Statistical Theory of Nuclear Collision Cross Sections*

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A formalism is developed for a statistical treatment of the energy variations of nuclear scattering and reaction cross sections. A statistical collision matrix $\mathbf{U}^S$ is defined which has the form of an energy-independent direct-transition matrix plus a fixed simple resonance-pole expansion, the matrix residues of which are products of complex channel-width amplitudes. By direct comparison with the Wigner-Eisenbud and Kapur-Peierls collision matrices it is found that under widely applicable conditions the statistical collision matrix may be used to calculate averages of observables over energy intervals containing many resonances and many total widths. The problem of determining the statistical properties of the parameters of $\mathbf{U}^S$ is defined and is solved for several special cases by relating it to the statistics of R-matrix parameters. Using these methods averages and mean-square fluctuations of total and reaction cross sections are calculated under general conditions admitting direct and compound processes and arbitrary average values of the total widths Γ and the resonance spacings D. The results are expressed in terms of the direct-reaction matrix elements and the statistical properties of resonance parameters appropriate to the energy region under consideration and are related to the locally applicable optical-model phase shifts and transmission coefficients. Simplifications are obtained under special assumptions such as uncorrelated width amplitudes, small and large Γ/D, pure compound-nucleus reactions, many competing open channels, and many competing direct processes. In the limit of small Γ/D one obtains the leading terms of an expansion of the average cross section which had previously been derived from R-matrix theory directly. In the limit of large Γ/D, many competing channels, but no direct reactions, the nonelastic fluctuation (or average compound nucleus) cross sections approach the Hauser-Feshbach formula. Except in this limit, corrections due to partial-width fluctuations and resonance-resonance interference are applicable. The former are sensitive to the magnitudes of direct reaction matrix elements, the latter to the correlations of resonance energies. Competing direct reactions are shown to require reductions of the transmission coefficients. The mean-square fluctuations of cross sections are found to approach Ericson's results in the limit of large Γ/D and many competing channels, but are in general much larger for moderate Γ/D and few channels. They are also sensitive to the details of resonance parameter statistics.

I. INTRODUCTION

THE local energy variations of nuclear cross sections may be thought of as arising from two sources: first, there are kinematical or size effects typified by surface-barrier penetration factors and hard-sphere phase shifts. These introduce dependences on the asymptotic relative momenta in all channels which are predictable and smoothly varying with energy except for singularities at thresholds. Second, there are dynamical effects due to the details of the nuclear interactions which produce strong and varied energy fluctuations, the detailed prediction or discussion of which is often difficult and may be irrelevant to the problems under investigation.

In a statistical theory of nuclear cross sections we forego a detailed description of the dynamical fluctuations in favor of a statistical description in terms of the statistical properties of the many-body scattering system or of models describing it. Specifically we wish to obtain energy averages of cross sections and statistical descriptions of their fluctuations. The principal problem one encounters in such a program is due to the difficulty of disentangling the dynamical fluctuations from the kinematical energy dependences.

This separation is accomplished most completely in the R-matrix theory of Wigner and Eisenbud[1] in which the dynamical fluctuations arise from the properties of the R matrix, its resonance energies and reduced-width amplitudes, while the kinematic variations reside chiefly in the $\mathbf{L}$ and $\boldsymbol{\Omega}$ matrices. The success of the separation depends on the judicious choice of a boundary in configuration space and of boundary conditions which specify the self-adjoint boundary value problem whose eigenfunctions provide the basis for an expansion of the wave function. The R matrix is also most suitable for the introduction of statistical models of the interaction.[2,3] Unfortunately, in the general many-channel case the functional dependences of the cross sections on the R matrix are so complicated, that explicit formulas for the energy dependences of cross sections have been obtained only in certain special cases involving either few open channels or few nearby resonances.[1,4,5] The same is true of energy averages which have been obtained successfully only in the form of approximations based on the one or two channel case or on the small width-to-spacing-ratio limit.[5,6] Several excellent reviews of R-matrix theory exist.[7–9]

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[1] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

[2] E. P. Wigner, Ann. Math. **55**, 7 (1952); *Fourth Canadian Mathematical Congress Proceedings* (University of Toronto Press, Toronto, 1957), p. 174.

[3] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).

[4] T. Teichmann, Phys. Rev. **77**, 506 (1950).

[5] R. G. Thomas, Phys. Rev. **97**, 224 (1955).

[6] P. A. Moldauer, Phys. Rev. **123**, 968 (1961).

[7] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).

[8] G. Breit, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41/1.

[9] H. B. Willard, L. C. Biedenharn, P. Huber, and E. Baumgartner, in *Fast Neutron Physics, Part II*, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1963), p. 1217.

The formalism of Kapur and Peierls[10] does not suffer from these difficulties. It leads to very simple functional expressions for the cross sections that are also formally easy to average. Unfortunately, the kinematic aspects of the energy variation are contained only implicitly in the parameters of the theory, having been introduced through an energy-dependent boundary condition. This fact precludes straightforward energy averaging of the Kapur-Peierls cross sections. The unified reaction theory of Feshbach[11] is closely related to the Kapur-Peierls theory in that the resonance parameters are expressed in terms of matrix elements of the interaction Hamiltonian with respect to single particle or quasiparticle states of the system and these matrix elements are implicitly energy-dependent. On the other hand, these parameters may be directly interpretable in terms of the detailed properties of nuclear interactions and nuclear structure.[12]

Finally, there is the pole expansion of Humblet and Rosenfeld[13] which shares with the Kapur-Peierls and Feshbach formalisms their simple functional form, but which also has energy-independent resonance parameters. Indeed, the adoption of this formalism would resolve most of the difficulties and would obviate the necessity for much of this discussion as well as most of Sec. II. There are, however, two reasons for not following that course. First, it does not appear to be known at present precisely what conditions must be imposed on the interaction Hamiltonian in order to justify the assumed existence of a Mittag-Leffler expansion of the multichannel collision matrix.[14] This difficulty could be avoided by regarding the Humblet-Rosenfeld series as a purely formal expansion valid in a restricted energy range.[15] That would, however, still leave a second disadvantage that at present there does not appear to exist a statistical theory of the resonance parameters in the Humblet-Rosenfeld formalism and that the methods employed for that purpose in R-matrix theory do not appear to be applicable.

In order to overcome these difficulties, we define in Sec. II a fictional *statistical-collision matrix* $\mathbf{U}^S$ which has the desired properties of complete separation of kinematic and dynamic aspects and of having a simple explicit energy dependence. It is shown that in a sufficiently small energy interval $\mathbf{U}^S$ is an arbitrarily good approximation ot the actual collision matrix $\mathbf{U}$ and on the basis of this comparison a connection is established between average cross sections and cross-section expectation values calculated by means of $\mathbf{U}^S$. Similar connections exist for other statistical properties.

The statistical-collision matrix $\mathbf{U}^S$ is derived in two ways: first, by means of an expansion in terms of the eigenstates of a general complex boundary-value problem which is developed in Appendix A, and then from R-matrix theory. The first derivation is simpler, the second leads to the discussion in Sec. III of the statistical properties of the parameters of $\mathbf{U}^S$ in terms of those of the R matrix which have been studied extensively.[2,3,16–20]

Section IV is devoted to the derivation of cross-section expectation values from $\mathbf{U}^S$ and their discussion, and in Sec. V the magnitudes of the mean-square fluctuations of the cross sections are obtained. Several integrals needed in Secs. IV and V are evaluated in Appendix B.

II. STATISTICAL-COLLISION MATRIX

A. Definitions

The various cross sections $\sigma(E)$ are functions of the elements of the collision matrix $\mathbf{U}(E)$ which characterizes the asymptotic form of the wave function of the collision process. The energy fluctuations of $\sigma(E)$ therefore reflect similar fluctuations of the elements of $\mathbf{U}(E)$. We shall assume it to be possible to specify the statistical properties of the fluctuations of $\mathbf{U}(E)$ in the vicinity of any specified total energy E_0,[21] so that we may define a uniform (or stationary) random matrix function[22] of E in terms of these same statistical properties. We shall call a realization of this uniform random function a *statistical-collision matrix* $\mathbf{U}^S(E;\ E_0)$ if $\mathbf{U}^S(E_0; E_0) = \mathbf{U}(E_0)$ and if the absolute values of the elements of

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[10] P. L. Kapur and R. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938).

[11] H. Feshbach, Ann. Phys. (N.Y.) **5**, 357 (1958); **19**, 287 (1962).

[12] B. Block and H. Feshbach, Ann. Phys. (N.Y.) **23**, 47 (1963); C. M. Shakin, *ibid.* **22**, 373 (1963); R. H. Lemmer, Phys. Letters **4**, 205 (1963); A. K. Kerman, L. S. Rodberg, and J. E. Young, Phys. Rev. Letters **11**, 422 (1963).

[13] J. Humblet and L. Rosenfeld, Nucl. Phys. **26**, 529 (1961).

[14] This fact was emphasized by Professor G. Breit and Professor H. Feshbach at the Topical Conference on Compound Nuclear States, Gatlinburg, October 1963 [Rev. Mod. Phys. (to be published)]. I am indebted to Professor Breit and Professor Feshbach for helpful discussions on this point.

[15] This indeed was the point of view I adopted for the derivation of some of the results of Secs. IV and V as presented at the Topical Conference on Compound Nuclear States, Gatlinburg, October 1963 [P. A. Moldauer, Rev. Mod. Phys. (to be published)].

[16] C. E. Porter and N. Rosenzweig, Ann. Acad. Sci. Fennicae Ser. **A.VI** No. 44 (1960).

[17] M. L. Mehta, Nucl. Phys. **18**, 395 (1960); M. L. Mehta and M. Gaudin, *ibid.* **18**, 420 (1960); M. Gaudin, *ibid.* **25**, 447 (1961).

[18] Freeman J. Dyson, J. Math Phys. **3**, 140 157, 166 (1962).

[19] T. J. Krieger and C. E. Porter, J. Math. Phys. **4**, 1272 (1963).

[20] N. Rosenzweig, Phys. Letters **6**, 123 (1963).

[21] As an example, one way of specifying these local statistical properties would be to give the statistical properties of R-matrix resonance parameters for resonances in the vicinity of E_0 and the average contribution of distant resonances to the R matrix at E_0.

[22] For definitions see, for example, A. M. Yaglom, *An Introduction to the Theory of Stationary Random Functions*, translated by Richard A. Silverman (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962). Uniform random functions are ensembles of functions the values of which are specified by distribution functions which are invariant under all translations of the argument, in this case E. A particular sample function, or realization, may therefore be said to fluctuate with uniform statistical properties for all values of E from $-\infty$ to $+\infty$.

$(E_0-E_0)^{-1}[\mathbf{U}^S(E,E_0)-\mathbf{U}(E)]$ are bounded in the vicinity of E_0, so that $\mathbf{U}^S$ is an arbitrarily good approximation to $\mathbf{U}$ sufficiently near E_0.

To obtain the statistical properties of σ, say its average at E_0, we must average $\sigma(\mathbf{U})$ over the ensemble of its values which is characteristic of the dynamical fluctuations of $\mathbf{U}$ in the vicinity of E_0. By definition this is equivalent to averaging over the ensemble of random matrix functions of which $\mathbf{U}^S$ is an element. We shall call such ensemble averages *expectation values* at E_0 and denote them by brackets: $\langle\sigma\rangle_{E_0}$. Moments and correlation functions of cross-section distributions are similarly defined in terms of expectation values of appropriate functions of σ. By the ergodic theorem[22] the expectation values $\langle\sigma\rangle_{E_0}$, etc., are equal to energy averages of $\sigma(\mathbf{U}^S)$, etc., in the limit of an arbitrarily large averaging interval Δ. In practice we expect such averages to converge quite rapidly when Δ becomes large compared to the mean resonance spacing D and the average total width Γ which are the two characteristic fluctuation periods of the problem and will be defined later. Therefore, if there exists an interval Δ around E_0 such that $\mathbf{U}^S(E,E_0)$ is a good approximation of $\mathbf{U}(E)$ within Δ and $\Delta\gg\Gamma,D$, then the expectation values $\langle\sigma\rangle_{E_0}$, etc., are equivalent to ordinary energy averages which may be compared directly with the results of relevant measurements. The relative error incurred is at most of the order of the maximum relative deviation of the elements of $\mathbf{U}^S$ from those of $\mathbf{U}$ within Δ.

At some energies E_0 the above two conditions on the interval Δ may be inconsistent with one another because of rapid energy variations of kinematic factors in $\mathbf{U}$. This may be expected to occur in the vicinity of a threshold. In that case the ensemble average definition tells us that we must consider $\langle\sigma\rangle_{E_0}$, etc., as expectation values of $\sigma(E_0)$, etc., which are realized by the mean values of $\sigma(E_0)$, etc., with respect to a large number of different values of E_0 all having the same expectation values $\langle\sigma\rangle_{E_0}$, etc., and separated from one another by many correlation distances Γ or D. It follows therefore that if $\langle\sigma\rangle_E$ is constant over values of E occupying an interval $W\gg\Gamma,D$ we may write for the energy average of σ over W

$$\bar{\sigma}\equiv\frac{1}{W}\int_W dE\sigma(E)=\frac{1}{W}\int_W dE\langle\sigma\rangle_E=\langle\sigma\rangle_{E_0}, \qquad (1)$$

where E_0 may be taken as the center of the interval W. Similar results apply to averages of other quantities. In many circumstances the conditions required for Eq. (1) to be valid are well satisfied. Expectation values as a rule are fairly constant over regions large compared to Γ,D even if these contain thresholds. Meyerhof[23] has shown that threshold effects on average cross sections are confined to decreases in partial reaction cross sections above thresholds due to the average additional competition of the new channel. In the presence of many competing channels this effect is slight. Equation (1) should however not be relied upon in the cases of partial cross sections for reactions near their own thresholds and nonthreshold cross sections in the vicinity of a threshold if there are very few (2 or 3) strongly competing open channels, for in these cases $\langle\sigma\rangle_E$ may vary too rapidly with energy.[23] The possibility afforded by Eq. (1) to obtain averages over intervals containing thresholds is particularly important since in many applications the spacing of thresholds may be comparable to Γ or even smaller.

One way to generate $\mathbf{U}^S$ is to insert Wigner's statistical R matrix[24] into the expression for the collision matrix (A23) and to maintain $\mathbf{L}$, $\mathbf{P}$, $\boldsymbol{\Omega}$ constant. But the resulting expression for cross sections are mathematically very difficult to average. In the following two subsections, we therefore proceed by other methods to define $\mathbf{U}^S$ and to estimate how its deviation from $\mathbf{U}$ depends on the size Δ of the averaging interval about E_0.

B. Complex Boundary Conditions

We first derive the statistical-collision matrix from a resonance formalism employing an expansion of the wave function in terms of the eigenfunctions of a boundary-value problem with arbitrary complex boundary conditions. This generalization of both the Wigner-Eisenbud and Kapur-Peierls formalisms is developed in Appendix A where it is shown that the collision matrix may be written in the form (A23):

$$\mathbf{U}=\boldsymbol{\Omega}\{\mathbf{1}+2i\mathbf{P}^{1/2}[\mathbf{1}-\mathfrak{R}(\mathbf{L}-\mathbf{B})]^{-1}\mathfrak{R}\mathbf{P}^{1/2}\}\boldsymbol{\Omega},$$

where the complex R matrix is given by Eq. (A21) as

$$\mathfrak{R}_{cc'}=\sum_\mu\frac{\theta_{\mu c}\theta_{\mu c'}}{\mathcal{E}_\mu-E-\frac{1}{2}i\Gamma_\mu},$$

and the meaning of the matrix index c is defined in the beginning of Appendix A as specifying completely a particular partial wave in a particular channel. The $\theta_{\mu c}$ are complex while the $\mathcal{E}_\mu$ and Γ_μ are real and $\boldsymbol{\Omega},\mathbf{L}=\mathbf{S}+i\mathbf{P}$ are the usual diagonal channel matrices defined in (A24) and (A25). The boundary conditions are specified by the arbitrary complex diagonal matrix $\mathbf{B}$.

