CONCLUSIONS

It appears that consistent quantitative agreement between the experimental angular distributions of (d,p) reactions with medium light nuclei and the DWBA predictions for such reactions is unlikely. The difficulty seems to arise from the contribution to the experimental differential cross section from compound-nucleus formation. The angular distributions of this compound-nucleus component are fluctuating and asymetric and hence difficult to disentangle from the stripping distribution. At the low energies employed in this experiment, the direct interaction mechanism accounts for only approximately one-half of the observed cross section. The magnitudes of the CN effects are such that various anomalies observed in (d,p) reactions at considerably higher energies can probably be explained as large CN contributions to the stripping distribution.[31,35]

With observations at a number of bombarding energies, individual experimental angular distributions may be obtained which apparently are relatively slightly affected by CN contamination. These experimental distributions can be fairly well approximated by DWBA predictions in a consistent manner. Multiple observations are especially necessary at low bombarding energies where the stripping distribution is not so sharply peaked. With these precautions, DWBA analysis of low-energy (d,p) distributions would appear to yield valid spectroscopic data. Detailed confirmation of the proper values of the optical-model parameters remains, however, a difficult and uncertain problem because of the residual CN effects. An uncritical application of plane-wave Born-approximation analysis to (d,p) reactions initiated with low-energy deuterons can yield misleading information concerning l_n.

There is some evidence that the rapid variations in cross sections and angular distributions observed in this experiment can be explained as Ericson-type fluctuations in the compound-nucleus component of the reactions. The role of interference between the compound nucleus and direct modes of interaction is difficult to assess, however, and may account for a considerable portion of the variation.

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Approximation for the Phase Shifts Produced by Repulsive Potentials Strongly Singular in the Origin

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An approximate expression for the partial-wave scattering phase shifts produced by a potential strongly divergent at the origin is introduced. This approximation is applicable over the whole energy range; it yields the correct threshold behavior of the phase shifts and becomes exact at high energy. The high-energy behavior of the phase shifts δ_l is ascertained; we find, as k diverges (l fixed), $\delta_l=-Ak^{1-2/m}$, where m characterizes the behavior of the potential at the origin, $V(r)\sim r^{-m}$, $m>2$, and A is a constant independent of l which is evaluated explicitly. Numerical tests of the accuracy of the approximation are given, including comparisons with the WKB approximation.

I. INTRODUCTION

In this paper we introduce and discuss an approximate expression for the scattering phase shifts produced by a repulsive potential strongly divergent in the origin. We concentrate our attention on S-wave scattering; the generalization to all partial waves is given at the end of the paper.

Singular potentials are important in phenomenological treatments of nuclear and atomic interactions. A second motivation for discussing scattering on singular potentials lies in the connection of this problem with that of the divergences in (renormalizable or unrenormalizable) field theories.

It is well known that for strongly divergent potentials scattering theory makes sense only if the divergent core is repulsive, because only in this case may the normalization of the wave function in the origin be maintained. This implies that the scattering amplitude will not depend in an analytic way upon the strength of the interaction. Thus, the Born approximation, being the first term in a power expansion in the strength of the potential, fails to converge; nor is Fredholm theory applicable. This breakdown of the traditional perturbative approaches is in contrast with the simplicity of the physical situation, in which the effect of the repulsive core is simply to reflect the scattering particles, forbidding their penetration in the inner region. Our treatment is based on the phase approach to scattering theory[1]; in fact, this approach appears naturally appropriate in this case, because it deals only with the physically relevant quantity—the phase shift—rather than the wave function (whose detailed behavior in the core region, while physically irrelevant, is in fact responsible for the mathematical difficulties).

An approximate expression for the scattering phase shift which retains its validity for singular potentials is the WKB approximation $\delta_{\rm WKB}$. This approximation, however, because of its semiclassical nature, is only valid provided the wavelength associated with the scattering particles is small as compared with the radius of the repulsive core; thus it breaks down completely at low energy. The approximation we introduce maintains, instead, its validity over the whole energy range, and it yields the correct threshold behavior of the phase shifts; it is, however, more reliable at higher energies, and it becomes exact as the energy diverges.

In the following section, the approximate expression $\delta_{\rm app}$ for the S-wave phase shift is derived. In Sec. III, the high-energy limit of the phase shift is ascertained. In Sec. IV, we consider the low-energy limit and obtain an approximate expression for the scattering length. This expression is compared with the exact expression, for one class of potentials whose scattering length may be evaluated exactly. In Sec. V, a numerical comparison is made of $\delta_{\rm app}$ and $\delta_{\rm WKB}$ with the exact phase shift for the class of potentials

$$V(r)=r^{-m}e^{-\mu r}. \tag{1.1}$$

In Sec. VI, we introduce the generalization to all partial waves. Section VII contains some concluding remarks. Throughout this paper, units are chosen so that $\hbar=2M=1$, M being the mass of the scattering particles.

II. INTRODUCTION OF THE APPROXIMATION

The S-wave scattering phase shift δ may be obtained from the phase equation[1]

$$\delta'(r)=-k^{-1}V(r)\sin^2[kr+\delta(r)],\qquad \delta(0)=0,\quad \delta=\delta(\infty). \tag{2.1a}$$

Here $V(r)$ is the potential, and k is the momentum of the scattering particles. The phase function $\delta(r)$ measures the phase accumulated by the potential up to the distance r; specifically, $\delta(r')$ is the scattering phase shift produced by the potential $V(r)\theta(r'-r)$, i.e., the actual potential amputated of its part extending beyond r'. Here and in the following, $\theta(x)$ is the step function, $\theta(x)=0$ for $r<0$, $\theta(x)=1$ for $x>0$.

