Approximate Independence of Optical-Model Elastic Scattering Calculations on the Potential at Small Distances*

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In optical-model calculations of the elastic scattering of alpha particles or other heavier nuclei on nuclei, it has been observed that the cross section is rather insensitive to variation of the optical potential at small values of the interaction distance. In the present note it is pointed out that in many cases this effect can be explained with the aid of the JWKB approximation. If the real part V of the optical potential $V+iW$ is sufficiently deep and if W produces sufficient absorption, then at some point R_b contained inside the interaction region, the outgoing branch of the JWKB expression for the radial wave function becomes negligible compared to the ingoing branch. In this case the choice of the potentials for $r<R_b$ does not affect the phase shifts, and the use of an ingoing wave boundary condition applied at R_b to the radial wave function is nearly equivalent to the use of conventional optical-model procedures. Under such circumstances the phase shifts are in essence determined by the penetration of the radial wave function through the barrier formed by the combined action of the Coulomb, nuclear, and centrifugal potentials.

I. INTRODUCTION

THE optical model has been extensively used in the description of the interaction between a light and a heavy nucleus, as for example, in distorted-wave Born approximation calculations of stripping or pickup reactions. In many cases it was found that the final results are insensitive to certain combinations of optical-model parameters. Indeed Igo,[1] in analyzing elastic α-nucleus scattering, finds that the depth of the optical-model potential in the "interior" of the interaction region does not affect the angular distribution and he is led to optical potential parameters which characterize only the "surface" of the interaction region. The optical model is also coming into use for the description of the interaction of heavy ions. McIntosh, Park, and Rawitscher[2] theoretically obtain the long-range part of the interaction between N^{14}, C^{12}, and O^{16}, which, when combined with a phenomenological part of the interaction at small distance lead to cross sections in reasonable agreement with experiments. A purely phenomenological optical-model potential was employed in 1958 by Porter[3] in the study of $N^{14}-N^{14}$ elastic scattering, and recently also by Kuehner and Almqvist[4] in a study of $O^{16}-C^{12}$, $C^{12}-N^{14}$, and $N^{14}-Be^{9}$ scattering. In all these optical-model calculations it is assumed that the nucleus-nucleus interaction can be represented by a two-body potential, to be employed for all values of the interaction distance r. At distances such that the two nuclei overlap considerably, the optical potential is interpreted as a mathematical device used to obtain wave functions which can be given physical meaning only at large distances. This interpretation would become questionable if the cross section should depend on the choice of the optical potential in the region where the overlap of nuclear matter is pronounced. It is of interest, therefore, to have an understanding of the mathematical conditions under which the cross section is expected to be insensitive to the small distance potentials. A study of this question had led to the JWKB consideration outlined below, which may serve as a partial guide in the interpretation of optical-model calculations.

II. JWKB CONSIDERATIONS

In this section the solution of the usual radial differential equation

$$(\hbar^2/2\mu)(d^2\mathfrak{F}_L/dr^2)+(E-V_{\text{eff}}{}^L-iW)\mathfrak{F}_L=0 \qquad (1)$$

will be discussed in terms of the JWKB approximation. Here $\mathfrak{F}_L(r)$ is the partial wave corresponding to angular momentum $\hbar L$, V_{eff} contains the real part V^N of the optical-model potential, the Coulomb potential V^c and the centrifugal term

$$V_{\text{eff}}{}^L=V^c+V^N+(\hbar^2/2\mu)L(L+1)/r^2, \qquad (2)$$

W is the imaginary part of the optical-model potential, μ is the reduced mass, $\hbar$ is Planck's constant divided by 2π, and r is the center-to-center distance between the two interacting nuclei. Equation (1) is usually solved by numerical integration, with the boundary condition that $\mathfrak{F}_L$ vanishes at the origin of r, and a nuclear phase shift, $K_L=K_L{}^R+iK_L{}^I$ is obtained in the usual way. The JWKB approximation becomes valid if the effective wavelength

$$k_L(r)=(2\mu/\hbar^2)^{1/2}(E-V_{\text{eff}}{}^L-iW)^{1/2} \qquad (3)$$

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[1] G. Igo, Phys. Rev. **115**, 1665 (1959). For a review of α-particle scattering on nuclei see R. M. Eisberg and C. E. Porter, Rev. Mod. Phys. **33**, 190 (1961).

[2] J. S. McIntosh, S. C. Park, and G. H. Rawitscher, Phys. Rev. **134**, B1010 (1964). A preliminary account of this work is given in *Proceedings of the Second Conference on Reactions between Complex Nuclei*, edited by A. Zucker, F. T. Howard, and E. Halbert (John Wiley & Sons, Inc., New York, 1960), p. 127; and also in the *Proceedings of the Third Conference on Reactions between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, California, 1963).

[3] C. E. Porter, Phys. Rev. **112**, 1722 (1958).

[4] J. A. Kuehner and E. Almqvist, *Proceedings of the Third Conference on Interactions between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, California, 1963).

varies sufficiently slowly with distance so that

$$|(dk_L/dr)/2k_L^2| \ll 1 \tag{4}$$

as is frequently the case at sufficiently large distances from a turning point r_T. The JWKB approximation for $\mathfrak{F}_L$ then is given by

$$(\mathfrak{F}_L)_{\text{JWKB}} = A_L k_L^{-1/2} \exp\left(i\int_{r_T}^{r} k_L dr\right) + B_L k_L^{-1/2} \exp\left(-i\int_{r_T}^{r} k_L dr\right). \tag{5}$$

The term with the positive exponential is the outgoing branch, the other is the ingoing branch. The ratio of the constants A_L and B_L depends on the nature of the potentials near r_T and on the condition imposed on $\mathfrak{F}_L$ for $r \ll r_T$.