If one specifies the boundary conditions to be

$$B_c=L_c(E_0), \qquad (2)$$

then at $E=E_0$ the collision matrix becomes exactly the Kapur-Peierls collision matrix[10]

$$\mathbf{U}=\boldsymbol{\Omega}(\mathbf{1}+2i\mathbf{P}^{1/2}\mathfrak{R}\mathbf{P}^{1/2})\boldsymbol{\Omega}, \qquad (3)$$

and at $E=E_0+\delta E$ we may expand $\mathbf{U}$ as follows

$$\mathbf{U}(E_0+\delta E)=\boldsymbol{\Omega}\{\mathbf{1}+2i\mathbf{P}^{1/2}\times[\mathfrak{R}+\mathfrak{R}\mathbf{L}^0\mathfrak{R}+\mathfrak{R}\mathbf{L}^0\mathfrak{R}\mathbf{L}^0\mathfrak{R}+\cdots]\mathbf{P}^{1/2}\}\boldsymbol{\Omega}, \qquad (4)$$

[23] W. E. Meyerhof, Phys. Rev. **128**, 2312 (1962); **129**, 692 (1963); J. T. Wells, A. B. Tucker, and W. E. Meyerhof, *ibid.* **131**, 1644 (1963).

[24] E. P. Wigner, Ann. Math. **53**, 36 (1951).

where

$$L_c^0(E_0+\delta E)=L_c(E_0+\delta E)-L_c(E_0). \quad (5)$$

We shall assume that within the range of values of δE of interest the energy variation of $\mathbf{L}$ is adequately represented by its linear dependence upon E, so that

$$L_c^0(E+\delta E)\approx\delta E L_c'(E_0)=\delta E[S_c'(E_0)+iP_c'(E_0)], \quad (6)$$

the prime denoting an energy derivative. Away from thresholds the boundedness of the elements of $\mathfrak{R}$ and of the energy derivatives of $\mathbf{L}$ assures the existence of some energy interval of size Δ around E_0 such that within Δ the expansion (4) not only converges but also the elements of $\mathfrak{R}\mathbf{L}^0\mathfrak{R}$ and all higher order terms are very small compared to those of $\mathfrak{R}$. We may therefore consider the collision matrix throughout Δ to be given by Eq. (3) plus a small correction of the order of at most

$$\boldsymbol{\Omega}\mathbf{P}^{1/2}\mathfrak{R}\mathbf{L}^0\mathfrak{R}\mathbf{P}^{1/2}\boldsymbol{\Omega},$$

with L^0 evaluated at $E=E_0\pm\frac{1}{2}\Delta$.

In order to estimate the magnitude of this error for a given interval size Δ, we consider the quantities

$$d_{cc'}=\tfrac{1}{2}\Delta|(\mathfrak{R}\mathbf{L}'\mathfrak{R})_{cc'}/\mathfrak{R}_{cc'}|, \quad (7)$$

which, when evaluated at E_0, give the relative errors introduced by the second term in the series (4) at the ends of the interval Δ. If E_0 is in a region of well-separated resonances where $\Gamma\ll D$, the maximum errors will occur if a resonance energy $\mathcal{E}_\mu$ lies at the edge of Δ and then the $d_{cc'}$ become at most

$$d_{cc'}{}^{(\mu)}<(\Delta/\Gamma_\mu)|\textstyle\sum_{c''}\theta_{\mu c''}{}^2L_{c''}'|, \quad (8a)$$

where the sum is taken over all channels which compete with c and c'.

For the case of overlapping resonances we evaluate $d_{cc'}$ under the assumption that the $\theta_{\mu c}$ for different channels are uncorrelated. Then we find that the average value of $d_{cc'}$ at the edge of the interval Δ is given by

$$\langle d_{cc'}\rangle_{E_0\pm\frac{1}{2}\Delta}\approx\tfrac{1}{2}\pi(\Delta/D)\delta_{cc'}|\langle\theta_{\mu c}{}^2\rangle L_c'|. \quad (8b)$$

The root-mean-square deviation of the error from its mean is obtained for large width to spacing ratios by the methods of Appendix B using the results of Sec. III which yields at the edge of the interval Δ

$$\frac{\Delta}{2}\left[\frac{\langle|(\mathfrak{R}\mathbf{L}'\mathfrak{R})_{cc'}|^2\rangle}{\langle|\mathfrak{R}_{cc'}|^2\rangle}\right]^{1/2}\approx\left\langle\frac{\Delta}{\Gamma_\mu}\textstyle\sum_{c''}|\theta_{\mu c''}|^2|L_{c''}'|\right\rangle_\mu. \quad (8c)$$

For the positive energy channels $c+$ the energy derivative L_{c+}' is generally dominated by its imaginary part $\mathbf{P}_{c+}'$, the real part being at the very most of the order of an MeV^{-1}.[7,25] Sufficiently far above threshold P_{c+} approaches $k_\alpha a_\alpha$ (see Appendix A) and hence P_{c+}' approaches $(2E_{c+})^{-1}P_{c+}$, where E_c is the energy measured from the threshold of channel c. Then using Eqs. (2) and (A15) we find from (8b) that

$$\langle d_{cc}\rangle<\tfrac{1}{8}\pi\frac{\langle N_\mu\Gamma_{\mu c}\rangle}{D}\frac{\Delta}{E_c}, \quad (9a)$$

$$\langle d_{cc'}\rangle=0, \quad c\neq c'.$$

From the results of Secs. III and IV we may expect $\pi\langle N_\mu\Gamma_{\mu c}\rangle/8D$ to be at most of the order of unity, and much less than that for channels c with nearby thresholds and large Coulomb or centrifugal barriers. Hence $\langle d_{cc}\rangle$ is at most of the order of Δ/E_c which is expected to impose a significant limitation on Δ through the nearest lower thresholds involving the emission of neutrons with low orbital angular momenta. To evaluate the expressions (8a) and (8c) we need to consider contributions from both the open and the closed channels c''. We consider these two contributions to the channel sums separately and then add their magnitudes, thus increasing the error estimate. The same method which led to the estimate (9a) shows that the contribution of the open channels to the sums in (8a) and (8c) is at most of the order

$$\textstyle\sum_{c+}|\theta_{\mu c+}|^2|L_{c+}'|<(N_\mu\Gamma_\mu/4\bar{E}_{c+}), \quad (9b)$$

where we have employed the definition

$$1/\bar{E}_{c+}=(1/\Gamma_\mu)\textstyle\sum_{c+}(\Gamma_{\mu c+}/E_{c+}). \quad (9c)$$

In order to obtain a rough estimate of $\bar{E}_{c+}$ we assume that a channel c contributes substantially to the sum in (9c) only when E_c exceeds the channel's surface barrier height and that then $\Gamma_{\mu c}$ can be approximated by $\Gamma_{\mu c}{}^0E_c^{1/2}$. Assuming the same reduced width $\Gamma_{\mu c}{}^0$ for all channels and a uniform effective threshold density beginning at a lowest threshold with $E_c=E_c{}^{(\max)}$, one finds that $\bar{E}_{c+}=\frac{1}{3}E_c{}^{(\max)}$. This is probably an overestimate. However, other more realistic (and more involved) estimates yield values of $\bar{E}_{c+}$ which are at least of the order of the smallest s-wave neutron E_c and substantially greater at energies above large numbers of thresholds. In view of (8a), (8c), (9b), and (9c) as well as (9a), we therefore conclude that so far as the contribution of the open channels to the error terms in Eq. (4) are concerned, Δ must be kept small compared to the energy measured to the nearest lower s-wave neutron threshold.

For the closed channels $c-$ only the real part S_{c-} of L_{c-} is nonvanishing. For s-wave neutrons its energy derivative is

$$S_{c-}'=(M_\alpha a_\alpha{}^2/2\hbar^2E_{c-})^{1/2}\ \text{(no barrier)}, \quad (10)$$

where E_{c-} is the energy difference between the threshold of $c-$ and E_0. For channels having surface barriers, S_{c-}' is much smaller near threshold and approaches the value (10) at energies below threshold which are large compared to the surface barrier height. Of the channels

[25] Numerical confirmation can be obtained from J. E. Monahan, L. C. Biedenharn, and J. P. Schiffer, Argonne National Laboratory Report ANL-5846, 1958 (unpublished).

with nearby thresholds above E_0 again only those involving s-wave neutron emission are ordinarily important and their contributions to (8a) and (8c) are of the same order of magnitude as those of channels with nearby thresholds below E_0. We therefore conclude that Δ must also be small compared to the smallest s-wave neutron E_{c-}. However, we must also discuss the possible effects of the many closed channels with distant thresholds. We limit the discussion to discrete channels and in analogy with the arguments of Teichmann and Wigner[26] and Lane, Wigner, and Thomas,[27] we estimate $\sum_c|\theta_{\mu c}|^2$ to be of the order of a single nucleon reduced width $\hbar^2/M_n a_n^2$, where the subscript n refers to a nucleon channel. Assuming arbitrarily that the thresholds of two-fragment channels are distributed with uniform density up to a maximum threshold energy equal to the total mass number A times the binding energy per nucleon, that E_0 is very far below this maximum threshold, and that (10) applies to all closed channels, we obtain the estimate

$$\sum_{c-}|\theta_{\mu c-}|^2|S_{c-}'|<2A^{-5/6}, \tag{11a}$$

which, in view of (8a) and (8c), imposes the requirements that

$$\Delta\ll\tfrac{1}{2}\langle\Gamma_\mu\rangle A^{5/6}. \tag{11b}$$

This could be a much more restrictive condition than those imposed by the contributions of the open channels, particularly in the case of light nuclei. It seems likely, however, that (11a) constitutes a substantial overestimate,[7] as can be gathered from empirical evidence. The left-hand side of (11a) is of the order of the rate of the relative shift of observed resonance levels due to the closed channels. If this rate of shift were as large as indicated in (11a), the observed resonance level densities in the lighter nuclei would have to be much smaller than those computed statistically on the basis of observed or theoretical bound-state levels which correspond to solutions of the Schrödinger equation without artificial boundary conditions. We shall therefore assume that the conditions imposed on Δ by the contribution to Eq. (4) of the open channels and the closed channels with nearby thresholds will be governing. In Sec. II., we shall deal with the closed-channel effect in another way which will support this conclusion.

Having established estimates for the size of the interval Δ within which (3) is a good approximation to the collision matrix, we now suppose that there exists an interval of size I around E_0, where $I\gg\Delta$, Γ, and D, such that for all $\mathcal{E}_\mu$ in I the corresponding values of $\theta_{\mu c}$, Γ_μ, and $\mathcal{E}_\mu$ can be considered a typical sample of the appropriate stationary ensemble of resonance parameters. By "typical sample" we shall mean that according to an appropriate statistical test the distribution of the above sample does not deviate significantly from the ensemble distribution and that the same is true of samples contained in all subintervals of I. The latter requirement implies for example that there are no strong systematic variations of D, Γ, or $\langle\theta_{\mu c}\rangle_\mu$ within I. These assumptions are in part physical and in part merely formal requirements in the sense that they are affected by the choice of channel radii. The "appropriate stationary ensemble" mentioned above may be considered as provided in part either by theoretical considerations or empirically by the study of resonance parameters in a very large collection of intervals. We shall call energy intervals I which satisfy the above conditions *intervals of uniform resonance statistics*.

We now separate the complex R matrix (A21) into two parts

$$\mathfrak{R}=\mathfrak{R}^{(0)}+\mathfrak{R}^{(1)}, \tag{12}$$

where $\mathfrak{R}^{(1)}$ contains all those terms in the sum (A21) for which $\mathcal{E}_\mu$ lies within I and $\mathfrak{R}^{(0)}$ contains the remainder. The variation of $\mathfrak{R}^{(0)}$ within Δ is at most of the order of $2(\Delta/I)\langle\boldsymbol{\theta}_\lambda\times\boldsymbol{\theta}_\lambda\rangle/D$ and we may therefore approximate $\mathfrak{R}^{(0)}$ by a constant matrix in Δ. We further consider a complex matrix $\mathfrak{R}^{(2)}$ the $\mathcal{E}_\mu$ of which are distributed outside I from $E=-\infty$ to $+\infty$ and the resonance parameters of which follow the same distribution law as those of $\mathfrak{R}^{(1)}$ inside I. In particular the values of D, Γ, and $\langle|\theta_{\mu c}|^2\rangle$ for $\mathfrak{R}^{(2)}$ are everywhere the same as those for $\mathfrak{R}^{(1)}$. The variation of $\mathfrak{R}^{(2)}$ within Δ is of the same order as that of $\mathfrak{R}^{(0)}$. Adding and subtracting $\mathfrak{R}^{(2)}$ in Eq. (3), we finally obtain the statistical collision matrix

$$\mathbf{U}^S(E,E_0)=\mathbf{U}^0(E_0)-i\sum_\mu\frac{\mathbf{g}_\mu\times\mathbf{g}_\mu}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}, \tag{13}$$

where the sum is extended over all resonance terms of $\mathfrak{R}^{(1)}$ and $\mathfrak{R}^{(2)}$,

$$g_{\mu c}=\Omega_c(2P_c)^{1/2}\theta_{\mu c}, \tag{14}$$

and

$$\mathbf{U}^0=\boldsymbol{\Omega}[\mathbf{1}+2i\mathbf{P}^{1/2}(\mathfrak{R}^{(0)}-\mathfrak{R}^{(2)})\mathbf{P}^{1/2}]\boldsymbol{\Omega}, \tag{15}$$

and where all Ω_c and P_c may be evaluated at E_0 without introducing errors greater than those due to the omission of the higher order terms in Eq. (4). Though $\mathbf{U}^S$ is a good approximation to $\mathbf{U}$ in Δ, it is defined for all energies $(-\infty,+\infty)$ and satisfies the definition of a statistical collision matrix given at the beginning of this section. Therefore, $\mathbf{U}^S$ may be employed not only within Δ, but energy averages using $\mathbf{U}^S$ are ergodically equivalent to resonance-parameter ensemble averages defining expectation values at E_0.

At present the definition of the statistical-collision matrix in terms of the eigenstates of a complex boundary-value problem is somewhat inconvenient because the statistical properties of such states have not been studied, while those arising from a real Hermitian boundary-value problem have been studied intensively.

[26] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).
[27] A. M. Lane, R. G. Thomas, and E. P. Wigner, Phys. Rev. **98**, 693 (1955).

