{01} have resonances near these values of ω_c.

Having determined $\mathbf{G}_{mn}$, a particular solution to Eq. (2) is then calculated by the method presented in I. The amplitude of the complementary solution is then determined from the boundary conditions. The complete solution to Eq. (2) is now known. Since the propagation constants of the complementary solutions are resonant near the plasma resonances

$$\omega_c=[(m\omega_1+n\omega_2)^2-\omega_p{}^2]^{1/2}, \quad (8)$$

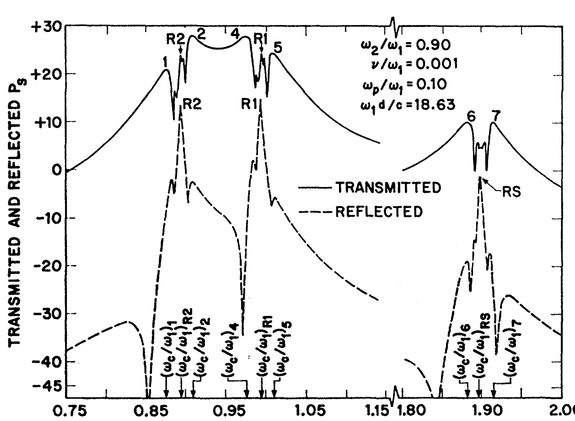

FIG. 1. Transmitted and reflected P_S versus ω_c/ω_1.

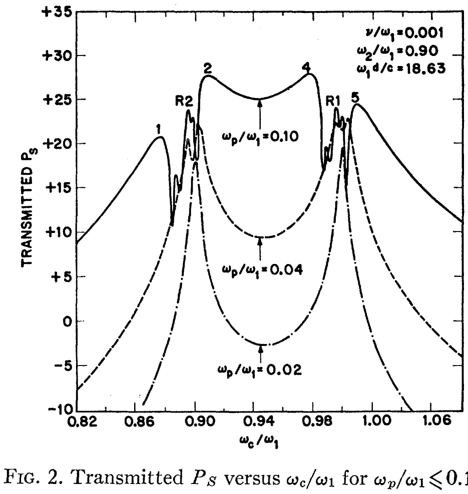

FIG. 2. Transmitted P_S versus ω_c/ω_1 for $\omega_p/\omega_1\leqslant0.1$.

the combination waves should also have resonances in these regions.

The amplitudes of the transmitted and reflected combination waves can be written as algebraic functions of the plasma parameters ω_p/ω_1, ω_c/ω_1, ν/ω_1, $\omega_1 d/c$, and the input wave parameters ω_2, ω_1, E_{10}, and E_{01}. Since the final solution is extremely complex, it will not be presented here. However, the results of computer calculations of the solution will be summarized.

Following I and II the power density in the waves at the combination frequencies, $P_{\omega_1+\omega_2}$, and $P_{\omega_1-\omega_2}$, respectively, can be written as

$$P_{\omega_1\pm\omega_2}=(2\mu_0e^2/m_e{}^2c)(P_1P_2/\omega_1{}^2) \times Q_{\pm}(\omega_p/\omega_1,\omega_c/\omega_1,\nu/\omega_1,\omega_1 d/c,\omega_2/\omega_1), \quad (9)$$

where P_1 and P_2 are the incident power densities at frequencies ω_1 and ω_2, respectively, μ_0 is the permeability of free space and $Q_{\pm}$ is a dimensionless function. Equation (9) indicates that the power in the combination waves is proportional to the product of the powers of the incident waves and inversely proportional to the square of one of the frequencies. Only the quantity $\omega_1{}^2$ appears in the denominator of Eq. (10) rather than the quantity $(\omega_1\pm\omega_2)^2$, because the dimensionless factor $1\pm(\omega_2/\omega_1)^2$ was incorporated into the quantity $Q_{\pm}$. The following discussion will be limited to a study of the quantities

$$P_{S(D)}=10\log_{10}Q_{\pm}, \quad (10)$$

where the subscript S, representing the sum frequency, corresponds to the plus sign, and the subscript D, representing the difference frequency, corresponds to the minus sign in Eq. (9).

THE SUM FREQUENCY

A typical curve of P_S versus ω_c/ω_1 is shown in Fig. 1. A number of resonances occur near values of ω_c/ω_1 corresponding to plasma resonance for the waves at frequencies ω_1, ω_2, and $\omega_1+\omega_2$ as given by Eqs. (7) and

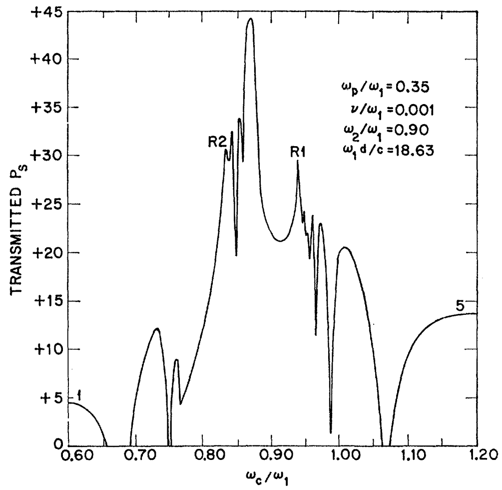

FIG. 3. Transmitted P_S versus ω_c/ω_1 for $\omega_p/\omega_1=0.35$.

