PHYSICAL REVIEW VOLUME 135, NUMBER 3B 10 AUGUST 1964

New Symmetry Group for Elementary Particles. I. Generalization of Lorentz Group Via Electrodynamics*†‡

BEHRAM KURŞUNOĞLU
Department of Physics, University of Miami, Coral Gables, Florida

(Received 22 January 1964)

By using the definition of the photon angular momentum a connection between the Lorentz group and the unitary symmetry group of the strong interaction is established. The new group (to be called ILU_4) is a twenty-parameter group containing SU_3 and the inhomogeneous Lorentz group as its subgroups. The space-time and internal symmetries of dynamical systems may be described by a single symmetry group. The new symmetry group imparts a unitary content to every Lorentz frame of reference.

I. INTRODUCTION

THE role of symmetry principles in describing elementary particle events has gained great impetus from the recent generalizations of the isotopic spin concept for strong interactions. The models used in introducing larger groups are either Sakata's symmetrical theory[1] of strong interactions or the so-called "Eightfold Way" proposed by Gell-Mann[2] and also by Ne'eman.[3] The invariance of the pion-nucleon interaction under unitary spin (or SU_3) transformations has replaced charge independence or invariance under isotopic spin (or SU_2) transformations of the forces between nucleons. The experimental facts demonstrated the conservation of two quantum numbers, hypercharge, and isotopic spin in place of the isotopic spin conservation alone in Kemmer's symmetrical theory.[4] Despite the great success of the "Eightfold Way" it suffers from the same drawback as its predecessor the isotopic spin invariance of strong interactions. In the first place, full unitary symmetry requires a mass degeneracy, mass differences lead to symmetry-breaking interactions. Furthermore, the introduction of fictitious spaces like isotopic spin space or unitary space as distinct from the space-time structure of elementary events has long been recognized to be quite unsatisfactory for further progress towards a real understanding of the dynamical principles underlying elementary particle interactions. The group that describes the isotopic spin symmetry is a continuous subgroup of the full unitary group in two dimensions. It is denoted by SU_2. The only part of the isotopic spin group that correlates with physical facts corresponds to rotations by 180° and 360°. In this sense "observed" isotopic spin symmetry is isomorphic to the permutation group rather than to the three-dimensional rotation group. A further extension of the charge-independence hypothesis (i.e., hypercharge and charge independence), requiring greater symmetry beyond that contained in the isotopic spin group, needs the introduction of a larger group. The symmetry group in question is the three-dimensional unitary unimodular group, SU_3. In this case also only discrete SU_3 operations are relevant for elementary particle events. Therefore the observed part of SU_3 is also isomorphic to a permutation group. In the existing formalism there is no particular selection rule excluding continuous operations in unitary spin space. The choice of the observed part of the group is made as a result of comparing the facts and the mathematical formalism. In short, the group SU_3 is not like the inhomogeneous Lorentz group in which all Lorentz transformations are physically acceptable.

In the following we propose a generalization of the inhomogeneous Lorentz group to describe all symmetry properties of all elementary particles as embedded in only one symmetry group, ILU_4. In this approach the unitary properties of the elementary particle events are not separable from their space-time properties. Every Lorentz frame of reference, in addition to spin and parity, shall assume finite dimensional unitary properties. It must be pointed out that what we propose to discuss is only a certain symmetry group, not the actual dynamics of the particles. As in the case of the Lorentz group itself, to some as yet unknown quantum-mechanical system of equations there will correspond a representation of ILU_4. The representations of ILU_4 can, to a large extent, replace quantum-mechanical equations. This means that at this stage we do not know the possible connections between observables at a given instant of time.

II. THE UNITARY GROUP IN ELECTRODYNAMICS

In this section, to prepare the ground for the introduction of LU_4 (homogeneous part of ILU_4), we shall give a different discussion of the basic elements of SU_3

* This research is supported by U. S. Air Force Office of Scientific Research, Washington, D. C., U. S. Air Force Contract No. 49(638)-1260.

† The basic ideas of this paper were presented in the 1963 Eastern Theoretical Physics Conference, at the University of North Carolina in Chapel Hill. In its present form, the paper was read at the *Conference on Symmetry Principles at High Energy, January 1964, Coral Gables, Florida* (W. H. Freeman and Company, San Francisco, 1964).

‡ This work was begun at Argonne National Laboratory during the author's visit there in summer 1963.