In the presence of a negative imaginary potential W, the imaginary part of k_L, $\text{Im}k_L$, is positive, and hence the ratio $\mathfrak{R}_L$ of the magnitudes of the outgoing to the ingoing branch

$$\mathfrak{R}_L = |A_L/B_L| \exp\left(-2\int_{r_T}^{r} \text{Im}k_L dr\right) \tag{6}$$

decreases with increasing r. If the ratio $\mathfrak{R}_L$ should become negligible compared to one beyond a certain value r_L', then the outgoing branch is negligible compared to the ingoing one and the logarithmic derivative of $\mathfrak{F}_L$,

$$(d\mathfrak{F}_L/\mathfrak{F}_L dr)_{\text{JWKB}} = -(dk_L/dr)/(2k_L) - ik_L, \tag{7}$$

depends only on the local value of k_L and its derivative and not on the potentials present for smaller values of r_L'. In this case the nuclear phase shift is independent of the potentials for $r < r_L'$, as long as the validity of Eq. (7) at r_L' continues to hold. If the JWKB approximation is valid beyond r_L', i.e., for $r_L' < r < r_L''$, then the nuclear phase shift depends only on the potentials for $r > r_L''$.

From the above discussion, the following procedure denoted by the abbreviation IWB is suggested. At some value R_b of r the logarithmic derivative of the wave function is taken equal to the right-hand side of Eq. (7), and the wave function is continued from R_b to the matching point R_m by numerical integration of the Schrödinger equation. The usual optical-model procedure, in which the wave function $\mathfrak{F}_L$ is set equal to zero at the origin, is denoted by OM. The IWB and OM nuclear phase shifts are nearly the same if Eq. (7) is a good approximation for the OM logarithmic derivatives at R_b. If OM and IWB logarithmic derivatives disagree at R_b, then OM and IWB nuclear phase shifts may nevertheless be nearly the same if the increase of the magnitude of the wave functions between R_b and the matching point R_m is sufficiently large. The reason is as follows: The nuclear phase shift is related to the difference of the logarithmic derivatives of Coulomb and nuclear wave functions

$$\Delta_L = [F_L'/F_L - \mathfrak{F}_L'/\mathfrak{F}_L]_{r=R_m}$$

according to the well-known relation

$$[\exp(2ik_L) - 1]/2i = [F_L^2 \Delta_L/(1 - F_L H_L \Delta_L)]_{r=R_m},$$

where F_L and G_L are the usual regular and irregular Coulomb wave functions, $H_L = iF_L + G_L$, and the prime denotes differentiation with respect to kr. An error in the logarithmic derivative at R_b gives rise to an error at R_m according to the relation

$$(\mathfrak{F}_1'/\mathfrak{F}_1 - \mathfrak{F}_2'/\mathfrak{F}_2)_{R_m} = -(\mathfrak{F}_1'/\mathfrak{F}_1 - \mathfrak{F}_2'/\mathfrak{F}_2)_{R_b}[(\mathfrak{F}_1\mathfrak{F}_2)_{R_b}/(\mathfrak{F}_1\mathfrak{F}_2)_{R_m}], \tag{8}$$

where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two nuclear radial wave functions $\mathfrak{F}_L$ which obey different boundary conditions at R_b but satisfy the Schrödinger equation with the same potentials for $r > R_b$.

It is the purpose of this discussion to determine a position of R_b such that the IWB and OM procedures give nearly the same results for all relevant values of L. It should be pointed out that the use of form (7) as a boundary condition is not the only possible one which leads to ingoing waves. A simpler L-independent form has been employed by Feshbach, Weisskopf, and coworkers[5] in calculations of neutron-nucleus interactions; a similar form has been employed by Ebel and Becker[6] in a study of $N^{14}-N^{14}$ scattering, and an L-dependent ingoing wave boundary condition has been under recent consideration by Strutinsky.[7] The choice of Eq. (7) is motivated here[8] by the desire of establishing a connection to optical-model calculations.

For the usual choice of an attractive nuclear potential of a Woods-Saxon form, the plot of V_{eff} versus r, for low values of L, has a valley which is contained between the repulsive centrifugal part near the origin and the repulsive hump, to be denoted in what follows as the barrier region, where the Coulomb and centrifugal potentials dominate over the nuclear potential. As the value of L increases the valley and barrier in V_{eff} become less pronounced, until they disappear entirely.

[5] H. Feshbach and V. F. Weisskopf made extensive use of an ingoing wave boundary condition (IWBC) for the calculation of reactions involving nucleons. Phys. Rev. **71**, 145 (1947); and **76**, 1550 (1949). However, the continuum model, which is based on the IWBC, has lately fallen into disuse because it gives a wrong energy dependence of the reaction cross section, as described by G. E. Brown, Rev. Mod. Phys. **31**, 893 (1959). The above-mentioned authors assume an angular momentum independent wave number for the ingoing wave boundary condition and also take no account of the rounding of the Coulomb barrier due to the diffuse nature of the nuclear potential.

[6] M. E. Ebel and R. L. Becker, following a suggestion by G. Breit, have applied the ingoing wave boundary condition, in much the same spirit as described in footnote 5, to the elastic scattering of N^{14} on N^{14}, and have compared the resulting phase shifts to several optical-model results.

[7] V. M. Strutinsky, paper read at the *Third Conference on Reactions between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, California, 1963), and recent preprint from the I. V. Kurtchatov Institute of Atomic Energy, Moscow, 1963 (unpublished).

[8] The IWB procedure as outlined here has been used in part by the authors in the publications listed in Ref. 2, and the connection to optical-model calculations was apparent to the present author since April 1961.