For this reason we now give a slightly more complicated derivation of (13) in terms of the real eigenvalues and eigenfunctions of the Wigner-Eisenbud R-matrix theory.[1]

C. Real Boundary Conditions

As in Appendix A, we follow the notation of Lane and Thomas[7]; but now, following Wigner and Eisenbud,[1] we choose real boundary conditions, for example

$$B_c = S_c(E_0)\,, \tag{16}$$

though we shall not restrict ourselves to this choice [see Eq. (37) below]. The eigenvalues E_μ are now real and $\gamma_{\mu c}$ can likewise be chosen real so that the real R matrix becomes

$$R_{cc'} = \sum_\mu \gamma_{\mu c}\gamma_{\mu c'}/(E_\mu - E)\,. \tag{17}$$

We immediately regard (17) as the reduced R matrix from which reference to all closed channels has been eliminated.[26] In order that the $\gamma_{\mu c}$ and E_μ still be constant within Δ, we require that all S_{c-} may be represented by linear functions of E in Δ.[7] In practice this imposes the same closed channel condition on Δ which was assumed in Sec. IIB, namely that Δ be small compared to the separation of E, from the nearest higher s-wave neutron threshold.

We again separate R into two parts $\mathbf{R}^{(0)}$ and $\mathbf{R}^{(1)}$ as in Eq. (12), where $\mathbf{R}^{(1)}$ again contains those terms of the sum (17) whose E_μ lie in the interval $I \gg \Delta$, D, and $\langle\Gamma_\mu\rangle$, which is an interval of uniform statistics with respect to the E_μ and the $\gamma_{\mu c}$. Again $\mathbf{R}^{(0)}$, which contains the remaining terms of (17), is constant within Δ to within an error of at most $2(\Delta/I)\langle\gamma_\lambda \times \gamma_\lambda\rangle/D$. Following the procedure of Wigner and Eisenbud[1] as generalized by Lane and Thomas,[7] we now expand the collision matrix (A23) in terms of the level matrix $\mathbf{A} = \{A_{\mu\nu}\}$.

$$\mathbf{U}(E) = \mathbf{U}^0(E) + 2i\boldsymbol{\Omega}\mathbf{P}^{1/2}\sum_{\lambda\mu}(\boldsymbol{\alpha}_\lambda \times \boldsymbol{\alpha}_\mu)A_{\lambda\mu}\mathbf{P}^{1/2}\boldsymbol{\Omega}\,, \tag{18}$$

where

$$\mathbf{U}^0(E) = \boldsymbol{\Omega}[\mathbf{1} + 2i\mathbf{P}^{1/2}(\mathbf{1} - \mathbf{R}^{(0)}\mathbf{L}^0)^{-1}\mathbf{R}^{(0)}\mathbf{P}^{1/2}]\boldsymbol{\Omega}\,, \tag{19}$$

and

$$\boldsymbol{\alpha}_\lambda = (\mathbf{1} - \mathbf{R}^{(0)}\mathbf{L}^0)^{-1}\boldsymbol{\gamma}_\lambda\,, \tag{20}$$

and

$$\mathbf{A} = (\mathbf{e} - \mathbf{E} - \boldsymbol{\xi})^{-1}. \tag{21}$$

The elements of the matrices $\mathbf{e}$, $\mathbf{E}$, and $\boldsymbol{\xi}$, are given by

$$e_{\lambda\mu} = E_\lambda\delta_{\lambda\mu}\,, \quad E_{\lambda\mu} = E\delta_{\lambda\mu}\,, \tag{22}$$

$$\xi_{\lambda\mu} = \sum_{cc'}\gamma_{\lambda c}[\mathbf{L}^0(\mathbf{1} - \mathbf{R}^{(0)}\mathbf{L}^0)^{-1}]_{cc'}\gamma_{\mu c'}\,. \tag{23}$$

The double sum in Eq. (18) is carried over those level indices μ whose E_μ lie in I. We separate the level matrix $\boldsymbol{\xi}$ into two parts

$$\boldsymbol{\xi}(E) = \boldsymbol{\xi}^0 + \boldsymbol{\xi}^1(E)\,, \tag{24}$$

where

$$\boldsymbol{\xi}^0 = \boldsymbol{\xi}(E_0)\,, \tag{25}$$

and diagonalize the complex symmetric matrix $(\mathbf{e} - \boldsymbol{\xi}^0)$, supposing it to have distinct roots, by means of the complex orthogonal transformation $\mathbf{T}$.

$$\mathbf{T}(\mathbf{e} - \boldsymbol{\xi}^0)\mathbf{T}^{-1} = \mathbf{T}(\mathbf{e} - \boldsymbol{\xi}^0)\mathbf{T}^T = \boldsymbol{\mathcal{E}} - \tfrac{1}{2}i\boldsymbol{\Gamma}\,, \tag{26}$$

where $\mathbf{T}^T$ is the transpose of $\mathbf{T}$ and

$$\mathcal{E}_{\lambda\mu} = \mathcal{E}_\lambda\delta_{\lambda\mu}\,, \quad \Gamma_{\lambda\mu} = \Gamma_\lambda\delta_{\lambda\mu}\,. \tag{27}$$

Within a sufficiently small interval Δ around E_0 we may then expand Eq. (21)

$$\mathbf{A}(E) = \mathbf{T}^T[\mathfrak{C}^{-1} + \mathfrak{C}^{-1}\mathbf{X}\mathfrak{C}^{-1} + \mathfrak{C}^{-1}\mathbf{X}\mathfrak{C}^{-1}\mathbf{X}\mathfrak{C}^{-1} + \cdots]\mathbf{T}\,, \tag{28}$$

where

$$\mathfrak{C}_{\mu\nu} = (\mathcal{E}_\mu - E - \tfrac{1}{2}i\Gamma_\mu)\delta_{\mu\nu}\,, \tag{29}$$

and

$$\mathbf{X} = \mathbf{T}\boldsymbol{\xi}^1(E)\mathbf{T}^T. \tag{30}$$

Using Eq. (28) we now obtain for the resonance terms in (18)

$$\sum_{\lambda\mu}(\boldsymbol{\alpha}_\lambda \times \boldsymbol{\alpha}_\mu)A_{\lambda\mu} = \sum_{\lambda\mu}(\boldsymbol{\theta}_\lambda \times \boldsymbol{\theta}_\mu) \times [\mathfrak{C}_\lambda^{-1}\delta_{\lambda\mu} + \mathfrak{C}_\lambda^{-1}X_{\lambda\mu}\mathfrak{C}_\mu^{-1} + \cdots]\,, \tag{31}$$

where the complex amplitudes

$$\boldsymbol{\theta}_\mu = \sum_\nu T_{\mu\nu}\boldsymbol{\alpha}_\nu \tag{32}$$

are analogous to the complex amplitudes of Appendix A and as has been shown by Lane and Thomas[7] they satisfy a relation analogous to (A15).

$$\Gamma_\mu = \sum_c \Gamma_{\mu c}\,, \qquad \Gamma_{\mu c} = N_\mu^{-1}2P_c|\theta_{\mu c}|^2\,, \tag{33}$$

where

$$N_\mu = \sum_\nu |T_{\mu\nu}|^2 \geq 1\,. \tag{34}$$

Since the $X_{\lambda\mu}$ vanish at E_0, only the first term of the series (31) contributes there and upon substitution into Eq. (18) we again obtain the Kapur-Peierls collision matrix (3). We again adopt this collision matrix throughout Δ and estimate the resulting error precisely as in the preceding subsection by evaluating the magnitudes of the elements of the second term of (31) relative to the magnitudes of the corresponding elements of the first term. To first order in the interval size Δ and using the fact that $\mathbf{R}^{(0)}$ is symmetric we obtain for a maximum magnitude of $X_{\lambda\mu}$ in Δ

$$\tfrac{1}{2}\Delta(\boldsymbol{\theta}_\lambda \cdot [d\mathbf{L}^0/dE) + \mathbf{L}^0(d\mathbf{R}^0/dE)\mathbf{L}^0]\boldsymbol{\theta}_\mu)\,, \tag{35}$$

where we have used the notation $(\mathbf{a}\cdot\mathbf{b}) = \sum_c a_c b_c$. The first term in the bracket yields exactly the same relative correction $\mathbf{d}$ to $\mathbf{U}$ as given in Eq. (7). For the positive energy channels under consideration here this correction was found there to be at most of the order of the ratio of Δ to the separation of E_0 from the nearest lower

s-wave neutron threshold and usually much less than that. Using the estimate of the variation of $\mathbf{R}^{(0)}$ within Δ which was given below Eq. (17) and the magnitude of the variation in $\mathbf{L}^0$ used in connection with Eqs. (9), we find that the diagonal components of the second term of (35) contribute at most of the order of $8(\langle\Gamma_{\mu c}\rangle/I)\times(E_c/D)$ times the corresponding contribution of the first term. The relative contributions to (35) of any off-diagonal terms of R^0 may be taken to be similarly small and of varying sign.

Under our assumptions, Eq. (31) is therefore well approximated in Δ by the first term on the right-hand side. Substituting this first term into Eq. (18) one obtains again an expression of the form (13) for the collision matrix in Δ, except that the $\mathcal{E}_\mu$ and Γ_μ belong to a finite set arising from the transformation (26). In order to obtain the statistical collision matrix we make some observations regarding the distribution of the $\mathcal{E}_\mu$. Lane and Thomas[7] have shown that

$$\mathcal{E}_\mu=\mathcal{E}_\mu{}^0+\mathcal{S}_\mu, \tag{36a}$$

where

$$\mathcal{E}_\mu{}^0=N_\mu{}^{-1}\textstyle\sum_\lambda E_\lambda|T_{\mu\lambda}|^2, \tag{36b}$$

and

$$\mathcal{S}_\mu=N_\mu{}^{-1}(\boldsymbol{\theta}_\mu{}^*\cdot[\mathbf{L}^{0*}\mathbf{R}^{(0)}\mathbf{L}^0-\mathbf{S}^0]\boldsymbol{\theta}_\mu). \tag{36c}$$

By substituting the upper and lower limits of the interval I containing all the E_λ into (36b) in place of E_λ and using (34), it is easy to see that all the $\mathcal{E}_\mu{}^0$ are contained in I. From the assumed symmetry of the distribution functions of the E_μ and $\gamma_{\mu c}$ with respect to E_0, it follows that also the $\mathcal{E}_\mu{}^0$, Γ_μ and $\theta_{\mu c}$ have distribution functions which are symmetric with respect to E_0. If we choose the boundary condition B_c so that

$$S_c{}^0=L_c{}^{0*}R_{cc}{}^{(0)}L_c{}^0, \tag{37}$$

instead of Eq. (16), then the shift $\mathcal{S}_\mu$ is given by the sum over any nonvanishing off-diagonal elements of $\mathbf{R}^{(0)}$.

$$\mathcal{S}_\mu=N_\mu{}^{-1}\textstyle\sum_{c\neq c'}(\theta_{\mu c}L_c{}^0)^*R_{cc'}{}^{(0)}(\theta_{\mu c'}L_{c'}{}^0), \tag{38}$$

which is likely to be small because of the probable alternation of signs of the various terms in the many open channel case and is as likely to be positive as negative for any given resonance μ. As a result, we expect the $\mathcal{E}_\mu$, Γ_μ, $\theta_{\mu c}$ to have statistical distributions which are symmetric with respect to E_0 and we expect the $\mathcal{E}_\mu$ to occupy an energy interval whose size is of the order of I. Though the density of the $\mathcal{E}_\mu$ may not be uniform throughout this interval, its symmetry about E_0 suggests that there exists an interval $I'\gg\Delta$ around E_0, which is an interval of uniform statistics with respect to the $\mathcal{E}_\mu$, Γ_μ, and $\theta_{\mu c}$. If there is no such satisfactory interval I' we may expect to produce one by enlarging the original interval I, adding resonances to maintain statistical uniformity and making a compensating change in $\mathbf{R}^{(0)}$.

The statistical collision matrix $\mathbf{U}^S$ of Eq. (13) is now obtained by incorporating in $\mathbf{U}^0$ the contributions of those terms with $\mathcal{E}_\mu$ outside I' and by adding resonance terms with uniform statistics throughout the infinite energy intervals $(-\infty,\infty)$ outside I', making another compensating change in $\mathbf{U}^0$ as described in connection with Eqs. (13) to (15).[28]

It is important to recognize that the partial widths (33), the resonance energies $\mathcal{E}_\mu$ defined in (26), and the background matrix $\mathbf{U}^0$ are not identical to the partial widths $2P_c\gamma_{\mu c}{}^2$, resonance energies E_μ and hard sphere scattering matrices $\Omega_c{}^2$ as customarily defined in the R-matrix expansion. They are in general not even the same within the interval Δ. The resonance parameters of $\mathbf{U}^S$ might be described as "observable" local resonance parameters at the energy E_0, while the "formal" R-matrix resonance parameters are valid at all energies, but less directly connected with observable quantities.

III. STATISTICS OF RESONANCE PARAMETERS

The statistical distributions of the resonance parameters $\mathcal{E}_\mu$, Γ_μ, $\theta_{\mu c}$ of $\mathbf{U}^S$ may be discussed in two ways. One method would be to deduce the statistical properties of the eigenvalues and eigenfunctions of the complex boundary value problem of Sec. II.B from plausible assumptions or models regarding the Hamiltonian of the system. Another method is to use the considerable amount of existing discussion of the statistical properties of the eigenvalues E_μ and eigenfunction coefficients $\gamma_{\mu c}$ of the real boundary value problem of Sec. II.C, and to deduce the distributions of the $\mathcal{E}_\mu$, Γ_μ, $\theta_{\mu c}$ in the vicinity of E_0 by studying the statistical properties of the transformation $\mathbf{T}$ of Eq. (26). We do not give here a complete solution to either of these problems, but will discuss by the second method some limiting cases and general trends.

A. Isolated Resonances

In the vicinity of an isolated resonance, the one-level approximation[1,7] gives a collision matrix $\mathbf{U}$ which is identical to $\mathbf{U}^S$, except that the $\boldsymbol{\theta}_\mu$ are replaced by the $\boldsymbol{\alpha}_\mu$ and for the boundary conditions yielding (37) the $\mathcal{E}_\mu$ are replaced by the E_μ; also $\Gamma_\mu=2\sum_c P_c|\alpha_{\mu c}|^2$. If the spacing of such isolated resonances is much larger than their widths Γ_μ, one may suppose that the $\mathbf{R}^{(0)}$ for each one-level approximation is due almost entirely to the influence of the very distant resonances and that therefore the same $\mathbf{R}^{(0)}$ applies to all these one-level approximations. In the limiting case of such noninterfering resonances the collision matrix has then the form

[28] The involved procedure described here could have been avoinded if, starting with Eq. (18), we had used Wigner's Statistical R Matrix (Ref. 24) for $\mathbf{R}^{(1)}$. This would, however, require restrictions on the statistics of $\mathbf{R}^{(1)}$ to insure the boundedness of the infinite dimensional transformation corresponding to Eq. (26) and may complicate the solution of the statistical problem discussed in Sec. III. Statistical properties of several kinds of matrix transformations have been studied by E. P. Wigner, Ann. Math. **62**, 548 (1955); **65**, 203 (1957).

of a sum over one-level approximations and is identical to $\mathbf{U}^S$ with $T_{\mu\nu}=\delta_{\mu\nu}$. Therefore also the statistical distributions of the $\mathcal{E}_\mu$ and $\theta_{\mu c}$ are the same as those of the E_μ and the $\alpha_{\mu c}$ and the N_μ are unity.