[1] F. Calogero, Nuovo Cimento **27**, 261 (1963).

The potential $V(r)$ is assumed to be positive definite; the extension of the results to the case when $V(r)$ is not positive definite is mentioned in Sec. VII. To be definite, we also assume the potential to be of the form

$$V(r)=r_0^{-2}(r_0/r)^mC(r),\qquad m>2, \tag{2.2a}$$

with

$$C(0)=1. \tag{2.2b}$$

It will be obvious that certain of our results—in particular, the approximate expressions for the phase shifts—apply also to potentials which diverge in the origin more strongly than any inverse power of r. The potential is also required to vanish at infinity, as usual in scattering theory.

To arrive at an approximate expression for the phase shift, we integrate Eq. (2.1a) and obtain for the phase shift

$$\delta=-k^{-1}\int_0^\infty dr V(r)\sin^2[kr+\delta(r)]. \tag{2.1b}$$

We now propose to substitute an approximate determination of the phase function $\delta(r)$ in the integrand in the right-hand side of this equation. Notice that the substitution of the zeroth-order approximation $\delta(r)=0$ would yield the Born approximation for δ (which, however, diverges for the class of potentials we are considering). Because the potential is very large close to the origin, it is from this region that the major contribution to the phase shift originates; therefore, it is in this region that the phase function must be approximated more carefully. Thus, we investigate the behavior of the phase function near the origin, and from Eq. (2.1a) we obtain

$$\delta(r)=-kr+k[V(r)]^{-1/2}+\tfrac{1}{4}kV'(r)V^{-2}(r)+O[(r/r_0)^{\frac{1}{2}(3m-4)}]. \tag{2.3}$$

The first term in the right-hand side corresponds to the phase shift produced by a hard sphere of radius r; the second and third terms behave, respectively, as $r^{m/2}$ and r^{m-1}.

We might then try to substitute in place of $\delta(r)$ in the right-hand side of Eq. (2.1b) the first two terms of this expansion. In this way, however, while accurately approximating the behavior of the phase function near the origin, we would approximate it very badly at large r (where the phase function should become asymptotically constant rather than diverge). When the energy is large, this would be acceptable; in fact, as will be shown in the next section, in this manner one may obtain the correct high-energy behavior of the phase shift. To obtain an expression valid at all energies, it is necessary to manufacture an approximate expression for $\delta(r)$ which, while reproducing the first two terms of Eq. (2.3) at small r, also yields a reasonable result at large r. One such expression, which reduces to the zeroth-order result $\delta(r)=0$ at large r, is

$$-kr+kr\{r[V(r)]^{1/2}+1\}^{-1}. \tag{2.4}$$

This expression, while relatively simple, has the additional feature of yielding a result, correct up to a constant factor, also for the third term in the expansion Eq. (2.3). Inserting this expression in Eq. (2.1b), we obtain the following approximate expression for the scattering phase shift:

$$\delta_{\text{app}}=-k^{-1}\int_0^\infty drV(r)\sin^2\{kr(r[V(r)]^{1/2}+1)^{-1}\}. \quad (2.5)$$

This may be viewed as a modified Born approximation, which takes into account the suppression of the wave function in the interior region. Note that this expression is real only for repulsive potentials, and that the nonanalyticity in the strength of the interaction appears as a built-in feature of this approximation.

As implied by Eqs. (2.3) and (2.4), the approximation Eq. (2.5) should be expected to be accurate only for $m\gg 2$ and for potentials which vanish fast at large distances. It is anyway amusing to apply it to the centrifugal potential $V(r)=l(l+1)r^{-2}$, in which case it yields $\delta_{\text{app}}=-\frac{1}{2}\pi\{l(l+1)/([l(l+1)]^{1/2}+1)\}$. The correct result is of course $-\frac{1}{2}\pi l$. The Born approximation value is $-\frac{1}{2}\pi l(l+1)$, and for large values of l is much worse than the one we obtain.

In the case of a hard sphere, $V(r)=\infty$ for $r<R$, $V(r)=0$ for $r>R$, Eq. (2.5) yields the correct result $\delta=-kR$.

III. HIGH-ENERGY BEHAVIOR OF THE PHASE SHIFT

In the preceding section, we have given an expansion Eq. (2.3) for the phase function in the neighborhood of the origin. But at high energy, it is just the behavior at the origin which determines the scattering phase shift. Thus, by substituting in the right-hand side of Eq. (2.1b) the first terms of the expansion Eq. (2.3), we get an asymptotic expansion for the phase shift at high energy. In this manner, we establish the high-energy behavior of the phase shift; obviously the dominant term coincides with that yielded by the approximate expression Eq. (2.5).