Accordingly, two extreme situations can be distinguished. For sufficiently low incident energies the values of L which contribute significantly to the scattering cross section are small enough so that the corresponding barriers in V_{eff} are well developed. The other case is that of high energies where only one turning point exists for the large values of L which are still significant.

The high-energy case will be discussed first, although the use of the IWB method is in this case not as useful as in the low-energy case. An imaginary potential of a Woods–Saxon-like r dependence is assumed present. One turning point, $r_T(L)$, exists for the high values of L and the corresponding reflection coefficient, $\exp(-2K_L{}^I)$ depends on the magnitude of the imaginary potential present for r larger than $r_T(L)$. In this case a small value of the reflection coefficient corresponds to a large increase of the magnitude of the wave function between turning and matching points, which can be seen as follows. If the JWKB approximation is valid at the matching point R_m, then $\mathfrak{R}_L$, as given by Eq. (6) and evaluated between r_T and R_m, is equal to the reflection coefficient. For small values of the reflection coefficient, the outgoing branch has become negligible compared to the ingoing one at $r=R_b$, and the quantity in square brackets of Eq. (8), with $\mathfrak{F}_1$ taken to be the same as $\mathfrak{F}_2$, is approximately as small as the part in $\mathfrak{R}_L$ containing the exponential. The position of R_b can now be chosen to occur near the turning point of the highest value of L for which the corresponding reflection coefficient is still as small as the desired order of the error made through the use of Eq. (7). If L_b denotes this value of L, then for $L\sim L_b$ the error in the phase shift is, according to Eq. (8) and the above considerations, of the order of the reflection coefficient. For $L>L_b$, R_b lies in the region where $E<V_{\text{eff}}$ and Eq. (7) is valid to within the error of the JWKB approximation at R_b. For $L<L_b$, $r_T(L)<R_b$, W should be large enough for $r\gtrsim r_T(L)$ so as to absorb the outgoing branch of wave function relative to the ingoing one at $r=R_b$, and Eq. (7) should again be valid. For the low values of L the "hump" in V_{eff}, mentioned previously, may be in evidence, and since, as illustrated in the Appendix for the case of a square-well complex potential, the absorption of the outgoing branch decreases the larger the value of $E-V_{\text{eff}}$, a part of the outgoing wave may still be present at R_m. In this case the reflection coefficient fluctuates as a function of L, as can also be seen from the illuminating considerations of Austern.[9] The independence of the phase shifts of the potentials for $r<R_b$, is in the case of high energies to a large extent a result of the fact that the wave functions are very small for $r<R_b$, and the position of R_b depends sensitively on the amount of imaginary potential present. This will be illustrated by a numerical example in Sec. III.

The case of low energies differs from the one just discussed mainly because the presence of the additional barrier destroys the validity of the JWKB approximation in the "barrier" region. As a result a great deal of reflection is introduced, and the reflection coefficient depends to a large extent on the penetrability through the barrier. The position of R_b now depends on the condition that for all values of L less than a certain L', (a) R_b be contained in the validity region of the JWKB approximation and (b) the value of $\mathfrak{R}_L$ calculated from r_T to R_b, be $\ll 1$. Fulfillment of conditions (a) and (b) depends on the presence of sufficient imaginary potential for $r_T(L)<r<R_b$ and on the validity of Eq. (4) near R_b. The value of L' is again determined by the requirement, in addition to (a) and (b) above, that the square bracket in Eq. (8) be sufficiently small so as to eliminate the error introduced at R_b for $L>L'$.

This condition is expected to be satisfied for values of L' such that the corresponding reflection coefficient is close to unity, as can be seen from considerations of barrier penetrability. Accordingly, the value of R_b can be taken close to the middle one of the three turning points which occur in $V_{\text{eff}}{}^L$ for $L=L'$.

Whenever Eq. (7) can be applied as a boundary condition for the nuclear wave functions, then in the low-energy case the nuclear phase shifts are essentially determined by the amount of penetration of the wave functions through the barrier region. By approximating the values of V_{eff} in the barrier region by means of parabolas, the penetrability can be estimated in terms of parabolic Weber functions, as has been done several years ago by Wheeler and collaborators.[10] This procedure can be extended[11] to obtain an approximation for the real part of the nuclear phase shifts as well, and serves to give an insight into the L dependence of optical-model nuclear phase shifts at low energies. In the cases where the outgoing branch of the JWKB approximation to the radial wave functions in the valley region is not negligible compared to the ingoing one, the IWB considerations alone do not apply, and the additional interference "wiggles" which then appear in the cross section depend on the nature of the potentials in the valley region. The considerations of Austern[9] and of Drisco *et al.*[12] are very helpful for these cases.

It also should be kept in mind that the use of Eq. (7) at $r=R_b$ implies either that the optical-model description of the elastic scattering of heavy ions is a valid procedure for all values of r, the optical-model potentials being such that the outgoing waves disappear at R_b, or

[9] N. Austern, Ann. Phys. (N. Y.) **15**, 299 (1961).

[10] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953) particularly on p. 1141, and also K. W. Ford, D. L. Hill, M. Wakano, and J. A. Wheeler, Ann. Phys. (N. Y.) **7**, 239 (1959) make use of Weber function results for parabolic barriers to estimate the L dependence of nuclear phase shifts. T. D. Thomas, Phys. Rev. **116**, 703 (1959) has used parabolic transmission coefficients in the calculation of compound nucleus formation in heavy ion induced reactions. Comparisons between parabolic barrier transmission coefficients and optical model results were carried out for alpha particles by J. R. Huizenga and G. Igo, Nucl. Phys. **29**, 462 (1962).

[11] G. H. Rawitscher (to be published).