This result may also be obtained with the help of matrix perturbation methods. If it is assumed that the off-diagonal elements of ξ^0 are small, then we may hope to expand the diagonalization (26) as follows

$$\mathcal{E}_\mu-\tfrac{1}{2}i\Gamma_\mu=E_\mu-\xi_{\mu\mu}{}^0+\textstyle\sum_{\nu\neq\mu}((\xi_{\mu\nu}{}^0)^2/E_\mu-E_\nu)+\cdots, \quad (39a)$$

$$T_{\mu\nu}=\delta_{\mu\nu}+(1-\delta_{\mu\nu})(\xi_{\mu\nu}{}^0/E_\mu-E_\nu)+\cdots, \quad (39b)$$

and by Eq. (32)

$$\theta_{\mu c}=\alpha_{\mu c}+\textstyle\sum_{\nu\neq\mu}(\xi_{\mu\nu}{}^0\alpha_{\nu c}/(E_\mu-E_\nu))+\cdots. \quad (39c)$$

If the conditions

$$|\xi_{\mu\nu}{}^0|\ll|E_\mu-E_\nu|\,, \quad \mu\neq\nu \quad (40)$$

hold, then the series (39) will converge rapidly. In the case of the boundary condition (37) we have

$$\xi_{\mu\nu}{}^0=i\textstyle\sum_c P_c\alpha_{\mu c}{}^*\alpha_{\nu c}-\sum_{c'\neq c}\alpha_{\mu c}{}^*L_c{}^{0*}R_{cc'}{}^{(0)}L_{c'}{}^0\alpha_{\nu c'}. \quad (41)$$

Assuming $\mathbf{R}^{(0)}$ to be diagonal, Eq. (39a) gives in zeroth order

$$\mathcal{E}_\mu\approx E_\mu, \quad \Gamma_\mu\approx 2i\textstyle\sum_c P_c|\alpha_{\mu c}|^2. \quad (42)$$

The magnitudes of the off-diagonal elements of ξ^0 are then also of the order of the total widths Γ_μ and if the signs of the real and imaginary parts of the $\alpha_{\mu c}$ are random, we may make the estimate

$$|\xi_{\mu\nu}{}^0|\approx 2(\Gamma_\mu\Gamma_\nu/n)^{1/2}, \quad \mu\neq\nu \quad (43)$$

where n is the number of important competing open channels. This, together with the repulsion of the R-matrix poles E_μ (see below), leads us to conclude that the condition (40) will be satisfied when the ratio of the average total widths to the average resonance spacing Γ/D is small compared to $\frac{1}{2}n^{1/2}$. Under these conditions, (42) is a good approximation, $\theta_{\mu c}\approx\alpha_{\mu c}$ and $N_\mu\approx 1$. These conclusions remain valid whenever the second sum in Eq. (41) does not greatly exceed the first sum in magnitude, either because the off-diagonal elements of $\mathbf{R}^{(0)}$ are sufficiently small or because of the effects of the random signs of the $\alpha_{\mu c}$.

B. Spacing Distribution

The correlation of resonance energies E_μ has been much discussed by a number of authors.[2,16–18] For our purposes we shall need mainly the two-level correlation function $R_2(E)$ which gives the probability that any resonance level E_μ will be found in a unit energy interval a distance E from a given resonance energy E_ν. For the customary assumptions regarding the Hamiltonian[2] this function has been shown by Dyson[29] to be given by

$$DR_2(E)=1-[s(y)]^2-\frac{ds(y)}{dy}\int_y^\infty s(t)dt, \quad (44)$$

where

$$s(y)=\sin y/y,$$

and

$$y=\pi|E|/D,$$

and D is the mean spacing of the E_μ. Contrary to its apparent oscillatory nature, $\mathbf{R}_2$ increases monotonely to its asymptotic value D^{-1} which it approaches to within 1% in a distance of less than three mean spacings D. There is thus no pronounced long-range correlation of resonance levels.

C. Width Distribution: Direct Reaction Effect

We shall assume with Porter and Thomas,[3] as well as more recent work by Krieger and Porter[19] and by Rosenzweig[20] that the $\gamma_{\mu c}$ for fixed c are normally distributed. From this follows that the $\gamma_{\mu c}{}^2$ follow the Porter-Thomas distribution law

$$P_{\text{P.T.}}(x)dx=(2\pi x)^{-1/2}e^{-x/2}dx \quad (45a)$$

for $x=\gamma_{\mu c}{}^2/\langle\gamma_{\mu c}{}^2\rangle_\mu$. If $\mathbf{R}^{(0)}$ is diagonal, then by Eq. (20) the $\alpha_{\mu c}$ are also normally distributed along some line in the complex plane passing through the origin. Consequently, the $|\alpha_{\mu c}|^2$ also follow the Porter-Thomas distribution law (45a).

We shall consider nonvanishing off-diagonal elements of $\mathbf{R}^{(0)}$ as responsible for "direct reactions" since they generate transitions between channels via the nonresonant portion of the collision matrix $\mathbf{U}^0$. Such off-diagonal elements of $\mathbf{R}^{(0)}$ will mix different $\gamma_{\mu c'}$ linearly into $\alpha_{\mu c}$ so that the latter are no longer restricted to the above mentioned line but have a certain presumably normal distribution also perpendicular to that line. This will have the effect that the distribution of the $|\alpha_{\mu c}|^2$ is less peaked at zero than in Eq. (45a). In the extreme case where very many channels are connected by sizable direct reaction matrix elements, the $\alpha_{\mu c}$ may be expected to be distributed normally and isotropically with respect to the origin of the complex plane. The $|\alpha_{\mu c}|^2$ are then distributed exponentially according to

$$P_E(x)dx=e^{-x}dx, \quad (45b)$$

where

$$x=|\alpha_{\mu c}|^2/\langle|\alpha_{\mu c}|^2\rangle_\mu.$$

Because of the derivations of $\mathbf{U}^S$, the presence of nonvanishing off-diagonal elements of $R_{cc'}{}^{(0)}$ might tend to suggest correlations between the $\gamma_{\mu c}$ and $\gamma_{\mu c'}$. However, this is not necessary, since the appropriate choice of channel radii resulting in uncorrelated channel width amplitudes for resonances near E_0 may imply

[29] Freeman J. Dyson, J. Math. Phys. 3, 166 (1962).

correlations for the distant resonances which contribute to $\mathbf{R}^{(0)}$. In fact it seems unlikely that any single choice of channel radii should be capable of eliminating correlations everywhere. Moreover, it seems probable that if $\mathbf{R}^{(0)}$ has many appreciable off-diagonal elements, any local correlations in the $\gamma_{\mu c}$ may not carry over into the $\alpha_{\mu c}$, but would be washed out by the transformation (20).

Assuming the $\theta_{\mu c}$ and hence the width amplitudes $g_{\mu c}$ of the statistical collision matrix to have the same distributions as the $\alpha_{\mu c}$ we shall need to define some parameters which characterize the possible distributions, namely,

$$\begin{aligned} &b_c=\langle g_{\mu c}{}^2\rangle_\mu/\langle|g_{\mu c}|^2\rangle_\mu, \quad B_c=|b_c|^2. \\ &A_c=\langle|g_{\mu c}|^4\rangle_\mu/\langle|g_{\mu c}|^2\rangle_\mu{}^2. \end{aligned} \tag{46}$$

For the statistical conditions leading to the Porter-Thomas distribution of the partial width (45a) the parameters (46) assume the values $A_c=3$, $B_c=1$, while in the opposite extreme when many direct processes lead to the Gaussian distribution (45b), the values of the parameters are $A_c=2$, $B_c=0$. In general, A_c and B_c may be expected to lie between these limits.

IV. CROSS-SECTION EXPECTATION VALUES

A. General Theory

In accordance with the definitions introduced in Sec. II, we now compute cross-section expectation values at E_0 by averaging the appropriate functions of $\mathbf{U}^S(E,E_0)$ over an energy interval the width of which may be allowed to grow beyond all bounds, but must certainly be very large compared to the average spacing D of the $\mathcal{E}_\mu$ and the average total width Γ. We denote such averages by $\langle\ \rangle_{\rm av}$. In particular, we are here interested in the averages of the elements of $\mathbf{U}^S$ and of their absolute squares. The former will yield the expectation values of the total cross section and the optical-model-shape scattering and absorption cross sections, while the latter yield expectation values of reaction cross sections which we classify into direct and fluctuation cross sections. Specifically, we have the following expectation values

$$\langle\sigma_c{}^{\rm total}\rangle_{E_0}=2\pi\lambda\!\!\!^{-}{}_c{}^2(1-{\rm Re}\langle U_{cc}{}^S(E,E_0)\rangle_{\rm av}), \tag{47}$$

$$\langle\sigma_{cc'}\rangle_{E_0}=\pi\lambda\!\!\!^{-}{}_c{}^2\langle|\delta_{cc'}-U_{cc'}{}^S(E,E_0)|^2\rangle_{\rm av}, \tag{48}$$

and employ the following additional definitions

$$\sigma_{cc'}{}^{\rm direct}(E_0)=\pi\lambda\!\!\!^{-}{}_c{}^2|\delta_{cc'}-\langle U_{cc'}{}^S(E,E_0)\rangle_{\rm av}|^2, \tag{49}$$

$$\sigma_{cc}{}^{\rm direct}(E_0)=\sigma_c{}^{\rm shape\ elastic}(E_0), \tag{50}$$

$$\begin{aligned} \sigma_c{}^{\rm absorption}(E_0)&=\pi\lambda\!\!\!^{-}{}_c{}^2T_c(E_0) \\ &=\pi\lambda\!\!\!^{-}{}_c{}^2(1-|\langle U_{cc}{}^S(E,E_0)\rangle_{\rm av}|^2), \end{aligned} \tag{51}$$

$$\begin{aligned} \sigma_{cc'}{}^{\rm fluctuation}(E_0)&=\pi\lambda\!\!\!^{-}{}_c{}^2 \\ &\times\{\langle|U_{cc'}{}^S|^2\rangle_{\rm av}-|\langle U_{cc'}{}^S\rangle_{\rm av}|^2\}, \end{aligned} \tag{52}$$

which lead to relations such as

$$\langle\sigma_c{}^{\rm total}\rangle=\sigma_c{}^{\rm shape\ elastic}+\sigma_c{}^{\rm absorption}, \tag{53}$$

$$\langle\sigma_{cc'}\rangle=\sigma_{cc'}{}^{\rm direct}+\sigma_{cc'}{}^{\rm fluctuation}, \tag{54}$$

and by flux conservation

$$\textstyle\sum_{c'}\langle\sigma_{cc'}\rangle=\langle\sigma_c{}^{\rm total}\rangle. \tag{55}$$

With the collision matrix (13)

$$U_{cc'}{}^S=U_{cc'}{}^0-i\textstyle\sum_\mu\dfrac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu},$$

all of whose parameters are regarded as constants, we have

$$\langle U_{cc'}{}^S\rangle_{\rm av}=U_{cc'}{}^0-(\pi/D)\langle g_{\mu c}g_{\mu c'}\rangle_\mu, \tag{56}$$

where $\langle\ \rangle_\mu$ is an ensemble average over resonance parameters. The complex optical-model phase shifts[30] δ_c are then given by

$$e^{2i\delta_c}=U_{cc}{}^0-(\pi/D)\langle g_{\mu c}{}^2\rangle_\mu, \tag{57}$$

and the transmission coefficients are

$$\begin{aligned} T_c&=1-|U_{cc}{}^0|^2 \\ &+(2\pi/D)\,{\rm Re}(U_{cc}{}^{0*}\langle g_{\mu c}{}^2\rangle_\mu)-(\pi^2/D^2)|\langle g_{\mu c}{}^2\rangle_\mu|^2. \end{aligned} \tag{58}$$

From Eq. (B11), we obtain

$$\begin{aligned} &\langle|U_{cc'}{}^S|^2\rangle_{\rm av}-|\langle U_{cc'}{}^S\rangle_{\rm av}|^2 \\ &\qquad=\frac{2\pi}{D}\left\langle\frac{|g_{\mu c}|^2|g_{\mu c'}|^2}{\Gamma_\mu}\right\rangle_\mu-M_{cc'}, \end{aligned} \tag{59}$$

where

$$\begin{aligned} M_{cc'}=\frac{2\pi^2}{D^2}\Bigg\{&|\langle g_{\mu c}g_{\mu c'}\rangle_\mu|^2 \\ &-\left\langle g_{\mu c}g_{\mu c'}g_{\nu c}{}^*g_{\nu c'}{}^*\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)\right\rangle_{\mu\neq\nu}\Bigg\}. \end{aligned} \tag{60}$$

The function Φ_0 depends on the distributions of resonance spacings and is defined and discussed in Eqs. (B9) and (B10) and for the case of Eq. (44) it is plotted in Fig. 1, while for uncorrelated resonance levels $\Phi_0\equiv1$. With the aid of Eqs. (33) or (A15), we write

$$|g_{\mu c}|^2=N_\mu\Gamma_{\mu c}, \tag{61}$$

$$\Gamma_\mu=\textstyle\sum_c\Gamma_{\mu c}. \tag{62}$$

Defining

$$\Theta_{\mu c}=(2\pi/D)N_\mu{}^2\Gamma_{\mu c}, \tag{63}$$

$$\Theta_\mu=\textstyle\sum_c\Theta_{\mu c}, \tag{64}$$

we rewrite the first term on the right in Eq. (59) as

[30] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

follows:

$$\frac{2\pi}{D}\left\langle\frac{|g_{\mu c}|^2|g_{\mu c'}|^2}{\Gamma_\mu}\right\rangle_\mu=\left\langle\frac{\Theta_{\mu c}\Theta_{\mu c'}}{\Theta_\mu}\right\rangle_\mu. \quad (65)$$

To evaluate this resonance average we need to know the averages $\langle\Theta_{\mu c}\rangle_\mu$ and the distribution functions characterizing the fluctuations of the $\Theta_{\mu c}$ about their averages and their correlations. We shall assume that N_μ as given by the volume integral of the resonance-state wave function (A16) fluctuates much less widely than $g_{\mu c}$ which is determined by the surface value of the wave function. Hence we shall suppose that the fluctuations and correlations of the $\Theta_{\mu c}$ are governed chiefly by those of the $|g_{\mu c}|^2$ as discussed in Sec. III.

The average of $\Theta_{\mu c}$ is easily obtained by averaging the unitarity relation

$$\sum_{c'+}\langle|U_{cc'}{}^S|^2\rangle_{\rm av}=1, \quad (66)$$

which yields

$$\langle\Theta_{\mu c}\rangle_\mu=T_c-\sum_{c'\neq c}|U_{cc'}{}^0-(\pi/D)\langle g_{\mu c}g_{\mu c'}\rangle_\mu|^2+\sum_{c'}M_{cc'}. \quad (67)$$

By substituting (65) into Eq. (59), summing over all final channels c' and using Eq. (67), it is easily shown that

$$\sigma_c{}^{\rm absorption}=\sum_{c'}\sigma_{cc'}{}^{\rm fluctuation}+\sum_{c'\neq c}\sigma_{cc'}{}^{\rm direct}, \quad (68)$$

which confirms Eq. (55).