By inserting the first three terms in the right-hand side of Eq. (2.3) into Eq. (2.1b), we have

$$\delta=-k^{-1}\int_0^\infty drV(r)\sin^2\{k[[V(r)]^{-1/2}+\tfrac{1}{4}V'(r)V^{-2}(r)]\}. \quad (3.1)$$

The limit of this expression as k diverges is readily evaluated (see Appendix I). We find for the first two terms

$$\delta=-A(kr_0)^{1-(2/m)}[1+b(kr_0)^{-2/m}], \quad (3.2)$$

with

$$A=[2^{1-(2/m)}/(2-(2/m))!][(\pi/m)/\sin(\pi/m)], \quad (3.3)$$

$$b=\frac{1}{m}[r_0C'(0)]2^{-2/m}\left(2-\frac{2}{m}\right)!\left[\left(2-\frac{4}{m}\right)!\cos\left(\frac{\pi}{m}\right)\right]^{-1}. \quad (3.4)$$

This expression is valid for $m>4$. For $2<m\leq 4$ the second term in the right-hand side in Eq. (3.2) must be modified, as specified in Appendix I. The dominant term is the same.

Notice that the magnitude of the phase shift increases with energy as a result of the singularity of the potential; but it can never grow faster than linearly in k. This is consistent with the limitation implied by Wigner's theorem.[2]

It should also be mentioned that, while in this paper we have restricted out consideration to potentials singular in the origin as an inverse power of r, this same treatment could also be applied to more singular cases, for instance, to potentials behaving in the origin as $\exp(\alpha/r)$, $\alpha>0$. In such a case, the dominant term would behave asymptotically as $k(\ln k)^{-\beta}$, with $\beta>0$.

IV. APPROXIMATION FOR THE SCATTERING LENGTH

In the zero-energy limit, we get from Eq. (2.5) an approximate expression for the scattering length a, defined by

$$\lim_{k\to 0}[-\delta/k]=a. \quad (4.1)$$

This expression is

$$a_{\text{app}}=\int_0^\infty drr^2V(r)\{r[V(r)]^{1/2}+1\}^{-2}. \quad (4.2)$$

This may be integrated for the class of potentials

$$V(r)=r_0^{-2}(r_0/r)^m,\quad r<R,\qquad =0,\quad r>R, \quad (4.3)$$

and we find[3]

$$a_{\text{app}}=RF[2,2p;2p+1;-(R/r_0)^{1/(2p)}]. \quad (4.4)$$

Here $F(a,b;c;x)$ is the hypergeometric function,[4] and

$$p=(m-2)^{-1}. \quad (4.5)$$

It is convenient to rewrite this expression using Eq. (2.10.2) of Ref. 4. We then find

$$a_{\text{app}}=r_0(\pi p/\sin\pi p)(1-2p)/\cos\pi p-R[(r_0/R)^{1/p}/(m-3)]\times F[2,2-2p;3-2p;-(r_0/R)^{1/(2p)}]. \quad (4.6)$$

In the case $m=4$ the hypergeometric function in (4.6) is readily evaluated and we find

$$a_{\text{app}}=r_0R/(r_0+R),\quad (m=4). \quad (4.7)$$

In the limit of large m, we obtain:

$$a_{\text{app}}=R \text{ if } R\leq r_0,\quad (m\to\infty) \quad (4.8a)$$

$$a_{\text{app}}=r_0 \text{ if } R\geq r_0,\quad (m\to\infty). \quad (4.8b)$$

[2] E. P. Wigner, Phys. Rev. 98, 145 (1955).
[3] *Tables of Integral Transforms*, edited by E. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.
[4] *Higher Transcendental Functions*, edited by E. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.

TABLE I. A comparison of the approximate and exact values of the scattering length for the potential $V(r)=r_0^{-2}(r_0/r)^m$, for various values of m. The quantities in the table are the scattering lengths divided by r_0; they are computed from Eqs. (4.15) and (4.9).

m	a_{app}/r_0	a/r_0	m	a_{app}/r_0	a/r_0
4	1	1	7	0.793	0.666
4.5	0.855	0.806	8	0.806	0.670
5	0.806	0.729	10	0.833	0.688
6	0.785	0.676	20	0.907	0.773

The second term in the right-hand side of Eq. (4.6) vanishes in the long-range case $R=\infty$,[5] so that in this case we have simply

$$a_{\text{app}}=r_0(\pi p/\sin\pi p)(1-2p)/\cos\pi p\,, \quad (R=\infty)\,. \tag{4.9}$$

These results should be compared with the exact values of the scattering length, which may be obtained from the zero-energy radial wave function for $r<R$

$$u(r)=r^{1/2}K_p[2p(r_0/r)^{1/(2p)}] \tag{4.10}$$

and the matching condition

$$a=R-u(R)/u'(R)\,. \tag{4.11}$$

Here $K_p(x)$ is the modified Bessel function of the third kind.[6]

Again, this simplifies in the case $m=4$, in which case we find

$$a=a_{\text{app}}=r_0R/(R+r_0)\,, \quad (m=4)\,. \tag{4.12}$$

We thus see that the present approximation yields the exact value of the scattering length, for the class of potentials

$$V(r)=r_0^{-2}(r_0/r)^4\,, \quad r<R\,, \qquad =0\,, \quad r>R\,, \tag{4.13}$$

independently from the value of the ratio R/r_0.

In the limit of large m we find, after some algebra

$$a=R \quad \text{if} \quad R\leq r_0\,, \quad (m\to\infty)\,, \tag{4.14a}$$

$$a=r_0 \quad \text{if} \quad R\geq r_0\,, \quad (m\to\infty)\,, \tag{4.14b}$$

which again agrees with the exact result, Eqs. (4.8).