[12] R. H. Drisko, G. R. Satchler, and R. H. Bassel, Phys. Letters **5**, 347 (1963).

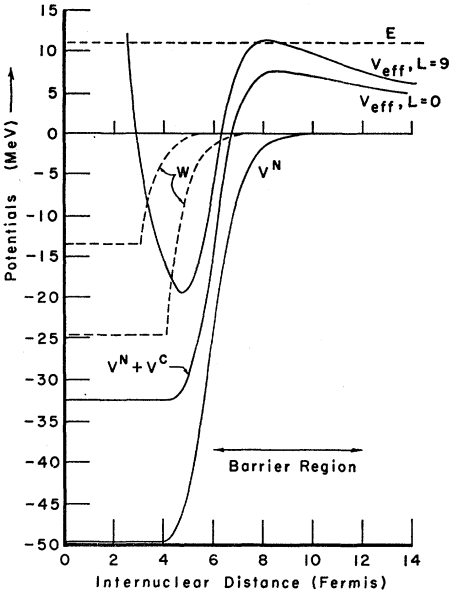

FIG. 1. Potentials used in the numerical example discussed in Sec. III. The nuclear potential is V^N+iW; V_{eff} denotes the sum of the real part of nuclear, Coulomb, and centrifugal potentials. These potentials are typical of low energy $C^{12}-N^{14}$, $N^{14}-N^{14}$, and $C^{12}-O^{16}$ optical models discussed by McIntosh *et al.* (Ref. 2). The barrier region extends from about 6.5 to 10 F. Two imaginary potentials are shown by dashed lines. They correspond to the Woods-Saxon form defined in Fig. 3, with R equal to 2.5 and 4 F, respectively.

else, that an optical-model description is valid for $r>R_b$ together with the assumed validity of the use of Eq. (7) at R_b. Neither of these assumptions is up to the present justifiable on physical grounds. A more physical procedure would consist in imposing a phenomenological boundary condition at a radius R_b' where the validity of the two-body description of the nucleus-nucleus system could be established without question.[13] This procedure then would be independent of the JWKB approximation, R_b' would be an essential parameter and the boundary parameters would play a role similar to the R-matrix elements.

The above-mentioned considerations require the validity of the JWKB approximation in the valley region, as well as the smallness of the quantity $\mathfrak{R}_L$ at R_b. In what follows it will be made plausible by means of examples that these conditions are likely to be satisfied for many cases of heavy nucleus-nucleus as well as alpha-nucleus interactions, but it should be emphasized that these are not necessary conditions for the optical-model cross sections to be insensitive to the values of the potential at small distances.

III. NUMERICAL EXAMPLES

For heavy ion scattering the interaction distances and the reduced masses are sufficiently large so that condition (4) is likely to be satisfied in the valley region mentioned previously. An example typical of the optical potentials required for the $N^{14}-C^{12}$, $N^{14}-N^{14}$, and $O^{16}-C^{12}$ cases considered by McIntosh *et al.*[2] is shown in Fig. 1 and will be discussed here in some detail. The value of $2\mu/\hbar^2$ is 0.335 F^{-2} MeV^{-1}, the energy is 11 MeV, and the left-hand side of relation (4) is less than 0.1 for 3.1 F$<r<$6.0 F and for $L=9$. The applicability of Eq. (7) depends on how small $\mathfrak{R}_L$, Eq. (6), is compared to unity. In the Appendix it is shown that for a square-well complex potential, the value of $|A_L/B_L|$ is less than unity, and hence the value of $\mathfrak{R}_L$ can be estimated from the exponential occurring in Eq. (6). This estimate will also hold for other, nonsquare-well cases, provided that the optical-model potential is nearly constant from the origin to a region beyond the first turning point r_T where the JWKB approximation is valid. The use of Eq. (A6) of the Appendix leads to $|A_9/B_9|\sim 0.7$, and evaluation of the integral in Eq. (6) from $r_T=3.1$ F to $r=6$ F leads to $\mathfrak{R}_9\sim 0.03$. The applicability of the boundary condition (7) is examined by comparing the results obtained from the left-hand side of Eq. (7) with logarithmic derivatives of the numerically calculated optical-model wave functions, both calculated at $r=7.08$ F. The comparison is shown in Fig. 2. The discrepancy between the optical-model re-

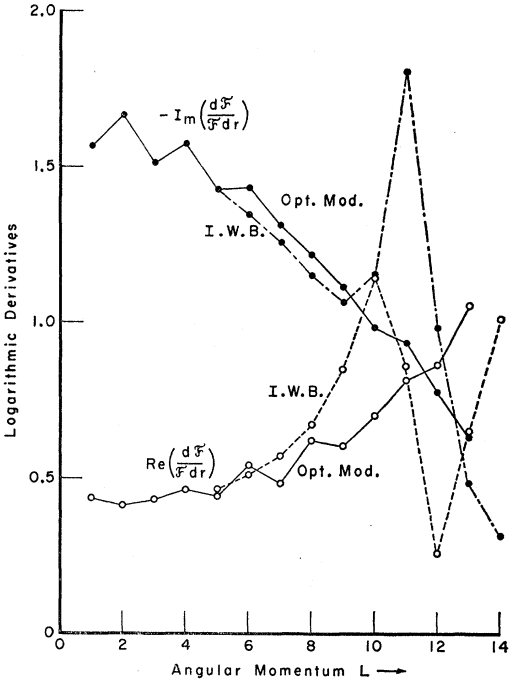

FIG. 2. Comparison between optical-model and JWKB logarithmic derivatives of nuclear wave functions. The corresponding optical-model potentials are indicated in Fig. 1. The JWKB results neglect the outgoing part of the wave function and make use of Eq. (7), text. Both results are calculated at $r=7.08$ F. The abscissa represents values of the angular momentum in units of $\hbar$. The disagreement for L larger than 8 is due to the nonvalidity of the JWKB approximation at $r=7$ F.