The transmission coefficient T_c gives the average fraction of the coherent incident flux in channel c which is removed by the interaction process. As seen from Eq. (67), we may write

$$T_c=T_c{}^{\rm D.R.}+T_c{}^{\rm C.N.}, \quad (69)$$

where $T_c{}^{\rm D.R.}$ measures the fraction of the flux which initiates direct reactions

$$T_c{}^{\rm D.R.}=\sum_{c'\neq c}|U_{cc'}{}^0-(\pi/D)\langle g_{\mu c}g_{\mu c'}\rangle_\mu|^2, \quad (70)$$

and $T_c{}^{\rm C.N.}$ measures the fraction of the flux which forms the compound nucleus

$$T_c{}^{\rm C.N.}=\langle\Theta_{\mu c}\rangle_\mu-\sum_{c'}M_{cc'}, \quad (71)$$

and which is further divided into an average resonance-absorption coefficient $\langle\Theta_{\mu c}\rangle_\mu$ and a resonance-interference contribution $-\sum_{c'}M_{cc'}$. Similarly the fluctuation cross section

$$\sigma_{cc'}{}^{\rm fluctuation}=\pi\lambda_c{}^2\left\{\left\langle\frac{\Theta_{\mu c}\Theta_{\mu c'}}{\Theta_\mu}\right\rangle_\mu-M_{cc'}\right\} \quad (72)$$

consists by Eqs. (59) and (65) of the average resonance contribution which is formally very reminiscent of the Hauser-Feshbach model[31] and a resonance interference term, $-M_{cc'}$.

We may also substitute into Eqs. (47) to (51) to obtain formal expressions for all other cross sections. However, further progress in the evaluation of cross sections will depend on additional models or assumptions regarding the $g_{\mu c}$ and $U_{cc'}{}^0$. We now turn to these.

[31] W. Hauser and H. Feshbach, Phys. Rev. 87, 366 (1952).

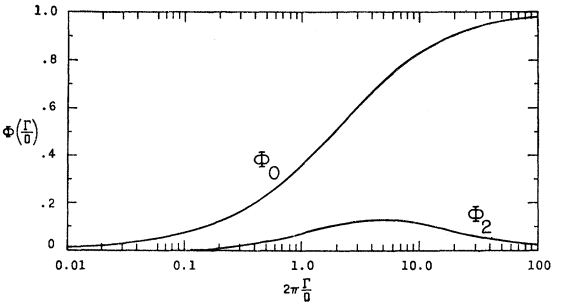

FIG. 1. The functions Φ_0 and Φ_2 evaluated with Dyson's two-level correlation function.

B. Simplifying Assumptions

Uncorrelated Amplitudes

The most helpful assumption is to say that the $g_{\mu c}$ for different channels are uncorrelated (see Sec. III):

$$\langle g_{\mu c}g_{\mu c'}\rangle_\mu=\delta_{cc'}\langle g_{\mu c}{}^2\rangle_\mu. \quad (73)$$

With the help of Eqs. (61) and (63) and the definitions (46), the assumption (73) leads to

$$\langle U_{cc'}{}^S\rangle_{\rm av}=U_{cc'}{}^0-\delta_{cc'}(\pi/D)\langle g_{\mu c}{}^2\rangle_\mu=(1-\delta_{cc'})U_{cc'}{}^0+\delta_{cc'}e^{2i\delta_c}=U_{cc'}{}^0-\tfrac{1}{2}\delta_{cc'}b_c\langle\Theta_{\mu c}/N_\mu\rangle_\mu, \quad (74)$$

and

$$M_{cc'}=\delta_{cc'}\frac{2\pi^2}{D^2}\left\{|\langle g_{\mu c}{}^2\rangle_\mu|^2-\left\langle g_{\mu c}{}^2 g_{\nu c}{}^{*2}\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)\right\rangle_{\mu\neq\nu}\right\}. \quad (75)$$

Considering the fact that in the presence of several channels the sum of two total widths fluctuates very little and furthermore that Φ_0 is a fairly slowly varying function of its argument, we may ordinarily approximate (75) by

$$M_{cc'}\approx\delta_{cc'}(2\pi^2/D^2)|\langle g_{\mu c}{}^2\rangle_\mu|^2[1-\Phi_0]=2\delta_{cc'}|U_{cc}{}^0-e^{2i\delta_c}|^2[1-\Phi_0]=\tfrac{1}{2}\delta_{cc'}B_c\langle\Theta_{\mu c}/N_\mu\rangle_\mu{}^2[1-\Phi_0], \quad (76)$$

where

$$\Gamma=\langle\Gamma_\mu\rangle_\mu. \quad (77)$$

Using the arguments below Eq. (65) to set

$$N_\mu\approx\langle N_\mu\rangle_\mu=N, \quad (78)$$

we obtain from Eqs. (71) and (76) the result

$$T_c{}^{\rm C.N.}=\langle\Theta_{\mu c}\rangle_\mu-\tfrac{1}{2}B_cN^{-2}(1-\Phi_0)\langle\Theta_{\mu c}\rangle_\mu{}^2, \quad (79)$$

which can be solved to give

$$\langle\Theta_{\mu c}\rangle_\mu = T_c^{\text{C.N.}} + Q_c^{-1}[1-(1-Q_c T_c^{\text{C.N.}})^{1/2}]^2, \quad (80)$$

where

$$Q_c = 2B_c N^{-2}(1-\Phi_0), \quad (81)$$

and from Eq. (72) we obtain

$$\sigma_{cc'}^{\text{fluctuation}} = \pi\lambda\!\!\!^{-}_c{}^2 \left\{ \left\langle \frac{\Theta_{\mu c}\Theta_{\mu c'}}{\Theta_\mu} \right\rangle_\mu + \delta_{cc'}(T_c^{\text{C.N.}} - \langle\Theta_{\mu c}\rangle_\mu) \right\}. \quad (82)$$

The requirement that Eq. (80) be real imposes restrictions on the averages of the partial widths defined in Eq. (61). In the single-channel case with $N=1$ this restriction is found with the help of Eq. (B10) or Fig. 1 to be

$$2\pi\langle\Gamma_{\mu c}\rangle_\mu/D \leq 2; \quad (83)$$

in the case of resonance-level repulsion. In the case of no repulsion with $\Phi_0 \equiv 1$, the limit is the usual $2\pi\langle\Gamma_{\mu c}\rangle_\mu D^{-1} \leq 1$ obtained by setting the left side of (83) equal to T_c. In general for vanishing Q_c, that is, if $\Phi_0=1$ or $B_c=0$, we have $\langle\Theta_{\mu c}\rangle_\mu = T_c^{\text{C.N.}}$ which by Eq. (63) gives

$$2\pi\langle\Gamma_{\mu c}\rangle_\mu/D = T_c^{\text{C.N.}}/N^2. \quad (84)$$

We see, therefore, that even though T_c may be close to unity, $\langle\Gamma_{\mu c}\rangle_\mu/D$ could be quite small either because of a large value of $T_c^{\text{D.R.}}$ or a large N or both.

Low Energies: No Direct Reactions

At low energies where all T_c are small and assuming no direct reactions, Eq. (82) becomes

$$\sigma_{cc'}^{\text{fluctuation}} \approx \pi\lambda\!\!\!^{-}_c{}^2 \times (T_c T_{c'}/\textstyle\sum_{c''} T_{c''}) \times W_{cc'}, \quad \Gamma \ll D, \quad (85)$$

where

$$W_{cc'} = \left\langle \frac{\Theta_{\mu c}\Theta_{\mu c'}}{\Theta_\mu} \right\rangle \Big/ \frac{\langle\Theta_{\mu c}\rangle\langle\Theta_{\mu c'}\rangle}{\langle\Theta_\mu\rangle}. \quad (86)$$

Without the factor $W_{cc'}$, Eq. (85) is the well-known Hauser-Feshbach formula[31] which expresses the independence of the decay-branching ratio $T_{c'}/\sum_{c''} T_{c''}$ from the formation cross section $\pi\lambda\!\!\!^{-}_c{}^2 T_c$ for compound-nuclear states. The additional factor $W_{cc'}$ is the width-fluctuation correction which arises from the fact that an average over many compound states is performed and that the partial widths for these states are distributed according to Eqs. (45) or (46) or some intermediate distribution law. This correction has been extensively discussed elsewhere.[6,32,33] We recall here only that for $c \neq c'$ the maximum corrections are $W_{cc'}=\frac{1}{2}$ in the case of the distribution (45), and $W_{cc'}=\frac{2}{3}$ in the case of the distribution (46), and that $W_{cc'}$ approaches unity for very large numbers of competing channels. Because of the channel self-correlation effect, W_{cc} is three times $W_{cc'}(c\neq c')$ with the same average parameters in the case of the distribution (45) and two times as much for the distribution (46). This is also true in the limit of very many competing channels.[34]

As T_c increases $\langle\Theta_{\mu c}\rangle_\mu$ rises more rapidly than T_c. In the limit of minimum resonance-resonance interference $\Phi_0=0$, $\langle\Theta_{\mu c}\rangle_\mu - T_c$ is greatest. This has the effect of making $\sigma_{cc'}^{\text{fluctuation}}$ greater than the approximation (85) when $c\neq c'$ and less when $c=c'$. Considered as a correction to the Hauser-Feshbach formula the sign of this effect is always opposite to that of the width fluctuation correction $W_{cc'}$, but ordinarily the two effects by no means cancel one another.

As Γ/D increases or as the resonance-level repulsion of the $\mathcal{E}_\mu$ is reduced, Φ_0 approaches unity and the fluctuation cross section again approaches Eq. (85). Since in the limit of very large Γ/D the number of competing channels is large and therefore the width-fluctuation effect is expected to be negligible, the simple uncorrected Hauser-Feshbach formula for $\sigma_{cc'}^{\text{fluctuation}}$ is expected to be applicable in that limit.

The relative enhancement of the compound elastic cross section $\sigma_{cc}^{\text{fluctuation}}$ with increasing Γ/D due to Eq. (80) may be undersotod qualitatively as follows. As the lifetime $\hbar/\Gamma$ of the compound system decreases compared to the characteristic period $\hbar/D$ of its internal motion, the time-dependent wave function retains a progressively larger component of the entrance-channel wave function (or "memory of the mode of formation") at the nuclear surface when the decay takes place, thus favoring re-emission into the entrance channel. Though the language of this explanation becomes less appropriate as Γ becomes large compared to D, the effect continues in the same direction. This point has been discussed by Feshbach.[11]

High Energies: Direct Reactions

As the total energy of the system increases beyond 20 MeV we expect more and more direct processes to play a dominant role. Therefore $\mathbf{U}^0$ is expected to have a substantial number of off-diagonal elements and, as discussed in Sec. III, we expect B_c and hence $M_{cc'}$ to approach zero. In the presence of many competing channels, $\sigma_{cc'}^{\text{fluct.}}$ is then given by the Hauser-Feshbach formula. However, as the values of the direct transmission coefficients (70) increase, the compound transmission coefficients $\langle\Theta_{\mu c}\rangle_\mu$ must decline by Eq. (69) since $T_c \leq 1$. Hence the magnitudes of the fluctuation cross sections will decrease as an ever greater portion of the incident flux initiates direct reactions. The latter must, of course, be discussed in terms of the correla-

[32] A. M. Lane and J. E. Lynn, Proc. Phys. Soc. (London) **A70**, 557 (1957).

[33] L. Dresner, Proceedings of the International Conference on Neutron Interactions with the Nucleus, Columbia University Report CU-175, 1957, p. 71 (unpublished).

[34] G. R. Satchler, Phys. Letters **7**, 55 (1963).

tions in the actual collision matrix, or preferably in terms dynamical models involving interactions of few degrees of freedom of the total system and therefore having smooth energy variations.

Expansion in R-Matrix Parameters

When Γ/D is small we can obtain an expansion of the fluctuation cross section in terms of the formal partial width to level spacing ratios of R-matrix theory by employing the expression[35]

$$T_c=\frac{\langle\tau_{\mu c}\rangle_\mu}{(1+\frac{1}{4}\langle\tau_{\mu c}\rangle_\mu)^2+(P_c\tilde{R}_c)^2}, \tag{87}$$

where

$$\tau_{\mu c}=4\pi P_c(\gamma_{\mu c}^2/D), \tag{88}$$

$$\tilde{R}_c=Pr\int dE\frac{\rho(E)\langle\gamma_{\mu c}^2\rangle_\mu}{E-E_0}, \tag{89}$$

and $\rho(E)$ is the density of formal resonance levels E_μ, $\langle\gamma_{\mu c}^2\rangle_\mu$ is the local resonance average evaluated at E_0 in Eq. (87) and at E in Eq. (89), and Pr stands for the principal value of the integral. Expanding (87) in powers of τ and assuming that $P\tilde{R}$ is small to the same order as τ, one obtains

$$T_c\approx\langle\tau_{\mu c}\rangle_\mu(1-\tfrac{1}{2}\langle\tau_{\mu c}\rangle_\mu-P_c\tilde{R}_c). \tag{90}$$

Substituting this into Eq. (80) with $B_cN^{-2}=1$, one finds that

$$\langle\Theta_{\mu c}\rangle_\mu\approx\langle\tau_{\mu c}\rangle_\mu(1-\tfrac{1}{2}\Phi_0\langle\tau_{\mu c}\rangle_\mu-P_c\tilde{R}_c), \tag{91}$$

where now the quadratic term in $\langle\tau_{\mu c}\rangle_\mu$ on the right-hand side is expected to be very small for $\Gamma\ll D$ because of the level repulsion effect (see Fig. 1). In that limit, we may therefore also drop the averages on the left-hand side and on the first factor on the right of Eq. (91) (see Sec. III) and substitute (91) into Eq. (82) to obtain

$$\sigma_{cc'}^{\text{fluct.}}\approx\pi\lambda\!\!\!^{-}_c{}^2\langle\tau_{\mu c}\tau_{\mu c'}/\textstyle\sum_{c''}\tau_{\mu c''}\rangle_\mu$$
$$\times\Big\{1-\tfrac{1}{2}\langle\tau_{\mu c}+\tau_{\mu c'}\rangle_\mu\Phi_0-\tfrac{1}{2}\delta_{cc'}\langle\tau_{\mu c}\rangle_\mu{}^2(1-\Phi_0)$$
$$+\frac{1}{2}\frac{\sum_{c''}\langle\tau_{\mu c''}\rangle_\mu{}^2(\Phi_0+P_{c''}\tilde{R}_{c''})}{\sum_{c'''}\langle\tau_{\mu c'''}\rangle_\mu}$$
$$-(P_c\tilde{R}_c+P_{c'}\tilde{R}_{c'})\Big\}, \tag{92}$$

which agrees with the leading terms of the expansion obtained in Ref. 6, Eqs. (38), (39), except for the terms involving $P_c\tilde{R}_c$ which were ignored there.[36]

[35] P. A. Moldauer, Phys. Rev. **129**, 754 (1963). This paper corrects the results of Sec. III of Ref. 6.

[36] There is one discrepancy. The term $-\frac{1}{2}\delta_{cc'}\langle\tau_{\mu c}\rangle_\mu{}^2$ has an erroneous additional factor of $\frac{1}{2}$ in Ref. 6, Eq. (39).