(8). Near the first two plasma resonances, the absorption of the incident waves is a maximum. The major resonances which appear in the curve of transmitted P_S are designated 1, $R2$, 2, 4, $R1$, 5, 6, and 7. $R1$ and $R2$ occur at the plasma resonances for the waves at frequencies ω_1 and ω_2, respectively. No peak occurs in the transmitted P_S at the plasma resonance point for the wave at frequency $\omega_1+\omega_2$. The resonances which occur on either side of the plasma resonance points are geometrical in nature since they occur for values of ω_c/ω_1 for which the electrical length of the plasma layer is approximately an integral number of quarter wavelengths thick for either of the incident waves. Such geometrical or boundary resonances have been discussed in relation to the second harmonic in I, Eqs. (40)–(43). The plasma resonances are more easily distinguished in the curve for the reflected P_S since the effective dielectric constant of the plasma near resonance deviates greatly from its free-space value, causing a large reflection at the first interface. The reflected P_S exhibits a plasma resonance corresponding to the frequency $\omega_1+\omega_2$ [Eq. (8)], which is labeled RS. The amplitude of P_S in the region of peaks 6 and 7 is lower than in the other resonance regions because the interaction of the incident waves with the plasma at this value of magnetic field is much weaker than when the magnetic field is close to plasma resonance for the incident signals. The remainder of this discussion will be devoted to describing the changes in structure, amplitude, and location of the resonance peaks in the curves of P_S versus ω_c/ω_1 as the parameters ω_p/ω_1, ν/ω_1, ω_2/ω_1, and $\omega_1 d/c$ assume different values.

The effect of the electron density on P_S is shown in Figs. 2 and 3. As ω_p/ω_1 increases, the outer geometrical resonances, peaks 1 and 2, and 4 and 5, move apart since the plasma becomes electrically thicker. Peaks $R2$ and $R1$ shift according to Eq. (7). The geometrical resonances become progressively broader as the electron density is increased. For values of ω_p/ω_1 in the neighborhood of 0.2, peaks 2 and 4 merge to form a single resonance as indicated by a comparison of Figures 2 and 3. Peaks 5 and 6 broaden and merge as ω_p/ω_1 increases and they are no longer identifiable as resonance peaks when ω_p/ω_1 exceeds approximately 0.4. As the density is increased further, the plasma becomes opaque to the lower frequency wave and only the resonances corresponding to the higher-frequency incident wave are present. The value of the magnetic field at which a particular resonance occurs, is a sensitive function of the electron density. This fact provides a technique for measuring the electron density. The amplitudes of the plasma resonance peaks increase, then level off, and eventually decrease as ω_p/ω_1 increases. In general, the amplitudes of all peaks decrease at sufficiently high values of ω_p/ω_1. The behavior of the plasma resonance peaks is similar to that shown for the peak amplitudes of the second, third, and fourth harmonic waves in I, Fig. 6, and II, Figs. 4 and 5.

The principal effect of increasing ν/ω_1 is to increase the linewidth and decrease the amplitude of all resonance peaks in both the transmitted and reflected P_S curves. Figure 4 shows this effect for the transmitted P_S curve in the main resonance region. The effect on peaks 6, RS, and 7, is similar. Fine structure is eliminated as the subsidiary peaks are damped out by the increasing ν/ω_1. Increasing ν/ω_1 has a more pronounced effect on the plasma resonance peaks than on the geometrical resonances, causing the former to decrease in amplitude more rapidly than the latter. This damping effect of ν/ω_1 on resonances of P_S is due to the decrease in the absorption of the incident waves at plasma resonance as the collisions are increased.

The structure of the P_S versus ω_c/ω_1 curves is a sensitive function of the incident frequency ratio ω_2/ω_1, since many of the resonances overlap as ω_2/ω_1 approaches 1.0. The curves for P_S transform continuously into the curves for the second harmonic as ω_2/ω_1 approaches unity. From Fig. 5, it can be seen that the change in the position of all the peaks is essentially a linear function of ω_2/ω_1 for $0.3\leqslant\omega_2/\omega_1\leqslant 0.95$. For values of $\omega_2/\omega_1\geqslant 0.95$, the plasma resonance peaks $R2$ and $R1$