[13] The author acknowledges helpful discussions with Professor G. Breit concerning this point.

sults and the results of Eq. (7), labeled IWB, for values of L larger than 8 is due to the lack of validity of the JWKB approximation at 7.08 F. The imaginary part of the potential employed in these calculations is the larger of the two W's shown in Fig. 1.

As another test of the applicability of Eq. (7) in the example under discussion, the imaginary potential was set equal to zero and the optical-model wave functions were started at $R_b \sim 4$ F by means of the boundary condition, Eq. (7), and then numerically integrated towards the matching point in the usual manner. The results are labeled with IWB and compared to optical-model results in Fig. 3. In order to investigate the effect of the presence of the imaginary potential in the barrier region, various imaginary potentials were employed in the optical-model calculations. The functional form of W is given in Fig. 3, where the significance of R becomes apparent. Two such W's are shown as dashed lines in Fig. 1, corresponding to $R=2.5$ F and 4 F, respectively.

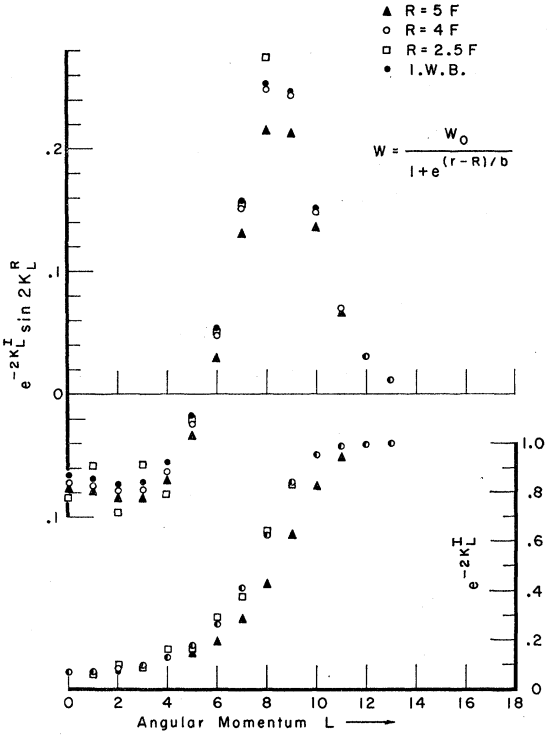

FIG. 3. Comparison of optical-model and IWB nuclear phase shifts $K_L = K_L^R + iK_L^I$, in order to test the validity of the ingoing wave boundary condition. The condition is applied to the wave function at the onset of the barrier region, as discussed in Sec. II. The abscissa represents values of the angular momentum in units of $\hbar$. Both calculations involve numerical solutions of the Schrödinger equation for the potentials shown in Fig. 1. In the optical-model calculations the wave function is started at the origin; in the "boundary condition" case, IWB, the imaginary potential is set equal to zero and the wave function is started at $r=4.0$ F with the logarithmic derivative given by Eq. (7). The various imaginary potentials W used for the optical-model calculations are given by the Woods-Saxon expression shown in the figure. Two such W's corresponding to $R=4$ and 2.5 F are shown by dashed lines in Fig. 1.

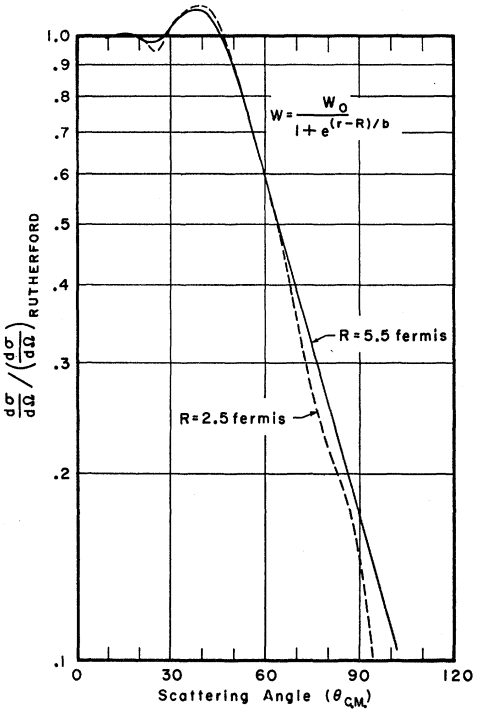

FIG. 4. Sensitivity of optical-model cross sections to the size of the imaginary potential. Ratios of the cross section to Rutherford cross section are plotted versus c.m. scattering angle. The nuclear phase shifts corresponding to one of the cases are shown in Fig. 3. The IWB cross section is not shown since it is nearly indistinguishable from the dashed curve, from which, however, the wiggles are averaged out.

Examination of Fig. 3 shows that as W is shifted toward smaller values of r, i.e., as R is decreased, the phase shifts move towards the IWB results, and nearly coincide with them for $R=4$ F. As R is reduced further, the phase shifts remain fairly stable until they begin to oscillate around the IWB values, as is illustrated by the case $R=2.5$ F. For such small values of W the outgoing branch is no longer negligible compared to the ingoing one at $r \sim R_b$ and the interference between the two is responsible for the oscillations. The corresponding cross sections are shown in Fig. 4. The IWB cross section is not shown since it lies very close to the dashed curve, obtained with $R=2.5$ F. It does not have the slight "wiggles" of the dashed curve, however.