C. Observable Cross Sections

The cross sections defined in Eqs. (47) to (52) involve transitions between asymptotic states c which are ordinarily not directly distinguishable by experimental methods. The observable cross sections are angular distributions, integrated and total cross sections involving initial and final states belonging to specified alternatives α which may have specified states of polarization, though we shall not concern ourselves with the latter possibility here. The average differential cross section for scattering with alternative α consisting of fragments with spins I and $\mathcal{I}$ in the incident beam and fragments α' emerging in a differential solid angle $d\Omega_{\alpha'}$ at the polar scattering angle $\theta_{\alpha'}$ is given by[37]

$$\langle d\sigma_{\alpha\alpha'}\rangle=(\lambda\!\!\!^{-}_\alpha{}^2/(2I+1)(2\mathcal{I}+1))$$
$$\times\textstyle\sum_L\langle B_L(\alpha,\alpha')\rangle_{\text{av}}P_L(\cos\theta_{\alpha'})d\Omega_{\alpha'}, \tag{93}$$

where $P_L(\cos\theta_{\alpha'})$ is the Legendre polynomial of order L and

$$B_L(\alpha,\alpha')=\textstyle\sum_{c_1c_1'c_2c_2'}\delta_{s_1s_2}\delta_{s_1's_2'}$$
$$\tfrac{1}{4}(-1)^{s_1-s_1'}\bar{Z}(l_1J_1l_2J_2;s_1L)$$
$$\times\bar{Z}(l_1'J_1l_2'J_2,s_1'L)\,\text{Re}(\mathcal{T}_{c_1c_1'}\mathcal{T}_{c_2c_2'}{}^*). \tag{94}$$

The summation over each c denotes a summation over the corresponding values of s, l, and J and c_1, c_2 belong to the alternative α while c_1', c_2' belong to α'. The $\bar{Z}$ coefficients are defined in Refs. 7–9. The transition amplitudes $\mathcal{T}_{cc'}$ are given by

$$\mathcal{T}_{cc'}=\delta_{cc'}-U_{cc'}=\mathcal{T}_{cc'}{}^0+\mathcal{T}_{cc'}{}^1, \tag{95}$$

where, employing the statistical collision matrix at E_0

$$\mathcal{T}_{cc'}{}^0=\delta_{cc'}-U_{cc'}{}^0 \tag{96}$$

is constant in energy and

$$\mathcal{T}_{cc'}{}^1=i\textstyle\sum_\mu\dfrac{g_{\mu c}g_{\mu c'}}{E-\mathcal{E}_\mu+\frac{1}{2}i\Gamma_\mu}. \tag{97}$$

Substituting these expressions into Eq. (94) we may write $\langle B_L\rangle_{\text{av}}$ in two parts

$$\langle B_L\rangle_{\text{av}}=B_L{}^{\text{direct}}+B_L{}^{\text{fluctuation}}, \tag{98}$$

where

$$B_L{}^{\text{direct}}=\textstyle\sum_{12}'\,\bar{Z}_{12L}\bar{Z}_{1'2'L}\,\text{Re}\{\mathcal{T}_{c_1c_1'}{}^0\mathcal{T}_{c_2c_2'}{}^{0*}$$
$$+\mathcal{T}_{c_1c_1'}\langle\mathcal{T}_{c_2c_2'}{}^{1*}\rangle_{\text{av}}+\langle\mathcal{T}_{c_1c_1'}{}^1\rangle_{\text{av}}\mathcal{T}_{c_2c_2'}{}^{0*}$$
$$+\langle\mathcal{T}_{c_1c_1'}{}^1\rangle_{\text{av}}\langle\mathcal{T}_{c_2c_2'}{}^{1*}\rangle_{\text{av}}\}. \tag{99a}$$

$$B_L{}^{\text{fluctuation}}=\textstyle\sum_{12}'\,\bar{Z}_{12L}\bar{Z}_{1'2'L}\,\text{Re}\{\langle\mathcal{T}_{c_1c_1'}{}^1\mathcal{T}_{c_2c_2'}{}^{1*}\rangle_{\text{av}}$$
$$-\langle\mathcal{T}_{c_1c_1'}{}^1\rangle_{\text{av}}\langle\mathcal{T}_{c_2c_2'}{}^{1*}\rangle_{\text{av}}\}. \tag{99b}$$

[37] John M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952); L. C. Biedenharn, in *Nuclear Spectroscopy, Part B*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960), p. 732.

We have used here the abbreviations

$$\bar{Z}_{12L}=\bar{Z}(l_1J_1l_2J_2;s_1L), \tag{100a}$$

$$\sum_{12}{}'=\sum_{c_1c_1'c_2c_2'}\delta_{s_1s_2}\delta_{s_1's_2'}\tfrac{1}{4}(-1)^{s_1-s_1'}. \tag{100b}$$

From Eq. (B3) we have

$$\langle \mathcal{T}_{cc'}{}^{1}\rangle_{\rm av}=\pi(\langle g_{\mu c}g_{\mu c'}\rangle_\mu/D), \tag{101}$$

while by Eq. (B11)

$$\langle \mathcal{T}_{c_1c_1'}{}^{1}\mathcal{T}_{c_2c_2'}{}^{1*}\rangle_{\rm av}=\pi^2\frac{\langle g_{\mu c_1}g_{\mu c_1'}\rangle_\mu\langle g_{\nu c_2}{}^*g_{\nu c_2'}{}^*\rangle_\nu}{D_1D_2}$$

$$+\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}\left\{\frac{2\pi}{D}\left\langle\frac{g_{\mu c_1}g_{\mu c_1'}g_{\mu c_2}{}^*g_{\mu c_2'}{}^*}{\Gamma_\mu}\right\rangle_\mu-(2\pi^2/D^2)\right.$$

$$\left.\times\langle g_{\mu c_1}g_{\mu c_1'}g_{\nu c_2}{}^*g_{\nu c_2'}{}^*[1-\Phi_0(\Gamma_\mu+\Gamma_\nu/2D)]\rangle_{\mu\neq\nu}\right\}, \tag{102}$$

where J_1 and Π_1 are the total angular momentum and parity of c_1 and c_1'. Under the assumption (73) of uncorrelated amplitudes, Eqs. (101) and (102) become

$$\langle \mathcal{T}_{cc'}{}^{1}\rangle=\tfrac{1}{2}\delta_{cc'}b_c\langle\Theta_{\mu c}/N_\mu\rangle_\mu. \tag{101a}$$

$$\langle \mathcal{T}_{c_1c_1'}{}^{1}\mathcal{T}_{c_2c_2'}{}^{1*}\rangle$$

$$=[\delta_{c_1c_2}\delta_{c_1'c_2'}+(1-\delta_{c_1c_1'})\delta_{c_1c_2'}\delta_{c_1'c_2}]\left\langle\frac{\Theta_{\mu c_1}\Theta_{\mu c_1'}}{\Theta_\mu}\right\rangle_\mu$$

$$+\delta_{c_1c_1'}\delta_{c_2c_2'}\left\{\tfrac{1}{4}b_{c_1}b_{c_2}{}^*\left\langle\frac{\Theta_{\mu c_1}}{N_\mu}\right\rangle_\mu\left\langle\frac{\Theta_{\nu c_2}}{N_\nu}\right\rangle_\nu\right.$$

$$\times[1-2\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}(1-\Phi_0)]$$

$$\left.+\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}\frac{2\pi}{D}\left\langle\frac{g_{\mu c_1}{}^2g_{\mu c_2}{}^{*2}}{\Gamma_\mu}\right\rangle_{\mu,c_1\neq c_2}\right\}. \tag{102a}$$

At low energies the expressions (101a) and (102a) may be rewritten by employing the relation

$$\pi/D\langle g_{\mu c}{}^2\rangle_\mu=b_c\langle\Theta_{\mu c}/N_\mu\rangle_\mu. \tag{103}$$

In the many-channel case of constant Γ_μ and uncorrelated $g_{\mu c}$ we may write

$$\frac{2\pi}{D}\left\langle\frac{g_{\mu c_1}{}^2g_{\mu c_2}{}^{*2}}{\Gamma_\mu}\right\rangle_\mu=b_{c_1}b_{c_2}{}^*\left\langle\frac{\Theta_{\mu c_1}\Theta_{\mu c_2}}{\Theta_\mu}\right\rangle_\mu. \tag{104}$$

In the case of nonelastic processes $\alpha'\neq\alpha$, $B_L{}^{\rm fluct.}$ consists according to Eqs. (101a) and (102a) of terms which are of the form of the Hauser-Feshbach formula, except that $\langle\Theta_{\mu c}\rangle_\mu$ replaces T_c and the width fluctuation correction is applicable. The compound elastic angular distribution contains additional correction terms, which in contrast to the result (76), do not vanish even for randomly distributed resonance energies, or in the limit $\Gamma\gg D$ (that is, when $\Phi_0=1$). These correction terms vanish only in the high-energy region when, due to many direct processes the statistics (45b) applies and hence the $b_c\to 0$. In that limit also the interference terms vanish and

$$\langle B_L\rangle_{\rm av}\to B_L{}^{\rm direct}+\tfrac{1}{4}\sum_{c_1c_1'}[(-1)^{s_1-s_1'}\bar{Z}_{11L}\bar{Z}_{1'1'L}$$
$$+(1-\delta_{c_1c_1'})\delta_{s_1s_1'}\bar{Z}_{11'L}{}^2]\times\sigma_{c_1c_1'}{}^{\rm fluct.}$$
(high-energy limit). (105)

The second term in the brackets affects only compound elastic processes.

For the average integrated cross section one obtains the familiar sum over all c belonging to α and all c' belonging to α'

$$\langle\sigma_{\alpha\alpha'}\rangle=\sum_{cc'}\mathfrak{g}_c\langle\sigma_{cc'}\rangle \tag{106}$$

with

$$\mathfrak{g}_c=[(2J_c+1)/(2I_c+1)(2\mathcal{I}_c+1)] \tag{107}$$

and the average total cross section for alternative α is

$$\langle\sigma_\alpha{}^{\rm total}\rangle=\sum_c\mathfrak{g}_c\langle\sigma_c{}^{\rm total}\rangle. \tag{108}$$

Similar "observable" expressions can be written down for the cross sections (49) through (52). In all cases these are just sums over the partial cross sections weighted with $\mathfrak{g}_c$ as in Eqs. (106), (108).

In the above results, as well as those of Sec. V, it should be borne in mind that by conservation of angular momentum and parity $U_{cc'}$ and $\sigma_{cc'}$ vanish unless $J=J'$ and $\Pi=\Pi'$.

V. CROSS-SECTION FLUCTUATIONS

The fluctuations of cross sections about their averages may be specified by correlation functions[38] which can be calculated by methods similar to those employed for the determination of average cross sections. We restrict ourselves here to a discussion of the simplest fluctuation problems, those dealing with the mean-square fluctuations in the total cross section and in nonelastic reaction cross sections.

The total cross section for an incident beam in alternative α is given by

$$\sigma_\alpha{}^{\rm total}=2\pi\lambda\!\!\!^{-}_\alpha{}^2\sum_c\mathfrak{g}_c\,{\rm Re}(\mathcal{T}_{cc}{}^0+\mathcal{T}_{cc}{}^1), \tag{109}$$

where the sum is extended over all c belonging to α. Following Ericson,[38] we define the mean-square fluctuation

$$F_\alpha=\langle(\sigma_\alpha{}^{\rm total})^2\rangle-\langle\sigma_\alpha{}^{\rm total}\rangle^2, \tag{110}$$

which by means of Eqs. (96), (97), (109), and (B11) is easily found to be

$$F_\alpha=\pi^2\lambda\!\!\!^{-}_\alpha{}^4\sum_{c_1c_2}\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}\mathfrak{g}_{c_1}{}^2$$
$$\times 2{\rm Re}\left\{\frac{2\pi}{D}\left\langle\frac{g_{\mu c_1}{}^2g_{\mu c_2}{}^{*2}}{\Gamma_\mu}\right\rangle_\mu\right.$$
$$\left.-\frac{2\pi^2}{D^2}\left\langle g_{\mu c_1}{}^2g_{\nu c_2}{}^{*2}\left[1-\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)\right]\right\rangle_{\mu\neq\nu}\right\}. \tag{111}$$

[38] T. Ericson, Ann. Phys. (N.Y.) 23, 390 (1963).

where c_1 and c_2 must both belong to α. In the many-channel limit where the Γ_μ are assumed not to fluctuate and when the width amplitudes are uncorrelated, Eq. (111) may be rewritten as follows:

$$F_\alpha = 2\pi\lambda\!\!\!^{-}_\alpha{}^2\{\sum_{c_1}\mathfrak{g}_{c_1}{}^2[\sigma_{c_1c_1}{}^{\text{fluct.}} + \sum_{c_2}(1-\delta_{c_1c_2})\,\mathrm{Re}b_{c_1}b_{c_2}{}^*\sigma_{c_1c_2}{}^{\text{fluct.}}]\}. \quad (112)$$

In the case of very many competing channels, we may suppose that the sign of $\mathrm{Re}b_{c_1}b_{c_2}{}^*$ fluctuates with c_2 so that the principal contribution to F_α comes from the first sum which in the case of nucleon scattering may then be estimated to yield

$$F_\alpha \sim (1/n)\langle\sigma_\alpha{}^{\text{total}}\rangle\sigma_{\alpha\alpha}{}^{\text{fluctuation}}, \quad (112a)$$

where n is a number of the order of the number of strongly competing channels c.

Next we evaluate the mean-square fluctuation of the reaction cross section

$$F_{\alpha\alpha'} = \langle\sigma_{\alpha\alpha'}{}^2\rangle - \langle\sigma_{\alpha\alpha'}\rangle^2, \quad (113)$$

for the case where $\alpha \neq \alpha'$ and the amplitudes are uncorrelated

$$\langle g_{\mu c}g_{\mu c'}\rangle = \delta_{cc'}\langle g_{\mu c}{}^2\rangle.$$

Under these circumstances

$$F_{\alpha\alpha'} = \pi\lambda\!\!\!^{-}_\alpha{}^2\sum_{c_1c_1'c_2c_2'}\mathfrak{g}_{c_1}\mathfrak{g}_{c_2}\{\langle|\mathcal{T}_{c_1c_1'}{}^1|^2|\mathcal{T}_{c_2c_2'}{}^1|^2\rangle - \langle|\mathcal{T}_{c_1c_1'}{}^1|^2\rangle\langle|\mathcal{T}_{c_2c_2'}{}^1|^2\rangle + 2\mathrm{Re}(\mathcal{T}_{c_1c_1'}{}^{0*}\mathcal{T}_{c_2c_2'}{}^{0*}\langle\mathcal{T}_{c_1c_1'}{}^1\mathcal{T}_{c_2c_2'}{}^1\rangle + \mathcal{T}_{c_1c_1'}{}^{0*}\mathcal{T}_{c_2c_2'}{}^0\langle\mathcal{T}_{c_1c_1'}{}^1\mathcal{T}_{c_2c_2'}{}^{1*}\rangle)\}, \quad (114)$$

where c_1, c_2 belong to α anc $c_{1'}$, $c_{2'}$ belong to α'.