In the long-range case, $R=\infty$, we have

$$a=r_0(\pi p/\sin\pi p)(p^p/p!)^2\,, \quad (R=\infty)\,. \tag{4.15}$$

In Table I, we give a numerical comparison of the approximate and exact expressions for the scattering length in the long-range case [Eqs. (4.9) and (4.15)]. Note that a_{app} and a have in this case the same qualitative behavior as functions of m, with a minimum between $m=6$ and $m=7$. Although the two expressions coincide in the limit of large m, the convergence is slow, as implied by the asymptotic expressions [derived from Eqs. (4.9) and (4.15)]:

$$a_{\text{app}}=r_0[1-2p+O(p^2)]\,, \quad (R=\infty)\,, \tag{4.16a}$$

$$a=r_0[1-2p(-\ln p-0.56)+O((p\ln p)^2)]\,, \quad (R=\infty)\,. \tag{4.16b}$$

V. COMPARISON OF THE APPROXIMATE PHASE SHIFTS WITH THE EXACT VALUES

In the general case, Eq. (2.5) must be evaluated numerically, and the exact phase shift must be obtained by numerical integration of Eq. (2.1a) or of the radial Schrödinger equation.[7] This has been done for the class of potentials

$$V(r)=r^{-m}\exp(-\mu r)\,, \tag{5.1}$$

for various values of the parameters m and μ and of the energy of the scattering particles. Note that these potentials correspond to those of Eq. (2.2), with $r_0=1$; this establishes the unit of length. For the purpose of comparison, we have also computed the value of the phase shift obtained in the WKB approximation, namely,[8]

$$\delta_{\text{WKB}}=\int_0^\infty dr\{[k^2-(2r)^{-2}-V(r)]^{1/2} \times\theta[k^2-(2r)^{-2}-V(r)]-[k^2-(2r)^{-2}]^{1/2} \times\theta[k^2-(2r)^{-2}]\}\,. \tag{5.2}$$

The values of δ_{WKB} and δ_{app}, Eq. (2.5), together with the exact values δ, are collected in Table II (actually for convenience of writing the phase shifts have been multiplied by the scale factor $-k^{-1}$).

From these data, the qualitative energy dependence of the phase shifts is manifest: they are proportional to k at low energy, and they grow with k slower than linearly at high energy. Note that the linear region extends to higher energies for more singular potentials. The validity of the determination of the high-energy behavior of the phase shifts given in Sec. III is borne out by a comparison of the data in the last two columns of Table II. The increase of the discrepancy with m and μ is also consistent with Eq. (3.3).

As regards the reliability of the present approximation, we observe that it appears to be more accurate at higher energies, in the sense that the relative error becomes smaller as the energy increases. However, because the phase shifts grow with energy, the absolute error in the determination of the phase shifts is actually

[5] We also require $m>3$; otherwise the asymptotic vanishing of the potential is too slow to allow the definition of a scattering length.

[6] *Higher Transcendental Functions*, edited by E. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II.

[7] Actually, it is convenient to use the radial Schrödinger equation rather than the phase equation (2.1a), because this equation, in the case of strongly singular potentials, is very unstable against errors introduced during the integration. The instability is caused by the large cancellation occurring in the argument of the sine function in the inner region, where the potential is very large.

[8] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 1101.

larger at intermediate energies than at low energies. But it never exceeds 30°, even when the phase shifts are very large. The present approximation almost invariably gives a determination for the absolute value of the phase shift, which is in excess of the exact value. The opposite happens for the WKB approximation.

Finally, we draw attention to the fact that the data in Table II span an energy variation of a factor 10^8, and that the corresponding phase shifts vary by factors considerably larger than 10^3 and attain very large values.

VI. GENERALIZATION TO ALL PARTIAL WAVES

One way to treat higher partial waves is by using the present approximation in order to evaluate also the phase accumulated by the centrifugal term. This leads to the approximate expression for the scattering phase shifts

$$\delta_{l,\mathrm{ap}}=\tfrac{1}{2}l\pi-k^{-1}\int_0^\infty dr W(r) \times\sin^2\{kr[r[W(r)]^{1/2}+1]^{-1}\}, \quad (6.1)$$

with

$$W(r)=V(r)+l(l+1)r^{-2}. \quad (6.2)$$

Because one is treating approximately the centrifugal potential, which has a long range and accumulates a large phase shift, this approximation should not be expected to be accurate. In particular, it will be incorrect at low energies, where it fails to reproduce the correct behavior of the phase shifts,[9] namely,

$$\delta_l \underset{k\to 0}{\longrightarrow} -a_l k^{2l+1}. \quad (6.3)$$

A better approximation is obtained, starting from the phase equation which expresses the accumulation of the phase shifts due only to the potential[1]

$$\delta_l'(r)=-k^{-1}V(r)\hat{D}_l^2(kr)\sin^2[\hat{\delta}_l(kr)+\delta_l(r)],\quad \delta_l(0)=0,\quad \delta_l=\delta_l(\infty). \quad (6.4)$$