The comparison of optical-model phase shifts to IWB results, the latter calculated with $W=0$, serve to emphasize that small amounts of imaginary potential present in the barrier region do not affect the main dependence of the complex nuclear phase shift on L, which is determined by the position and penetrability of the barrier in V_{eff}. The IWB method is, of course, not restricted to situations in which W is negligible in the barrier region. Calculations of the IWB type have been performed for the numerical example discussed above, in which various amounts of W are present for $r \gtrsim R_b$. The dependence of the phase shifts on W is similar to

that obtained in the optical-model case, i.e., the value of the maximum of $\sin(2k_L{}^R)\exp(2k_L{}^I)$ decreases and the differential cross section in the backward angles increases with an increase of W in the barrier region. In another series of calculations the sensitivity of the phase shifts to the choice of the position R_b of the ingoing wave boundary radius was tested. As R_b was changed from 5.0 to 6.0 F, the most sensitive phase shifts changed in the second place after the decimal point, while for $R_b=6.5$ F a change in the first place after the decimal point was noted. From the arguments made in the preceding section, one would expect 6.5 F to be an upper limit for R_b since for $L'=11$, $\exp(-2k_L{}^I)\sim0.991$ and the turning point $r_T'(11)\sim7.1$ F. At the distance of ~0.66 F to this turning point the JWKB approximation is valid to about 5% accuracy according to the Appendix and hence $R_b\sim7.1-0.66\sim6.5$ F. It is useful to note that 6.5 F is equal to 1.35 F$\times(A_1{}^{1/3}+A_2{}^{1/3})$.

For a comparison to experiment of the cross sections obtained by means of the IWB procedure at low energies the reader is referred to the paper of McIntosh *et al.*[2] In these comparisons the imaginary part of the potential is set equal to zero and the real part, having to be specified only in the barrier region, is given in terms of the parameters V_0 and a, by the expression

$$V^N=V_0\exp(-r/a)$$

thus reducing the total number of optical parameters to 2. The choice of R_b does not affect the results as long as it is taken inside the valley region, which corresponds to $R_b\sim1.20$ F$\times(A_1{}^{1/3}+A_2{}^{1/3})$ for the nuclei there discussed. By setting $W=0$, the simplifying but unnecessary assumption is made that the imaginary potential does not extend into or beyond the barrier region.

A high-energy case is obtained by increasing the center-of-mass energy to 60 MeV in the numerical example discussed above. The dependence on the imaginary potential is examined by the comparison of the two cases for which $R=4.0$ and 5.5 F, and the results contained in Table I show that for the high energy the sensitivity of the phase shift on W is more pronounced than it is for the low energy. A plot of $V_{\text{eff}}{}^L$ versus r shows that for $L\gtrsim25$, the valley and barrier has degenerated into curves decreasing monotonically with r. The values of R_b chosen for the case $R=4$ F are 4 and 5 F. The former is close to the turning point of $L=22$, for which the reflection coefficient is 0.038 and good agreement between OM and IWB results is expected. The agreement in the phase shifts is good to the third figure after the decimal point, and hence the OM values are not given in the table. The turning point for $L=27$ occurs near 5 F, and since the reflection coefficient for that L is ~0.3, poor agreement between IWB with $R_b=5$ F and OM results are obtained. For the case of larger imaginary potential, $R=5.5$ F, the curve of the reflection coefficient versus L has a marked shift towards higher L values, which is to be contrasted with the low-energy case. The choice of $R_b=4.6$ F now gives excellent agreement with the corresponding OM result since 4.6 F is close to the turning point for $L=26$, for which the reflection coefficient is ~0.02. Even for $R_b=5.6$ F good agreement is still obtained.

TABLE I. Dependence of $\exp(-2K_L{}^I)$ on W and R_b for two different energies. The real potentials are the same in all cases.

	$E_{\text{c.m.}}=10.8$ MeV					$E_{\text{c.m.}}=60$ MeV		
	OM[a]	IWB[c]				IWB[c]		
	$R=4$[b]	$W=0$	$R=4$[b]	$R=5.5$[b]		$R=4$[b]		$R=5.5$[b]
L		$R_b=5$[c]	$R_b=6$[c]	$R_b=6$[c]	L	$R_b=4$[c]	$R_b=5$[c]	$R_b=4.6$[b]
0	0.083	0.065	0.065	0.090	22	0.039	0.007	0.0006
1	0.067	0.069	0.070	0.096	23	0.069	0.011	0.0013
2	0.099	0.080	0.081	0.108	24	0.109	0.019	0.0029
3	0.085	0.098	0.100	0.129	25	0.160	0.043	0.0071
4	0.151	0.128	0.131	0.164	26	0.234	0.140	0.0175
5	0.153	0.180	0.185	0.220	27	0.314	0.356	0.044
6	0.293	0.269	0.275	0.308	28	0.337	0.352	0.106
7	0.383	0.416	0.423	0.444	29	0.615	0.616	0.234
8	0.649	0.630	0.634	0.624	30	0.888	0.888	0.425
9	0.828	0.842	0.841	0.803	31	0.958	0.958	0.620
10	0.958	0.956	0.953	0.919	32	0.979	0.979	0.763
11	0.989	0.991	0.990	0.971	33	0.988	0.988	0.852
12	0.998	0.999	0.998	0.990	34	0.993	0.993	0.906

[a] Optical-model results are denoted by OM.
[b] The imaginary potential is characterized by the value of R, defined in Fig. 3. The values of R and R_b are in units of 10^{-13} cm.
[c] The use of Eq. (7) at $r=R_b$ together with the usual optical-model procedures for $r>R_b$ is indicated by IWB.