Employing Eqs. (B11), (B13), (B14), and (B17) we obtain

$$F_{\alpha\alpha'} = \pi\lambda\!\!\!^{-}_\alpha{}^2\sum_{c_1c_1'c_2c_2'}\mathfrak{g}_{c_1}\mathfrak{g}_{c_2}\Bigg\{\frac{2\pi^2}{D_1D_2}\Bigg\langle\frac{|g_{\mu c_1}|^2|g_{\mu c_1'}|^2|g_{\nu c_2}|^2|g_{\nu c_2'}|^2}{\Gamma_\mu\Gamma_\nu}\Bigg[\Phi_0\Big(\frac{\Gamma_\mu+\Gamma_\nu}{2D_1}\Big)+\Phi_0\Big(\frac{\Gamma_\mu+\Gamma_\nu}{2D_2}\Big)-2\Bigg]\Bigg\rangle_{\mu\neq\nu}$$
$$+\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}\Bigg[\frac{4\pi}{D}\Bigg\langle\frac{|g_{\mu c_1}|^2|g_{\mu c_1'}|^2|g_{\mu c_2}|^2|g_{\mu c_2'}|^2}{\Gamma_\mu{}^3}\Bigg\rangle_\mu + \delta_{c_1c_2}\delta_{c_1'c_2'}\Bigg(\frac{4\pi^2}{D^2}\Bigg\langle\frac{|g_{\mu c_1}|^2|g_{\mu c_1'}|^2|g_{\nu c_1}|^2|g_{\nu c_1'}|^2}{\Gamma_\mu\Gamma_\nu}\Phi_0\Big(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\Big)\Bigg\rangle_{\mu\neq\nu}$$
$$-\frac{16\pi^2}{D^2}\Bigg\langle\frac{g_{\mu c_1}{}^2g_{\mu c_1'}{}^2g_{\nu c_1}{}^{*2}g_{\nu c_1'}{}^{*2}}{(\Gamma_\mu+\Gamma_\nu)^2}\Phi_2\Big(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\Big)\Bigg\rangle_{\mu\neq\nu}\Bigg) + 2\mathrm{Re}\mathcal{T}_{c_1c_1'}{}^{0*}\mathcal{T}_{c_2c_2'}{}^0\frac{2\pi}{D}\Bigg\langle\frac{|g_{\mu c_1}|^2|g_{\mu c_1'}|^2}{\Gamma_\mu}\Bigg\rangle_\mu\Bigg]\Bigg\}, \quad \alpha\neq\alpha', \quad (115)$$

where Φ_2 is defined in Eq. (B15) and plotted in Fig. 1. In the many-channel limit when the Γ_μ are fairly constatn (115) can be rewritten as

$$F_{\alpha\alpha'} = \sum_{c_1c_1'}\mathfrak{g}_{c_1}{}^2\sigma_{c_1c_1'}{}^{\text{fluct.}}\{2\sigma_{c_1c_1'}{}^{\text{direct}} + (\Phi_0 - B_{c_1}B_{c_1'}\Phi_2)\sigma_{c_1c_1'}{}^{\text{fluct.}}\} + \sum_{c_1c_1'c_2c_2'}\mathfrak{g}_{c_1}\mathfrak{g}_{c_2}\sigma_{c_1c_1'}{}^{\text{fluct.}}\sigma_{c_2c_2'}{}^{\text{fluct.}}$$
$$\times\{\tfrac{1}{2}[\Phi_0(\Gamma/D_1)+\Phi_0(\Gamma/D_2)]-1+\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}(D/\pi\Gamma)[1+\delta_{c_1c_2}(A_{c_1}-1)][1+\delta_{c_1'c_2'}(A_{c_1'}-1)]. \quad \alpha\neq\alpha' \quad (116)$$

The second sum clearly goes to zero in the limit of large Γ/D and the value of the first sum can in that limit be estimated to yield

$$F_{\alpha\alpha'} \sim (1/nn')\sigma_{\alpha\alpha'}{}^{\text{fluct.}}(\langle\sigma_{\alpha\alpha'}\rangle + \sigma_{\alpha\alpha'}{}^{\text{direct}}), \quad (116a)$$

where n and n' are the number of channels c competing strongly in the decays into alternatives α and α', respectively. Since Γ/D and n are expected to be roughly proportional, we see that we may expect $F_{\alpha\alpha'}$ to decline with increasing excitation energy even more rapidly in the region of large Γ/D than for isolated levels where the decline is governed by the factor $D/\pi\Gamma$ in Eq. (116). Further, discussions of the implications of Eq. (116) have been given elsewhere[39] and will not be repeated here.

Employing again Ericson's definitions we write

$$\langle(d\sigma_{\alpha\alpha'}/d\Omega_{\alpha'})^2\rangle - \langle d\sigma_{\alpha\alpha'}/d\Omega_{\alpha'}{}^2\rangle = \lambda\!\!\!^{-}_\alpha{}^4[(2I+1)^2(2\mathcal{J}+1)^2]^{-1}\sum_{LK}F_{LK}(\alpha,\alpha')P_L(\cos\theta')P_K(\cos\theta'), \quad (117)$$

where

$$F_{LK}(\alpha,\alpha') = \langle B_L(\alpha,\alpha')B_K(\alpha,\alpha')\rangle - \langle B_L(\alpha,\alpha')\rangle\langle B_K(\alpha,\alpha')\rangle. \quad (118)$$

For the case of nonelastic processes $\alpha\neq\alpha'$, we easily evaluate (118) with the help of Eqs. (B13), (B14), (B17) and obtain in the many-channel limit [see Eqs. (100)],

$$F_{LK}(\alpha,\alpha') = \sum_{12}{}'\sum_{34}{}'\bar{Z}_{12L}\bar{Z}_{1'2'L}\bar{Z}_{34K}\bar{Z}_{3'4'K}(f_{1234}(\alpha,\alpha')/\pi^2\lambda\!\!\!^{-}_\alpha{}^4), \quad (119)$$

where f_{1234} depends on the eight sets of channel parameters over which the summation extends and which occur in

[39] P. A. Moldauer, Phys. Letters 8, 70 (1964).

the expression

$$f_{1234}(\alpha,\alpha')=\sigma_{c_1c_1'}{}^{\text{fluct.}}\sigma_{c_3c_3'}{}^{\text{fluct.}}\{\delta_{12}\delta_{34}[\Phi_0-1+\delta_{J_1J_3}\delta_{\Pi_1\Pi_3}(D/\pi\Gamma)[1+\delta_{c_1c_3}(A_{c_1}-1)][1+\delta_{c_1'c_3'}(A_{c_1'}-1)]]$$
$$+\delta_{14}\delta_{23}[\Phi_0+\delta_{J_1J_3}\delta_{\Pi_1\Pi_3}(D/\pi\Gamma)[1+\delta_{c_1c_3}(A_{c_1}-1)+\delta_{c_1'c_3'}(A_{c_1}-1)-\delta_{13}A_{c_1}A_{c_1'}]]\}+\sigma_{c_1c_1'}{}^{\text{fluct.}}\sigma_{c_2c_2'}{}^{\text{fluct.}}$$
$$\times\delta_{13}\delta_{24}b_{c_1}b_{c_1'}b_{c_2}{}^*b_{c_2'}{}^*\{-\Phi_2+\delta_{J_1J_2}\delta_{\Pi_1\Pi_2}(D/\pi\Gamma)[1+\delta_{c_1c_2}(A_{c_1}-1)+\delta_{c_1'c_2'}(A_{c_1'}-1)-\delta_{12}A_{c_1}A_{c_1'}]\}$$
$$+\text{Re}\{\delta_{23}\mathcal{T}_{c_1c_1'}{}^0\mathcal{T}_{c_4c_4'}{}^{0*}\sigma_{c_2c_2'}{}^{\text{fluct.}}+\delta_{14}\mathcal{T}_{c_2c_2'}{}^{0*}\mathcal{T}_{c_3c_3'}{}^0\sigma_{c_1c_1'}{}^{\text{fluct.}}\}, \quad (120)$$

where we have used the notation

$$\delta_{12}=\delta_{c_1c_2}\delta_{c_1'c_2'}.$$

In the limit of large Γ/D this expression becomes

$$f_{1234}(\alpha,\alpha')\rightarrow\delta_{14}\delta_{23}\sigma_{c_1c_1'}{}^{\text{fluct.}}\sigma_{c_2c_2'}{}^{\text{fluct.}}+\text{Re}\{\delta_{23}\mathcal{T}_{c_1c_1'}{}^0\mathcal{T}_{c_4c_4'}{}^{0*}\sigma_{c_2c_2'}{}^{\text{fluct.}}+\delta_{14}\mathcal{T}_{c_2c_2'}{}^{0*}\mathcal{T}_{c_3c_3'}{}^0\sigma_{c_1c_1'}{}^{\text{fluct.}}\}, \quad \alpha\neq\alpha'. \quad (120a)$$

Comparing this result with Eq. (105) we find that in this limit for $\alpha\neq\alpha'$ and in the absence of direct reactions (all $\mathcal{T}_{cc'}{}^0=0$)

$$\frac{\langle(d\sigma_{\alpha\alpha'}/d\Omega)^2\rangle-\langle d\sigma_{\alpha\alpha'}/d\Omega\rangle^2}{\langle d\sigma_{\alpha\alpha'}/d\Omega\rangle^2}\rightarrow\frac{\sum_{LK}\sum_{12'}\bar{Z}_{12L}\bar{Z}_{1'2'L}\bar{Z}_{21K}\bar{Z}_{2'1'K}\sigma_{c_1c_1'}{}^{\text{fluct.}}\sigma_{c_2c_2'}{}^{\text{fluct.}}P_LP_K}{\sum_{LK}\sum_{12'}\bar{Z}_{11L}\bar{Z}_{1'1'L}\bar{Z}_{22K}\bar{Z}_{2'2'K}\sigma_{c_1c_1'}{}^{\text{fluct.}}\sigma_{c_2c_2'}{}^{\text{fluct.}}P_LP_K}, \quad (\alpha\neq\alpha'). \quad (121)$$

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APPENDIX A: COMPLEX BOUNDARY VALUE PROBLEM EXPANSION

In the various boundary condition formalisms[1,10] the configuration space of all nucleons in the scattering system is divided into an interior region and an exterior or channel region. The wave function in the interior region is expanded in the eigenfunctions of a boundary-value problem specified by the Hamiltonian operator and conditions on the normal logarithmic derivative at the surface dividing the two regions. The R-matrix theory of Wigner and Eisenbud[1] employs arbitrary real boundary values, while the Kapur-Peierls formalism[10] uses special energy-dependent complex boundary conditions. A particularly convenient derivation of the latter method has also been given by Bloch.[40] We require a formalism employing arbitrary complex boundary conditions. Since this slight generalization of the R matrix and Kapur-Peierls theories does not appear to be in the literature, we sketch here the derivation from the beginning employing mostly the same notation and conventions as Lane and Thomas used.[7] Omitted details may also be found there.

It is assumed that in the exterior region the system can be described by a collection of states or "channels" $c(\alpha,s,l,J,M)$ characterized by the specification of two fragments in specified states of internal excitation (with quantum numbers symbolized by α), and their angular momentum quantum numbers s(channel spin), l(relative orbital angular momentum), J, M(total angular momentum and its z component). The wave number of the relative motion, depending on the total energy E, is $k_\alpha(E)$. The wave function in the exterior is written as a sum over incoming and outgoing waves in the various channels

$$\Psi=\sum_c(x_c\mathcal{O}_c+y_c\mathcal{I}_c), \quad (A1)$$

and the collision matrix is defined by the channel matrix relation

$$\mathbf{x}=-\mathbf{U}\mathbf{y}. \quad (A2)$$

The wave functions $\mathcal{O}_c$ and $\mathcal{I}_c$ are assumed to be solutions of a Schrödinger equation with a spherically symmetric nonpolarizing potential operator $V(\alpha,s,l)$. The dependences of $\mathcal{O}_c$ and $\mathcal{I}_c$ on their internal coordinates ξ_α, their angular coordinates Ω_α, and their radial channel coordinates r_α may therefore be separated

$$\mathcal{I}_c=v_\alpha{}^{-1/2}\varphi_{\alpha slJM}(\xi_\alpha,\Omega_\alpha)I_{\alpha l}(r_\alpha), \quad (A3a)$$

$$\mathcal{O}_c=v_\alpha{}^{-1/2}\varphi_{\alpha slJM}(\xi_\alpha,\Omega_\alpha)O_{\alpha l}(r_\alpha), \quad (A3b)$$

where v_α is the relative velocity of fragments α and $I_{\alpha l}$ and $O_{\alpha l}$ are the incoming and outgoing solutions of the radial equation

$$[(d^2/dr_\alpha{}^2)-l(l+1)/r_\alpha{}^2-(2M_\alpha/\hbar^2)(V-E)]u=0, \quad (A4)$$

where M_α is the reduced mass. For the purpose of matching the logarithmic derivatives at the dividing surface one defines *value* and *derivative* quantities on the surface by

$$V_c=\left(\frac{\hbar^2}{2M_\alpha a_\alpha}\right)^{1/2}\int_{\text{surface}}\varphi_c{}^*\Psi d\mathcal{S}, \quad (A5a)$$

$$D_c=\left(\frac{\hbar^2}{2M_\alpha a_\alpha}\right)^{1/2}\int_{\text{surface}}\varphi_c{}^*\nabla_n(r_\alpha\Psi)d\mathcal{S}, \quad (A5b)$$

where a_α is the *channel radius* which defines a portion of

[40] C. Bloch, Nucl. Phys. 4, 503 (1957).

the dividing surface over the whole of which the integrations in Eqs. (A5) are carried. The normal gradient at the surface is ∇_n. The φ_c are a complete orthogonal set of functions on the surface, and therefore at $r_\alpha = a_\alpha$

$$\Psi = \sum_c (2M_\alpha a_\alpha/\hbar^2)^{1/2} V_c \varphi_c , \tag{A6a}$$

$$\nabla_n \Psi = \sum_c (2M_\alpha/a_\alpha \hbar^2)^{1/2} (D_c - V_c)\varphi_c . \tag{A6b}$$

In the interior Ψ is expanded in a complete set of functions X_μ

$$\Psi = \sum_\mu A_\mu X_\mu . \tag{A7}$$

The X_μ are defined as the eigenstates of the boundary-value problem

$$HX_\mu = W_\mu X_\mu , \tag{A8}$$

where H is the complete Hamiltonian and the boundary conditions at the dividing surface are specified by complex numbers B_c

$$\psi_{\mu c}/\theta_{\mu c} = B_c , \tag{A9}$$

where $\theta_{\mu c}$ and $\psi_{\mu c}$ are obtained by substituting X_μ for Ψ on the right-hand sides of Eqs. (A5a) and (A5b), respectively. Assuming H to be invariant under rotations and under time reversal, one obtains from (A8) and (A9)

$$H\tilde{X}_\mu = W_\mu^* \tilde{X}_\mu , \tag{A10}$$

$$\tilde{\psi}_{\mu c}/\tilde{\theta}_{\mu c} = B_c^* , \tag{A11}$$

where, denoting the time reversal operator by K,[41]

$$\tilde{X}_{\mu(J,M)} = (-1)^{J-M} K X_{\mu(J,J-M)} , \tag{A12}$$

and $\tilde{\psi}_{\mu c}$ and $\tilde{\theta}_{\mu c}$ are obtained by substituting $\tilde{X}_\mu$ for Ψ in the right-hand sides of Eqs. (A5). If we assume the eigenvalues W_μ of Eq. (A8) to be distinct except for the $(2J+1)$-fold M degeneracy for each μ, then the orthogonality of X_μ and $\tilde{X}_\nu$ for $\mu \neq \nu$ is easily demonstrated by operating on Eq. (A8) with $\int d\tau \tilde{X}_\mu^*$ and on the complex conjugate of Eq. (A10) with $\int d\tau X_\nu$, subtracting and employing the self-adjoint property of the interaction part of H, Green's theorem and Eq. (A9). For the normalization of the X_μ we choose

$$\int_{\text{interior}} \tilde{X}_\mu^* X_\nu d\tau = \delta_{\mu\nu} . \tag{A13}$$

Applying the same procedure to X_μ and X_μ^*, we find that

$$W_\mu = \mathcal{E}_\mu - \tfrac{1}{2} i \Gamma_\mu , \tag{A14}$$

with

$$\Gamma_\mu = \sum_c \Gamma_{\mu c} , \quad \Gamma_{\mu c} = 2N_\mu^{-1} \operatorname{Im} B_c |\theta_{\mu c}|^2 , \tag{A15}$$

where

$$N_\mu = \int_{\text{interior}} |X_\mu|^2 d\tau . \tag{A16}$$

In the special case of the Kapur-Peierls boundary conditions, $\operatorname{Im} B_c$ equals the penetration factor P_c corresponding to the channel radius a_α.