The functions $\hat{D}_l(x)$ and $\hat{\delta}_l(x)$ are the "amplitude" and "phase" of the Riccati-Bessel functions:

$$\hat{D}_l^2(x)=[\hat{\jmath}_l^2(x)+\hat{n}_l^2(x)], \quad (6.5a)$$

$$\hat{\delta}_l(x)=-\arctan[\hat{\jmath}_l(x)/\hat{n}_l(x)],\quad (\mathrm{mod}\,\pi). \quad (6.5b)$$

The Riccati-Bessel functions are the (regular and irregular) solutions of the radial Schrödinger equation in the absence of potential; they are, respectively, the spherical Bessel and Neumann functions multiplied by x. A list of properties of the functions $\hat{D}_l(x)$ and $\hat{\delta}_l(x)$ is given in Ref. 1. In addition to those properties, we have

$$\hat{\delta}_l'(x)=\hat{D}_l^{-2}(x), \quad (6.5c)$$

which follows from the Wronskian relation for the Riccati-Bessel functions. Thus, $\hat{\delta}_l(x)$ may also be defined as

$$\hat{\delta}_l(x)=\int_0^x dx'\hat{D}_l^{-2}(x'). \quad (6.5d)$$

This definition is free from the mod(π) ambiguity of Eq. (6.5b). An explicit expression for $\hat{D}_l^2(x)$ is the following:

$$\hat{D}_l^2(x)=\{\sum_{n=0}^{[l/2]}(l+2n)![(l-2n)!(2n)!]^{-1}(-2x)^{-2n}\}^2 +\{\sum_{n=0}^{[(l-1)/2]}(l+2n+1)![(l-2n-1)! \times(2n+1)!]^{-1}(-2x)^{-2n-1}\}^2. \quad (6.5e)$$

[9] For potentials which vanish asymptotically less rapidly than any exponential, the partial-wave scattering amplitude generally has a branch point at $k=0$. However, if the potential behaves asymptotically as r^{-m}, it is still true, for all the partial waves with $l<(m-3)/2$, that the behavior of the phase shifts at low energy is dominated by a term of the form Eq. (6.3).

TABLE II. A comparison of the exact and approximate values of the S-wave phase shift for the potential $V(r)=r^{-m}\exp(-\mu r)$, for various values of m, μ, and the momentum k. The quantities in the table are the phase shifts divided by $-k$. The exact phase shift δ is obtained numerically solving the radial Schrödinger equation; the approximate phase shifts δ_{app} and δ_{WKB} are computed from Eqs. (2.5) and (5.2), respectively. The symbol ~ 0 stands for all values smaller than 10^{-3}. The values in the last column are obtained from the asymptotic expression for the phase shift, Eqs. (I.16); the contribution of the two terms is explicitly displayed.

	$V(r)=r^{-m}\exp(-\mu r)$						
k	0.01	0.1	1	10	100	100	
$-\delta_{\mathrm{app}}/k$	0.987	0.922	0.654	0.3048	0.1106	0.1182	
$-\delta/k$	0.976	0.921	0.657	0.3088	0.1101	−0.0079	$m=4$, $\mu=0$
$-\delta_{\mathrm{WKB}}/k$	0.029	0.276	0.574	0.3055	0.1121	=0.1103	
$-\delta_{\mathrm{app}}/k$	0.516	0.516	0.477	0.2746	0.1070	0.118	
$-\delta/k$	0.486	0.486	0.459	0.2763	0.1085	−0.012	$m=4$, $\mu=1$
$-\delta_{\mathrm{WKB}}/k$	~ 0	0.001	0.342	0.2723	0.1084	=0.106	
$-\delta_{\mathrm{app}}/k$	0.191	0.191	0.191	0.1631	0.0862	0.118	
$-\delta/k$	0.169	0.169	0.169	0.1544	0.0855	−0.047	$m=4$, $\mu=10$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.009	0.1458	0.0860	=0.071	
$-\delta_{\mathrm{app}}/k$	0.806	0.806	0.779	0.566	0.3354	0.344	
$-\delta/k$	0.670	0.669	0.661	0.540	0.3355	−0.000	$m=8$, $\mu=0$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.447	0.535	0.3357	=0.344	
$-\delta_{\mathrm{app}}/k$	0.701	0.701	0.687	0.526	0.3217	0.344	
$-\delta/k$	0.582	0.582	0.579	0.498	0.3217	−0.015	$m=8$, $\mu=1$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.338	0.492	0.3217	=0.329	
$-\delta_{\mathrm{app}}/k$	0.391	0.391	0.390	0.348	0.2445	0.344	
$-\delta/k$	0.327	0.327	0.327	0.316	0.2428	−0.148	$m=8$, $\mu=10$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.038	0.304	0.2426	−0.196	
$-\delta_{\mathrm{app}}/k$	0.855	0.855	0.843	0.687	0.4825	0.492	
$-\delta/k$	0.709	0.709	0.767	0.649	0.4819	−0.000	$m=12$, $\mu=0$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.410	0.642	0.4819	=0.492	
$-\delta_{\mathrm{app}}/k$	0.786	0.786	0.777	0.649	0.4639	0.492	
$-\delta/k$	0.654	0.654	0.653	0.610	0.4631	−0.020	$m=12$, $\mu=1$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.339	0.603	0.4630	=0.472	
$-\delta_{\mathrm{app}}/k$	0.504	0.504	0.503	0.458	0.3568	0.492	
$-\delta/k$	0.427	0.427	0.427	0.420	0.3542	−0.200	$m=12$, $\mu=10$
$-\delta_{\mathrm{WKB}}/k$	~ 0	~ 0	0.073	0.405	0.3540	=0.292	

From the phase equation (6.4) and under the assumption that the potential is singular in the origin at least as implied by Eq. (2.2), we find that the first two dominant terms in the behavior of $\delta_l(r)$ at small values of r are

$$\delta_l(r)=-\hat{\delta}_l(kr)+k\hat{\delta}_l'(kr)[V(r)]^{-1/2}. \quad (6.6)$$

To obtain this equation we have also used Eq. (6.5c).