An example for alpha particles was carried out for the nucleus of calcium at 29.1-MeV c.m. energy. The potentials used were those obtained by Igo[1] for argon at 18 MeV, given by $(-87-15i)/\{1+\exp[(r-5.37)/0.6]\}$. The numbers are in units of either MeV or F. This example falls into the category of "low" energy, since for $L=15$ the reflection coefficient is 0.77 and the corresponding curve of V_{eff} versus r has a minimum of 9 MeV at 4.75 F and a maximum of 32 MeV at 6.9 F. The imaginary potential at that point is -1.1 MeV. The distances and the reduced mass occurring in this alpha-particle case are less than the corresponding quantities in the heavy ion case discussed above, so that the values of $\mathfrak{R}_L$ are not as small. The numbers are as follows. The distance from the turning point beyond which the JWKB approximation becomes valid is in this α-particle case very similar to the corresponding quantity in the heavy ion case. The reason is that the slope of V_{eff} with r is larger in the α-particle case and compensates for the decrease in the reduced mass, and hence the value of $c^{-1/3}$, defined in Eqs. (A7) and (A8) of the Appendix, remains approximately unaltered, equal to 0.72 F in the vicinity of $r_T'(15)\cong6.3$ F. The prime in r_T indicates that the middle one of the three turning points is being referred to. The value of 5.5 F is close to the upper limit for R_b since it is within 0.7 F of $r_T'(15)$. Evaluation of Eq. (6) for $\mathfrak{R}_{15}$ leads to ~0.16, which is about four times larger than the corresponding value of $\mathfrak{R}_9$ in the heavy ion case, because now the radial distance over which the integral is to be evaluated is smaller by a

TABLE II. Comparison of α-Ca differential scattering cross sections[a] for the OM[b] and IWB[b] calculations. The quantities listed[c] represent ratios to Rutherford scattering cross sections.

θ[d]	OM	IWB $R_b=3.5$[b]	IWB $R_b=5.0$[b]	θ	OM	IWB $R_b=3.5$[b]	IWB $R_b=5.0$[b]
30	0.15(0)	0.14(0)	0.14(0)	65	0.16(−1)	0.63(−2)	0.79(−2)
35	0.19(0)	0.20(0)	0.19(0)	70	0.18(−1)	0.39(−1)	0.41(−1)
40	0.56(−2)	0.75(−2)	0.69(−2)	75	0.22(−1)	0.35(−1)	0.40(−1)
45	0.18(0)	0.14(0)	0.14(0)	80	0.25(−3)	0.59(−2)	0.82(−2)
50	0.68(−1)	0.46(−1)	0.43(−1)	85	0.64(−2)	0.23(−1)	0.28(−1)
55	0.29(−1)	0.31(−1)	0.32(−1)	90	0.23(−2)	0.14(−1)	0.23(−1)
60	0.10(0)	0.87(−1)	0.86(−1)	95	0.41(−2)	0.63(−2)	0.66(−2)

[a] The center-of-mass energy is 29.1 MeV.
[b] The significance of the symbols is the same as in the previous table.
[c] The numbers in parenthesis represent the exponent of 10 by which the numbers in front of the parenthesis have to be multiplied.
[d] The scattering angle in the c.m. system is represented by θ and is given in degrees.

factor of about 2, and also because of the smaller value of the mass. Nevertheless, as Table II shows, the IWB method still gives reasonably good agreement with optical-model results, as would be expected from the calculations of Igo.[1]

SUMMARY AND CONCLUSIONS

A large class of absorbing optical potentials exist which lead to very similar phase shifts and cross sections. Some of the conditions to be satisfied for this to be true can be stated in terms of the JWKB considerations discussed in this paper. The potentials have in common the property of giving rise to only the ingoing branch of the JWKB wave function at the end of the region where they differ from each other. It appears that potentials of the above-mentioned type actually occur in alpha particle and heavy ion scattering calculations, and hence the considerations here presented should be a useful guide in such calculations.

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APPENDIX

It will now be shown that the ratio of the coefficients A_L and B_L, defined in Eq. (5), is less than unity for a complex square-well potential. According to Eq. (3), for constant nuclear potentials, k_L is the square root of e_L-iw and is given by

$$\sqrt{2}k_L=[(e_L^2+\omega^2)^{1/2}+e_L]^{1/2}+i[(e_L^2+\omega^2)^{1/2}-e_L]^{1/2}, \quad \text{(A1)}$$

where

$$e_L=e-l^2/r^2, \qquad l=[L(L+1)]^{1/2}, \quad \text{(A2)}$$

and where e and w are constants related to the reduced mass, the energy and the nuclear potentials. The integrals occurring in Eq. (5) can be evaluated exactly in this case and lead to the following expression for the radial wave function:

$$(\mathfrak{F}_L)_{\text{JWKB}}=k_L^{-1/2}\{A'e^{ik_Lr}[(k_Lr+il)/(k_Lr-il)]^{l/2} +B'e^{-ik_Lr}[(k_Lr+il)/(k_Lr-il)]^{-l/2}\}. \quad \text{(A3)}$$

The constants A' and B' can be shown to have the ratio

$$(A'/B')=(-1)^{L+1} \quad \text{(A4)}$$

from the comparison of the asymptotic expansion of Eq. (A3) with that of the exact expression given in terms of a Bessel function

$$\mathfrak{F}_L\propto z^{1/2}J_{L+1/2}(z). \quad \text{(A5)}$$

The two expansions agree in the terms containing l^2/z and l^4/z^2 but differ by a term $l^2/4z^2$ and terms in higher orders of z^{-2}. By evaluating (A2) at the turning point $r_T=l/\sqrt{e}$, and by combining relations (A4) and (A3) it can be shown that

$$\ln|A_L/B_L| =l\{-2y+\tfrac{1}{2}\ln[(1+2y+2y^2)/(1-2y+2y^2)]\}, \quad \text{(A6)}$$

where $y=|W/(E-V^N-V^c)|^{1/2}$. Since the right-hand side of the above expression is negative for all values of y, the ratio $|A_L/B_L|$ is less than unity. Expression (A3) is useful since it indicates how fast the outgoing branch disappears relative to the ingoing one. Since the expressions in square brackets approach unity as r recedes from r_T, the r dependence comes from the terms $\exp(\pm ik_Lr)$. When $w\ll e_L$, $\text{Im}k_L\approx\frac{1}{2}(\omega^2/e_L)^{1/2}$, which shows that the disappearance of the outgoing branch relative to the ingoing one occurs with distance more slowly the smaller the mass of the incident particle and the deeper the real potential.