Using Eq. (A13) we obtain for the expansion coefficients A_μ of Eq. (A7)

$$A_\mu = \int_{\text{interior}} \tilde{X}_\mu^* \Psi d\tau . \tag{A17}$$

Operating on the Schrödinger equation

$$H\Psi = E\Psi , \tag{A18}$$

with $\int d\tau \tilde{X}_\mu^*$ and subtracting the complex conjugate of Eq. (A10) after operating on it with $\int d\tau \Psi$, and employing the same methods as described above we obtain

$$A_\mu = (W_\mu - E)^{-1} \sum_c \tilde{\theta}_{\mu c}^* (D_c - B_c V_c) . \tag{A19}$$

Upon substituting this in (A17), one obtains the usual R-matrix relation

$$\mathbf{V} = \mathfrak{R}(\mathbf{D} - \mathbf{BV}) , \tag{A20}$$

where

$$\mathfrak{R}_{cc'} = \sum_\mu (\theta_{\mu c}\theta_{\mu c'}/W_\mu - E) , \tag{A21}$$

and where we have used the fact that

$$\tilde{\theta}_{\mu c}^* = \theta_{\mu c} , \tag{A22}$$

which follows from the properties of the time-reversal operator K and the choice of normalization (A13).[40]

By writing Eq. (A20) in terms of the incoming and outgoing waves of Eqs. (A3) and using (A1) and (A2), one obtains the usual expression for the collision matrix in terms of the R matrix which, for the elements connecting positive energy channels, may be written as

$$\mathbf{U} = \boldsymbol{\Omega}\{\mathbf{1} + 2i\mathbf{P}^{1/2}[\mathbf{1} - \mathfrak{R}(\mathbf{L} - \mathbf{B})]^{-1}\mathfrak{R}\mathbf{P}^{1/2}\}\boldsymbol{\Omega} , \tag{A23}$$

where the following diagonal matrices are used:

$$\Omega_c = [I_c(a_\alpha)/O_c(a_\alpha)]^{1/2} , \tag{A24}$$

$$L_c = k_\alpha a_\alpha O_c'(a_\alpha)/O_c(a_\alpha) = S_c + iP_c . \tag{A25}$$

The Kapur-Peierls theory corresponds to the choice $B_c = L_c$ at each energy which eliminates the matrix inversion indicated in Eq. (A23), but implies energy-dependent values of W_μ and $\theta_{\mu c}$. The Wigner-Eisenbud R-matrix theory encompasses any fixed real values of the B_c. As a result, the W_μ and $\theta_{\mu c}$ become the real parameters E_μ and $\gamma_{\mu c}$ used in Sec. II.C.

APPENDIX B: INTEGRALS

We are concerned here with functions of the form

$$F^{(j)}(E) = \sum_\mu \frac{a_\mu^{(j)}}{E - \mathcal{E}_\mu^{(j)} + \frac{1}{2} i \Gamma_\mu^{(j)}} ,$$

and

$$F^{(j)*} , \quad F^{(j)}F^{(k)*} , \quad F^{(j)}F^{(k)*}F^{(l)}F^{(m)*} , \text{ etc.} , \tag{B1}$$

where the real and distinct $\mathcal{E}_\mu^{(j)}$ are assumed to be distributed with uniform density D_j^{-1} and with uniform

[41] A full discussion of the properties of the time inversion operator has been given by Eugene P. Wigner, *Group Theory*, translated by J. J. Griffin (Academic Press Inc., New York, 1959), Chap. 26.

correlation of $\mathcal{E}_{\mu+n}^{(j)}$ and $\mathcal{E}_{\mu}^{(j)}$ from $E=-\infty$ to $+\infty$. The complex $a_{\mu}^{(j)}$ and the real and positive $\Gamma_{\mu}^{(j)}$ also are uniformly distributed in μ. We further assume that there is an effective maximum value $\Gamma_j^{\max}$ of the $\Gamma_{\mu}^{(j)}$ and we call the average $\langle\Gamma_{\mu}^{(j)}\rangle_{\mu}=\Gamma_j$.

We wish to obtain averages of the functions (B1) (collectively called F) over energy intervals which are allowed to grow beyond all bounds:

$$\langle F\rangle_{\mathrm{av}}=\lim\frac{1}{W}\int_W dEF(E). \tag{B2}$$

For the distributions of interest here the limiting value will be approached when $W\gg\Gamma_j, D_j$ for all j involved in F. We evaluate the integral in (B2) by considering the contour integral of F along a rectangle in the upper half-plane of the complex E plane with its base of length W along the real axis and of height $\mathfrak{W}\gg D_j, \Gamma_j^{\max}$ for all j involved in F. In fact, we make $\mathfrak{W}$ large enough so that along the top of the rectangle F has reached the limiting constant value which it approaches in the upper half-plane. The expectation values of the contributions from the two vertical sides of the contour integral are equal and opposite, and the root-mean-square fluctuation of the net contribution to $\langle F\rangle_{\mathrm{av}}$ from these sides goes to zero as W^{-1}. Their possible contribution can be further reduced toward zero by considering averages over many $\langle F\rangle_{\mathrm{av}}$ whose intervals are slightly displaced. This is equivalent to the use of other than rectangular resolution functions. These conclusions regarding the vanishing of the contribution from the vertical sides clearly hold equally whether we are dealing with $F^{(j)}$ or with functions in which $F^{(j)*}$ contributes poles in the upper half-plane. We conclude therefore that

$$\langle F\rangle_{\mathrm{av}}=F(i\mathfrak{W})+\lim(2\pi i/W)\sum_{\varepsilon_{\mu}\text{ in }W}R_{\mu}^{+}, \tag{B2}$$

where R_{μ}^{+} is the residue of any pole of F which lies in the upper half-plane and whose real coordinate is $\mathcal{E}_{\mu}$.

The average of $F^{(j)}$ is easily found by evaluating the first term in Eq. (B2)

$$\langle F^{(j)}\rangle_{\mathrm{av}}=F^{(j)}(i\mathfrak{W})=-i\pi(\langle a_{\mu}^{(j)}\rangle_{\mu}/D_j). \tag{B3}$$

When $F=F^{(j)}F^{(k)*}$, we find from (B3) that

$$F(i\mathfrak{W})=-\pi^2(\langle a_{\mu}^{(j)}\rangle_{\mu}/D_j)(\langle a_{\nu}^{(k)*}\rangle_{\nu}/D_k), \tag{B4}$$

and that

$$\sum R_{\mu}^{+}=\sum_{\xi_{\mu}\text{in}W}\sum_{\nu}\frac{a_{\nu}^{(j)}a_{\mu}^{(k)*}}{(\mathcal{E}_{\mu}^{(k)}-\mathcal{E}_{\nu}^{(j)})+\frac{1}{2}i(\Gamma_{\mu}^{(k)}+\Gamma_{\mu}^{(j)})}. \tag{B5}$$

The value of this sum will depend on whether the poles of $F^{(j)}$ and $F^{(k)}$ have the same coordinates or not. We denote the parameter determining the pole coordinates of $F^{(j)}$ by J_j (total angular momentum and parity of the system) and assume that if $J_j\neq J_k$ the positions of the $\mathcal{E}_{\mu}^{(k)}$ and $\mathcal{E}_{\nu}^{(j)}$ are completely uncorrelated. In that event (B5) yields for the residue contribution to the average

$$\frac{2\pi i}{W}\sum R_{\mu}^{+}=2\pi^2\frac{\langle a_{\nu}^{(j)}\rangle_{\nu}}{D_j}\frac{\langle a_{\mu}^{(k)*}\rangle_{\mu}}{D_k}, \quad J_j\neq J_k. \tag{B6}$$

If, however, $J_j=J_k$, and hence $D_j=D_k=D$, then we obtain a contribution from those terms in (B5) for which $\nu=\mu$ which is

$$(2\pi/D)\langle a_{\mu}^{(j)}a_{\mu}^{(k)*}/\Gamma_{\mu}\rangle_{\mu}, \tag{B7}$$

and a contribution from the terms with $\nu\neq\mu$:

$$(2\pi^2/D^2)\langle a_{\nu}^{(j)}a_{\mu}^{(k)*}\Phi_0[(\Gamma_{\mu}+\Gamma_{\nu})/2D]\rangle_{\mu\neq\nu}, \tag{B8}$$

where

$$\Phi_0\left(\frac{\Gamma}{D}\right)=-i\frac{D}{\pi}\int_{-\infty}^{\infty}\frac{d\epsilon R_2(\epsilon)}{\epsilon-i\Gamma}, \tag{B9}$$

and R_2 has been defined in connection with Eq. (44). For uncorrelated $\mathcal{E}_{\mu}^{(j)}$ the function R_2 has the constant value D^{-1} and then $\Phi_0\equiv 1$. For Dyson's expression (44) of R_2 we find that

$$\Phi_0\left(\frac{\Gamma}{D}\right)=1-(1/x)[1-(1/x)e^{-x}\sinh x]-(1/x)\mathrm{Ei}(-x)[\cosh x-(1/x)\sinh x], \tag{B10}$$

$$x=\pi\frac{\Gamma}{D}, \quad -\mathrm{Ei}(-x)=\int_x^{\infty}e^{-t}t^{-1}dt,$$

which is plotted in Fig. 1. For large Γ/D the function Φ_0 must always approach unity. For small Γ/D the deviation of Φ_0 from unity depends on the degree of the mutual "repulsion" of the neighboring $\mathcal{E}_{\mu}^{(j)}$ and on the range of this resonance level correlation effect. Combining the results (B4) through (B8), we obtain

$$\langle F^{(j)}F^{(k)*}\rangle_{\mathrm{av}}=\pi^2\frac{\langle a_{\mu}^{(j)}\rangle_{\mu}}{D_j}\frac{\langle a_{\nu}^{(k)*}\rangle_{\nu}}{D_k}+\delta_{J_jJ_k}\left\{\frac{2\pi}{D}\left\langle\frac{a_{\mu}^{(j)}a_{\mu}^{(k)*}}{\Gamma_{\mu}}\right\rangle_{\mu}-\frac{2\pi^2}{D^2}\left\langle a_{\mu}^{(j)}a_{\nu}^{(k)*}\left[1-\Phi_0\left(\frac{\Gamma_{\mu}+\Gamma_{\nu}}{2D}\right)\right]\right\rangle_{\mu\neq\nu}\right\}. \tag{B11a}$$

The function $F^{(j)}F^{(k)}$ has no poles in the upper half-plane and hence

$$\langle F^{(j)}F^{(k)}\rangle_{\mathrm{av}}=-\pi^2\frac{\langle a_{\mu}^{(j)}\rangle_{\mu}}{D_j}\frac{\langle a_{\nu}^{(k)}\rangle_{\nu}}{D_k}. \tag{B11b}$$

For more complicated functions of the $F^{(j)}$, the expressions for the averages become more and more lengthy. We shall here still evaluate the average of $F^{(j)}F^{(k)*}F^{(l)}F^{(m)*}$ under the restrictions

$$\langle a_{\mu}^{(i)}\rangle_{\mu}=0, \quad i=j,k,l,m. \tag{B12}$$

In that case $F(i\mathfrak{W})$ clearly vanishes and there remain three types of contributions to the sum over residues arising from terms with pairwise equal indices. The

terms with all four indices equal contribute

$$\delta_{J_jJ_k}\delta_{J_kJ_l}\delta_{J_lJ_m}\frac{4\pi}{D}\left\langle\frac{a_\mu^{(j)}a_\mu^{(k)*}a_\mu^{(l)}a_u^{(m)*}}{\Gamma_\mu^3}\right\rangle_\mu. \quad \text{(B13)}$$

A second contribution arises from the terms in which the indices arising from $F^{(j)}$ and $F^{(l)}$ coincide and the indices of $F^{(k)*}$ and $F^{(m)*}$ coincide but the two indices are different, from each other. This yields

$$-\delta_{J_jJ_l}\delta_{J_kJ_m}\times\frac{16\pi^2}{D^2}\left\langle\frac{a_\mu^{(j)}a_\mu^{(l)}a_\nu^{(k)*}a_\nu^{(m)*}}{(\Gamma_\mu+\Gamma_\nu)^2}\Phi_2\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D}\right)\right\rangle_{\mu\neq\nu}, \quad \text{(B14)}$$

where

$$\Phi_2\left(\frac{\Gamma}{D}\right)=-i\frac{D}{\pi}\Gamma^2\int_{-\infty}^{\infty}\frac{d\epsilon R_2(\epsilon)}{(\epsilon-i\Gamma)^3}, \quad \text{(B15)}$$

which vanishes identically for $R_2=D^{-1}$ when the $\mathcal{E}_\mu$ are uncorrelated and goes to zero for both very small and very large values of Γ/D in all circumstances. The functional form of Φ_2 for the case of Dyson's R_2 as given in Eq. (44) has been plotted in Fig. 1.

Finally, there are contributions from the poles of terms in which the indices arising from each of the unstarred factors coincide with the indices of one of the starred factors. These contribute to the average

$$(2\pi/W)\sum(\delta_{J_jJ_k}\delta_{J_lJ_m}a_\mu^{(j)}a_\mu^{(k)*}a_\nu^{(l)}a_\nu^{(m)*}+\delta_{J_jJ_m}\delta_{J_lJ_k}a_\mu^{(j)}a_\mu^{(m)*}a_\nu^{(l)}a_\nu^{(k)*})\{[\Gamma_\mu^{(j)}[\mathcal{E}_\mu^{(j)}-\mathcal{E}_\nu^{(l)}+\tfrac{1}{2}i(\Gamma_\mu^{(j)}+\Gamma_\nu^{(l)})][\mathcal{E}_\mu^{(j)}-\mathcal{E}_\nu^{(l)}+\tfrac{1}{2}i(\Gamma_\mu^{(j)}-\Gamma_\nu^{(l)})]]^{-1}+[\Gamma_\nu^{(l)}[\mathcal{E}_\nu^{(l)}-\mathcal{E}_\mu^{(j)}+\tfrac{1}{2}i(\Gamma_\nu^{(l)}+\Gamma_\mu^{(j)})][\mathcal{E}_\nu^{(l)}-\mathcal{E}_\mu^{(j)}+\tfrac{1}{2}i(\Gamma_\nu^{(l)}-\Gamma_\mu^{(j)})]]^{-1}\} \quad \text{(B16)}$$

where the sum is over all $\mu\neq\nu$ in W. In the limit of large W the expression (B16) is easily evaluated with the help of the antisymmetry property $\Phi_0(-x)=-\Phi_0(x)$ and yields

$$\frac{2\pi^2}{D_jD_l}\left\langle\frac{a_\mu^{(j)}a_\nu^{(l)}(\delta_{J_jJ_k}\delta_{J_lJ_m}a_\mu^{(k)*}a_\nu^{(m)*}+\delta_{J_jJ_m}\delta_{J_lJ_k}a_\mu^{(m)*}a_\nu^{(k)*})}{\Gamma_\mu\Gamma_\nu}\left[\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D_j}\right)+\Phi_0\left(\frac{\Gamma_\mu+\Gamma_\nu}{2D_l}\right)\right]\right\rangle_{\mu\neq\nu}. \quad \text{(B17)}$$

The sum of (B13), (B14), and (B17) equals $\langle F^{(j)}F^{(k)*}F^{(l)}F^{(m)*}\rangle_{\text{av}}$ under the restricting conditions (B12).