Proceeding in close analogy with the development of Sec. II we are thus led to the following approximate expression for the scattering phase shifts:

$$\delta_{l,\text{app}}=-k^{-1}\int_0^\infty drV(r)\hat{D}_l^2(kr)\sin^2\{\hat{\delta}_l(kr) \times[(\hat{\delta}_l(kr)/k)\hat{D}_l^2(kr)(V(r))^{1/2}+1]^{-1}\}. \quad (6.7)$$

As before, this may be viewed as a modified Born approximation, which takes into account the suppression of the wave function in the inner region. Again it yields the correct phase shifts, $\delta=-\hat{\delta}_l(kR)$, for a hard sphere of radius R.

In the low-energy region, this expression yields the correct behavior Eq. (6.3) for the phase shifts, with the following expression for the quantities a_l:

$$a_{l,\text{app}}=[(2l-1)!!]^{-2}\int_0^\infty drr^{2l+2}V(r) \times[r(V(r))^{1/2}+2l+1]^{-2}. \quad (6.8)$$

This expression may be integrated for the class of potentials Eq. (4.3), and we find[3]

$$a_{l,\text{app}}=R^{2l+1}[(2l+1)!!(2l-1)!!]^{-1} \times F[2,2q;2q+1;-(2l+1)(R/r_0)^{1/(2p)}], \quad (6.9)$$

with

$$q=(2l+1)p \quad (6.10)$$

and p defined in Eq. (4.5). As in Sec. IV, we rewrite this expression as follows

$$a_{l,\text{app}}=r_0^{2l+1}[(2l+1)!!(2l-1)!!(2l+1)^{2q}]^{-1}(\pi q/\sin\pi q) \times(1-2q)/\cos\pi q-R^{2l+1}[(2l+1)!!]^{-2} \times[(r_0/R)^{1/p}/(m-2l-3)]F[2,2-2q;3-2q; -(2l+1)^{-1}(r_0/R)^{1/(2p)}]. \quad (6.11)$$

The first term on the right-hand side is now recognized, as the value of $a_{l,\text{app}}$ for the long-range case $R=\infty$, in analogy to the S-wave case. We also have, in the limit of large m

$$a_{l,\text{app}}=R^{2l+1} \quad \text{if} \quad R\leq r_0, \quad (m\to\infty), \quad (6.12\text{a})$$

$$a_{l,\text{app}}=r_0^{2l+1} \quad \text{if} \quad R\geq r_0, \quad (m\to\infty). \quad (6.12\text{b})$$

In this case, we have also the limit of large l to consider. Applying the transformation of Eq. 2.9(3) of Ref. 4 to the hypergeometric function in Eq. (6.9), we easily obtain

$$a_{l,\text{app}}=R^{2l+1}[(2l+1)!!]^{-2}(r_0/R)^{1/p}(2l)^{-1} \times[1+O(1/l)]. \quad (6.13)$$

The exact expressions a_l may also be obtained explicitly, from the zero-energy radial wave function

$$u_l(r)=r^{1/2}K_q[2p(r_0/r)^{1/(2p)}] \quad (6.14)$$

and the matching condition

$$a_l=[(2l+1)!!]^{-2}(2l+1)R^{2l+1} \times(1-(l+1)\alpha_l)/(1+l\alpha_l), \quad (6.15)$$

where we have set

$$\alpha_l=u_l(R)/[Ru_l'(R)]. \quad (6.16)$$

In the long-range limit $R=\infty$, Eq. (6.15) becomes

$$a_l=[(2l+1)!!(2l-1)!(2l+1)^{2q}]^{-1} \times r_0^{2l+1}(\pi q/\sin\pi q)(q^q/q!)^2. \quad (6.17)$$

This expression is similar to the expression we had in the S-wave case, Eq. (4.15). The expression for $a_{l,\text{app}}$ in the case $R=\infty$ [see Eq. (6.11)] is also very similar to the corresponding equation for the S-wave case, Eq. (4.9). Thus, the numerical comparison of Table I applies also in the present case with appropriate changes. We emphasize that a_l may be defined, in the long-range case, only provided $l<(m-3)/2$ (see footnote 9).

It is also easily seen that in the limit of large m the exact values a_l coincide with the approximate ones, Eq. (6.12), in analogy to the S-wave case.

In the limit of large l, we find with some algebra that

$$a_l=[(2l+1)!!]^{-2}R^{2l+1}(r_0/R)^{1/p}(2l)^{-1}[1+O(1/l)]. \quad (6.18)$$

This coincides with the corresponding equation for $a_{l,\text{app}}$, Eq. (6.13). We thus see that even in the limit of large l and small k the present approximation yields the correct behavior. Notice that this situation is the most unfavorable from the point of view of the present approximation, because when the angular momentum is large and the energy is small, it is the outer part of the potential which plays the major role in determining the phase shifts. The success of the present approximation at large l may be understood remembering its similarity with the Born approximation.