[1] S. Sakata, Progr. Theoret. Phys. (Kyoto) **16**, 686 (1956).
[2] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[3] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).
[4] N. Kemmer, Proc. Roy. Soc. (London) **A166**, 127 (1938); Proc. Cambridge Phil. Soc. **34**, 354 (1938).

which is somewhat less *ad hoc* and also more pedagogical in nature than the usual treatments of this subject. Especially, it is rather heart warming to see that electrodynamics can manifest some aspects of the unitary symmetry group as it did in the case of Lorentz group. All that is required consists in recasting the equations of electrodynamics in terms of a spinor quantity or complex vector $\mathcal{E}+i\mathcal{H}$ where $\mathcal{E}$ and $\mathcal{H}$ refer to electric and magnetic vectors, respectively. It is a well known fact that the wave equation for a free photon uses the complex three-dimensional vector $\chi=\mathcal{E}+i\mathcal{H}$ as a wave function of the photon.[5] A group theoretical motivation for introducing the complex vector χ was discussed in connection with the three-dimensional complex orthogonal group.[6]

The six angular momentum operators $J_{\mu\nu}$ of the photon can be written in the form

$$J_{\mu\nu}=\frac{1}{2c}\int\langle\eta|(x_\mu B_{\nu\gamma}-x_\nu B_{\mu\gamma})E|\eta\rangle d\sigma^\gamma, \quad \text{(II.1)}$$

where the energy momentum tensor of the electromagnetic field is introduced by

$$T_{\mu\nu}=\tfrac{1}{2}\langle\chi|B_{\mu\nu}|\chi\rangle,\quad \mu,\nu=1,2,3,4. \quad \text{(II.2)}$$

Here the column vectors $|\chi\rangle$ and $|\eta\rangle$ are given by

$$|\chi\rangle=\begin{bmatrix}\chi_1\\ \chi_2\\ \chi_3\end{bmatrix},\quad \langle\chi|=[\chi_1^*,\chi_2^*,\chi_3^*],\quad |\chi\rangle=(8\pi E)^{1/2}|\eta\rangle,$$

E = photon energy, and the ten 3×3 matrices $B_{\mu\nu}$ can be defined as

$$\begin{aligned}&B_{\mu\nu}=B_{\nu\mu},\quad B_{i4}=B_{4i}=K_i,\\ &B_{44}=K_4,\quad B_{ij}=K_iK_j+K_jK_i-\delta_{ij},\\ &\qquad i,j=1,2,3,\end{aligned} \quad \text{(II.3)}$$

i.e., as bilinear combinations[7] of the generators of infinitesimal rotations in three dimensions (or spin-1 matrices),

$$K_1=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\quad K_2=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},$$
$$K_3=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix},\quad K_4=B_{44}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}. \quad \text{(II.4)}$$

The $d\sigma^\gamma$ in (II.1) represent the four-dimensional surface elements $(dx_2dx_3dx_4, dx_3dx_1dx_4, dx_1dx_2dx_4, dx_1dx_2dx_3)$.

In (II.1) the wave function $|\eta\rangle$ (or in vector notation η_i, $i=1,2,3$) is quantized and satisfies the wave equation

$$i\hbar(\partial/\partial t)|\eta\rangle=H|\eta\rangle, \quad \text{(II.5)}$$

and the supplementary condition

$$(\nabla\cdot\boldsymbol{\eta})|\Psi\rangle=0, \quad \text{(II.6)}$$

where

$$H=c\mathbf{K}\cdot\mathbf{p},\quad \mathbf{p}=-i\hbar\nabla,$$

and $|\Psi\rangle$ is the state vector of the quantized electromagnetic field. A detailed discussion of this quantization procedure can be found in MQT, p. 307 (see also Chap. II). From the above it follows that at any instant of time, the space and time components of $J_{\mu\nu}$ can be written as

$$T_i=\int\langle\eta|F_i|\eta\rangle d^3x,\quad Q_i=\int\langle\eta|H_i|\eta\rangle d^3x, \quad \text{(II.7)}$$

where

$$\begin{aligned}&F_i=L_i+\hbar K_i,\quad H_i=x_ip_4-x_4p_i+i\hbar K_i,\\ &T_i=\tfrac{1}{2}\epsilon_{iks}J_{ks},\quad Q_i=J_{i4}.\end{aligned} \quad \text{(II.8)}$$

The Hermitian operators T_i and Q_i are a set of generators for the homogeneous Lorentz group.