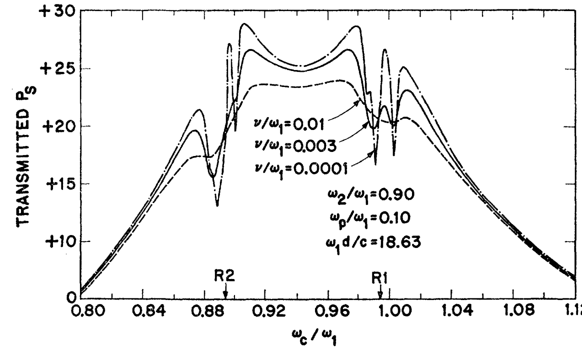

FIG. 4. Transmitted P_S versus ω_c/ω_1 for various values of ν/ω_1.

and the geometrical resonances between $R2$ and $R1$ (2 and 4) merge into a common peak located close to the plasma resonance point for the second harmonic. However, the outer geometrical resonances, peaks 1 and 5, retain their identity even in the limit of the second harmonic.

The effect of varying $\omega_1 d/c$ on the amplitudes and positions of the geometrical peaks is similar to the changes produced in these peaks by varying ω_p/ω_1. This similarity is understandable in terms of the geometrical nature of these peaks, since changes in the physical thickness of the layer are equivalent to changes in the electrical thickness of the layer.

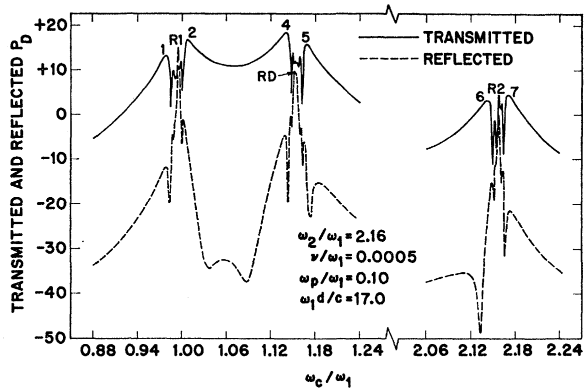

FIG. 6. Transmitted and reflected P_D versus ω_c/ω_1.

THE DIFFERENCE FREQUENCY

Figure 6 shows a typical curve of P_D versus ω_c/ω_1 for representative values of the plasma parameters and incident frequency ratio. The frequency ratio $\omega_2/\omega_1 = 2.16$ was chosen in order to separate the resonance regions, which for low-electron densities occur near $\omega_c/\omega_1 = 1.0$ and $\omega_c/(\omega_2-\omega_1) = 1.0$. As with the P_S curves, there are three principal resonance regions. These occur in the neighborhood of the plasma resonance points for the waves of frequency ω_1, ω_2, and the combination frequency $\omega_2-\omega_1$. The resonance peaks of the transmitted P_D are affected by changes in ω_p/ω_1 and ν/ω_1 in an analogous fashion to those of P_S. For similar conditions of electron density, collision frequency, and slab thickness, the maximum amplitude of P_D is approximately 10 to 15 dB less than that of P_S. This is due to the wide spacing between the incident frequencies, which allows only one of the waves to be close to plasma resonance for any given set of values of the plasma parameters.

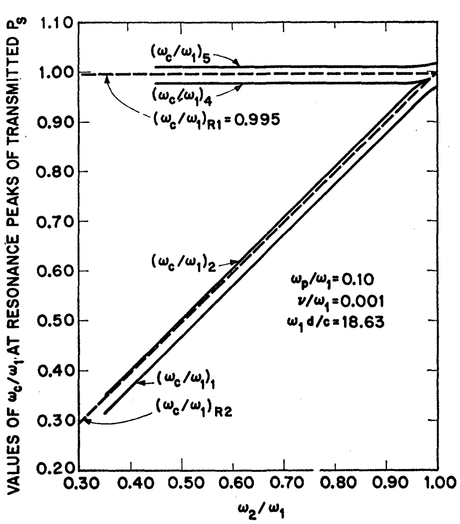

FIG. 5. Values of ω_c/ω_1 at resonance peaks of P_S versus ω_2/ω_1.

The resonance peaks of the reflected P_D in Fig. 6 are designated $R1$, RD, and $R2$. Peaks $R1$ and $R2$ are plasma resonance peaks occurring exactly at the plasma resonance points for the waves of frequency ω_1 and ω_2, respectively, and correspond to similar plasma resonance peaks on the reflected P_S curves. It can be seen that the $R1$ amplitude of the reflected P_D actually exceeds the $R1$ amplitude of the transmitted P_D. The RD peak occurs close to, but not exactly at, the plasma resonance point for a wave of frequency $\omega_2-\omega_1$, and it tends to broaden with increasing ω_p/ω_1. As was the case with the reflected P_S curve, the subsidiary geometrical resonance peaks in the reflected P_D curve increase in number as ω_p/ω_1 increases, and tend to spread away from the resonance peaks $R1$, RD, and $R2$.

ACKNOWLEDGMENT

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