Finally, the asymptotic behavior in energy of all partial-wave phase shifts may be discussed along the same lines as it was done in Sec. III for the S-wave phase shift. It is thus found that the exact dominant term for the high-energy behavior is yielded by the approximate expression for the phase shifts, Eq. (5.7), and that it is the same for all partial waves:

$$\delta_l\sim -Ak^{1-(2/m)}, \quad (6.19)$$

with A given in Eq. (3.3). A proof of this statement is given in Appendix II. We notice that this result is not surprising, once it is recognized that the high-energy behavior of the phase shifts is determined by the behavior of the potential in the origin, because the potentials we consider are more singular than the centrifugal term and thus dominate it completely.

It should be emphasized that the above conclusion refers to the asymptotic behavior in k of the phase shifts δ_l corresponding to a fixed finite value of l, but it does not imply anything about the behavior of the phase shifts as both k and l diverge. Thus, it would be incorrect to draw from Eq. (6.19) any conclusion concerning the angular dependence of the full scattering amplitude in the high-energy limit, because the partial-wave sum which defines the full scattering amplitude extends to infinity. Possibly the approximate expression Eq. (6.7) would also yield the correct asymptotic behavior as both k and l diverge, but this question will not be discussed in this paper.

VII. CONCLUSION

An approximation has been introduced for the evaluation of the scattering phase shifts produced by a repulsive potential strongly singular at the origin. The major advantage of this approximation is its validity over the whole energy range. In particular, it yields correctly the low-energy behavior and becomes exact at high energy.

A possible additional advantage of this approximation consists in the possibility of using this approach as the first step in a convergent iterative procedure.[10] This, however, may turn out to be of small practical value, because of the instability of the phase equation against errors in intermediate steps.[7]

In this paper, we have restricted our consideration to repulsive potentials. If one deals with a potential which is attractive in some region, the approximate expressions for the phase shifts, Eqs. (2.5) or (6.7), are inapplicable, because they are not real. It will, however, be obvious in every case how they should be modified; the simpler solution is to suppress the potential in the argument of the sine function whenever it becomes negative. Otherwise, by using techniques such as those described in Ref. 1, one may always reduce the problem to one in which only the repulsive part of the potential appears explicitly. One could then treat the rest of the potential exactly (or using some other approximation) and, proceeding as in the previous sections, one would obtain an approximate expression for the phase shifts due to the repulsive part of the potential in terms of the "amplitude" and "phase" of the radial wave functions apposite to the problem without repulsive contribution.

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APPENDIX I

In this Appendix, we evaluate the first terms in the large k behavior of the integral Eq. (3.1), with $V(r)$ given by Eqs. (2.2a) and (2.2b).

As will be clear below, the large k behavior is determined by the behavior of the integrand near the origin. Thus we expand the integrand around $r=0$, and we have

$$\delta=-(kr_0)^{-1}\int_0^\infty (dr/r_0)(r_0/r)^m(1+C'r) \times\sin^2\{kr_0[(r/r_0)^{m/2}(1+C'r)^{-1/2} -\tfrac{1}{4}m(r/r_0)^{m-1}]\}\,, \quad (\text{I.1})$$

where we write C' for $C'(0)$. Expanding again and keeping only the dominant terms, we find

$$\delta=-(kr_0)^{-1}\int_0^\infty (dr/r_0)(r_0/r)^m \times\sin^2\{kr_0(r/r_0)^{m/2}[1+\eta(r/r_0)^\epsilon]\} -(kr_0)^{-1}\int_0^\infty C'dr(r_0/r)^{m-1}\sin^2[kr_0(r/r_0)^{m/2}]. \quad (\text{I.2})$$

Here

$$\epsilon=1 \quad \text{for} \quad m\geq 4\,, \quad (\text{I.3a})$$

$$\epsilon=m/2-1 \quad \text{for} \quad 2<m<4\,, \quad (\text{I.3b})$$

and

$$\eta=-\tfrac{1}{2}r_0C' \quad \text{for} \quad m>4\,, \quad (\text{I.4a})$$

$$\eta=-\tfrac{1}{2}r_0C'-1 \quad \text{for} \quad m=4\,, \quad (\text{I.4b})$$

$$\eta=-(m/4) \quad \text{for} \quad 2<m<4. \quad (\text{I.4c})$$

Finally, expanding the sine function in the first term in the right-hand side of Eq. (I.2) and keeping only the dominant contributions, we obtain

$$\delta=-(kr_0)^{-1}I(m,m/2;kr_0)-\eta J((m/2)-\epsilon,m/2;2kr_0) -(kr_0)^{-1}(r_0C')I(m-1,m/2;kr_0)\,, \quad (\text{I.5})$$

with

$$I(s,t;z)=\int_0^\infty dx x^{-s}\sin^2 zx^t\,, \quad (\text{I.6a})$$

$$J(s,t;z)=\int_0^\infty dx x^{-s}\sin zx^t. \quad (\text{I.6b})$$

Now with the change of variable

$$y=zx^t\,, \quad (\text{I.7})$$

we find

$$I(s,t;z)=t^{-1}z^{(s-1)/t}I(s,t)\,, \quad (\text{I.8a})$$

$$J(s,t;z)=t^{-1}z^{(s-1)/t}J(s,t)\,, \quad (\text{I.8b})$$

[10] An investigation of the solution of the phase equation by successive approximations for nonsingular potentials is now in progress, in collaborations with M. Restignoli Tabet.