The distance to a turning point, beyond which the JWKB becomes applicable, can be examined easily for the case that the real potentials are linear near the turning point, i.e.,

$$e_L=c(r-r_T), \qquad \omega=\text{constant}. \quad \text{(A7)}$$

By comparison of the JWKB expression to a Bessel function of order $\frac{1}{3}$ of complex argument, in the asymptotic region, the error in lowest order is found equal to

$$(5/48)(c/\omega^{3/2})[(r-r_T)/(\omega/c)-i]^{-3/2}, \quad \text{(A8)}$$

which is the same as the left-hand side of Eq. (4) multiplied by 10/48. If $W\neq 0$, the distance to the turning point enters in terms of the ratio to ω/c, which is independent of the mass of the incident particle. When this ratio is sufficiently large so that the i in square brackets can be neglected, then the error is proportional to $[c(r-r_T)^3]^{-1/2}$, and $c^{-1/3}$ is the relevant unit of distance. It varies only slowly with the slope of V_{eff} and the mass of the incident particle. For the numerical heavy ion

example considered in the text, the following numbers result for $L=9$: $c/(\omega)^{3/2}\cong 1.60$, $\omega/c\cong 0.33$ F, and for distances to the right of the turning point ($r>r_T$) of 0, 0.33, 0.66, and 1 F, the corresponding error in the JWKB expression is found from Eq. (A8) to be equal to 17%, 10%, 5%, and 3%, respectively. It may be interesting to note that for $W\neq 0$ the quantity in (A8) does not become infinite at $r=r_T$, showing that the presence of W improves the validity of the JWKB approximation near the turning point.

A more general estimate of the error in the JWKB approximation can be obtained by utilizing the procedure outlined in the book on quantum mechanics by Kemble,[14] in which the difference between the actual potential and the potential for which the JWKB expression for $\mathfrak{F}$ is an exact solution, is treated as a perturbation.

[14] E. C. Kemble, *The Fundamental Principle of Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1937), Sec. 21.

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Nuclear Pair-Correlation Function via Electron Scattering

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A theoretical analysis is made of the differential cross section for the process of inelastic scattering of electrons by a nucleus accompanied by the emission of two nucleons. The analysis is twofold. First, assuming a purely electromagnetic interaction (of the semiclassical Möller type), and invoking closure and impulse approximations, the above cross section is shown to be simply proportional to the Fourier transform of the nucleon-nucleon correlation function in the nucleus (at a momentum determined experimentally). Secondly, following a physical idea due to Gottfried and employed in a similar context, the nuclear cross section is compared to that of the electrodisintegration of a deuteron. This has the advantage in that it eliminates the need to introduce an explicit Hamiltonian (as in the first case), and also it allows us to include some very important mesonic effects due to the virtual pion exchange between the two outgoing nucleons. This approach, therefore, shifts the problem to a theoretical understanding of the connection between the deuteron wave function and the pair-correlation function in a nucleus. A computation of the cross section is made, which is rather low, yet is seen to be within the reach of the present experimental techniques. Detailed kinematical questions are explored and the optimal experimental setup indicated. The corrections due to (1) the final-state interactions and (2) real-meson-production channels are also discussed. In conclusion, we surmise that there is a definite need for detailed experimental study of two or more nucleon emission phenomena for the determination of the correlation function at high momentum (or small distances). Beyond fixing some qualitative guidelines, the no-nucleon and one-nucleon emission cross sections seem to be less than adequate.

1. INTRODUCTION

WE would like to make a proposal regarding the measurement of the 2-nucleon correlation function at very small distances, in light/medium nuclei via inelastic electron-nucleus scattering accompanied by the emission of two nucleons. Some of the physical ideas presented here are due to Gottfried employed in a similar context.

First, before entering into the main theme, a few words about the pair-correlation function and the need for its evaluation are in order. Let $\psi_0(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_A)$ be the nuclear ground state in configuration space (assumed properly normalized to unity) then the pair-correlation function $C(\mathbf{r}_1,\mathbf{r}_2)$ is defined as

$$C(\mathbf{r}_1,\mathbf{r}_2)=\int|\psi_0(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_A)|^2(d\mathbf{r}_3)(d\mathbf{r}_4),\cdots,(d\mathbf{r}_A),$$

and thus gives the conditional probability density for finding a particle at $\mathbf{r}_2$ if another one is known to exist at $\mathbf{r}_1$. Of course, in any realistic situation, one can define spin- and isospin-dependent functions (which we consider later on), but for the time being let us disregard such inessential complications.

Fundamental theories of nuclear matter to date (for example, that due to Brueckner), are all based on the 2-nucleon interaction. From these nuclear theories one can say something directly about the properties of the infinite nuclear medium. One would also hope to be able to predict properties of the models of low-energy nuclear physics. However, this last step has not been very successful as yet. To consider an example, in Brueckner's theory it is difficult simply to justify the existence of various phenomenological models (like the shell model) much less to "predict" their properties in detail.

Now in so far as more than 2-body correlations be negligible, the pair-correlation function describes the nuclear matter and such associated properties as the

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