with

$$I(s,t)=\int_0^\infty dy y^{-1+(1-s)/t}\sin^2 y, \tag{I.9a}$$

$$J(s,t)=\int_0^\infty dy y^{-1+(1-s)/t}\sin y. \tag{I.9b}$$

These integrals are easily evaluated[11]:

$$I(s,t)=\pi 2^{-2+(s-1)/t}\{[(s-1)/t]! \times\sin[\pi(s-1)/(2t)]\}^{-1}, \tag{I.10a}$$

$$J(s,t)=\tfrac{1}{2}\pi\{[(s-1)/t]!\cos[\pi(s-1)/(2t)]\}^{-1}. \tag{I.10b}$$

Introducing these expressions in Eq. (I.5), we finally obtain

$$\delta=-A(kr_0)^{1-(2/m)}-B(kr_0)^{1-(4/m)} -\bar{B}(kr_0)^{1-(2/m)-(\epsilon/m)}, \tag{I.11}$$

with

$$A=[2^{1-(2/m)}/(2-(2/m))!](\pi/m)/\sin(\pi/m), \tag{I.12}$$

$$B=(r_0C'/2)[2^{1-(4/m)}/(2-(4/m))!] \times(2\pi/m)/\sin(2\pi/m), \tag{I.13}$$

$$\bar{B}=\eta\left[2^{(1+\epsilon)2/m}\left(1+\frac{(1+\epsilon)2}{m}\right)!\right]^{-1} \times(2\pi/m)\bigg/\sin\left(\frac{\pi}{m}(1+\epsilon)\right), \tag{I.14}$$

and ϵ, η defined in Eqs. (I.3), (I.4). Note that, for $m>4$

$$\bar{B}=-[1-(2/m)]B, \quad m>4, \tag{I.15a}$$

while for $m=4$

$$\bar{B}=-[\tfrac{1}{2}+1/(r_0C')]B, \quad m=4. \tag{I.15b}$$

We see that, while the dominant term in the asymptotic behavior has the same form in all cases, the next term has a different form depending upon whether the value of m is larger or smaller than 4. Thus, we have for the first two leading terms

$$\delta=-A(kr_0)^{1-(2/m)}-(2/m)B(kr_0)^{1-(4/m)}, \quad \text{for } m>4, \tag{I.16a}$$

$$\delta=-\tfrac{2}{3}\pi^{1/2}(kr_0)^{1/2}-\tfrac{1}{8}\pi(r_0C')+\tfrac{1}{4}\pi, \quad \text{for } m=4, \tag{I.16b}$$

$$\delta=-A(kr_0)^{1-(2/m)}-\bar{B}, \quad \text{for } 2<m<4. \tag{I.16c}$$

[11] Cf. Eqs. 6.5(15) and 6.5(1) of Ref. 3.

The equations given in the paper—Eqs. (3.2), (3.3), and (3.4)—are immediately derived from these equations in the case $m>4$.

APPENDIX II

In this Appendix, we show that the leading term in the large k behavior of all partial-wave phase shifts is the same. Because we are interested only in the leading term, we may keep only the first two terms for the behavior of the phase functions near the origin, Eq. (6.6). Thus, the problem is reduced to that of showing that the leading term in the large k behavior of the integral

$$-k^{-1}\int_0^\infty dr V(r)\hat{D}_l^2(kr)\sin^2[k\hat{D}_l^{-2}(kr)V^{-1/2}(r)] \tag{II.1}$$

is the same as that of the integral

$$-k^{-1}\int_0^\infty dr V(r)\sin^2[kV^{-1/2}(r)]. \tag{II.2}$$

Note that this expression is obtained from the previous one, substituting in place of $\hat{D}_l(x)$ its asymptotic value 1.

First, we divide the interval of integration into two parts, from zero to ρ and from ρ to infinity, with

$$\rho=r_0(kr_0)^{-2/m}. \tag{II.3}$$

It is then obvious that in the second integral, the asymptotic value may be substituted in place of $\hat{D}_l(kr)$, because the argument kr diverges at least as $(kr_0)^{1-(2/m)}$ when k diverges. As for the first integral, we may express the sine-square function by means of the corresponding power expansion and exchange the order of integration and sum. We get, thus, an expression of the form

$$\sum_{n=1}^\infty(-)^n[(2n)!]^{-1}(kr_0)^{(2n+m-2-mn)} \times\int_0^X dx x^{-m}\hat{D}_l^{2-4n}(x)x^{mn}, \tag{II.3}$$

with

$$X=(kr_0)^{1-(2/m)}. \tag{II.4}$$

Now to evaluate the leading term in the integral in the limit as k, or equivalently X, diverges, we may substitute the asymptotic value for $\hat{D}_l(x)$ as x diverges (note that the integral diverges as X diverges). But this shows that to get the leading term, all we have to do is substitute 1 in place of $\hat{D}_l(x)$ in Eq. (II.1), thus returning to the S-wave case, Eq. (II.2); Q.E.D.