Now let us consider the properties of the B matrices. By using the commutation and anticommutation relations

$$[K_i,K_j]=i\epsilon_{ijl}K_l, \quad \text{(II.9)}$$

$$K_iK_jK_l+K_lK_jK_i=\delta_{ij}K_l+\delta_{jl}K_i, \quad \text{(II.10)}$$

we further obtain the commutation relations

$$[K_i,B_{jl}]=i(\epsilon_{ijs}B_{sl}+\epsilon_{ils}B_{sj}), \quad \text{(II.11)}$$

$$[B_{ij},B_{ls}]=i(\delta_{is}\epsilon_{jlm}+\delta_{il}\epsilon_{jsm}+\delta_{js}\epsilon_{ilm}+\delta_{jl}\epsilon_{ism})K_m. \quad \text{(II.12)}$$

Hence, the system of Hermitian matrices $B_{\mu\nu}$ is closed under commutation. The commutator of any two of B's is a linear combination of the B's. These matrices are a set of generators of nine-parameter continuous group of unitary transformations.[8]

The B matrices further satisfy anticommutation relations of the form

$$[K_i,B_{jl}]_+=\delta_{ij}K_l+\delta_{il}K_j, \quad \text{(II.13)}$$

$$[B_{ij},B_{ls}]_+=\delta_{il}\delta_{js}+\delta_{is}\delta_{jl}-(\epsilon_{isk}\epsilon_{jlm}+\epsilon_{ilk}\epsilon_{jsm})B_{km}. \quad \text{(II.14)}$$

Hence

$$\operatorname{trace}(B_{ij}B_{ls})=2(\delta_{il}\delta_{js}+\delta_{is}\delta_{jl})-\delta_{ij}\delta_{ls}. \quad \text{(II.15)}$$

[5] B. Kurşunoğlu, *Modern Quantum Theory* (W. H. Freeman and Company, San Francisco, 1962). This will be referred to in this paper as MQT.

[6] B. Kurşunoğlu, J. Math. Phys. **2**, 22 (1961).

[7] We shall use the usual Lorentz metric $g_{\mu\nu}(g_{ij}=-\delta_{ij}, g_{i4}=g_{4i}=0, g_{44}=1)$, for raising or lowering indices. Throughout this paper Greek and Latin indices run through 1, 2, 3, 4 and 1, 2, 3, respectively. The notation is the same as in MQT.

[8] In connection with the three-dimensional complex orthogonal representation of the Lorentz group, these matrices were first introduced by the author in Ref. 6. See also MQT, Chap. VIII. A representation used more frequently in the SU_3 literature employs the normalization Norm $(B_{ij}')=1$, Norm $(B_{i4}')=1$, and trace $(B_{ij}')=0$. In this case the above definitions can be replaced by $B_{ij}'=\frac{1}{2}(\sqrt{\frac{3}{2}})(B_{ij}-\frac{1}{3}\delta_{ij})$, $B_{i4}'=K_i/\sqrt{2}$, $B_{44}'=B_{44}/\sqrt{3}$. The $B_{\mu\nu}$ satisfy the same commutation relations except for the factors $1/\sqrt{2}$ appearing on the right-hand sides of (II.9), (II.11), and the factor $3/4\sqrt{2}$ on the right-hand side of (II.12). For the present discussion the normalization of the B's is not relevant.

From the definitions (II.3) and (II.4) it follows that the particular set of generators for U_3 is given by

$$B_{11}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad B_{22}=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\quad B_{33}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \tag{II.16}$$
$$B_{23}=\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&0\end{bmatrix},\quad B_{31}=\begin{bmatrix}0&0&-1\\0&0&0\\-1&0&0\end{bmatrix},\quad B_{12}=\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix},$$

and $B_{i4}=K_i$ as defined by (II.4). Because of the identity

$$B_{44}=B_{11}+B_{22}+B_{33},$$

there are only nine linearly independent matrices. The latter fact is an expression of the well-known property

$$\mathrm{Trace}(T_{\mu\nu})=T_{44}-T_{11}-T_{22}-T_{33}=0. \tag{II.17}$$

Thus the B matrices defined via the definition (II.2) of $T_{\mu\nu}$ could not have been a complete set of generators for infinitesimal unitary transformations without the Lorentz-invariant property (II.17). In other words the unitary symmetry would be broken without (II.17). The B matrices as described above belong to the Lie algebra of U_3. For any unitary member of the group, one can put

$$U'=e^{i\delta}U, \tag{II.18}$$

where U is a unitary matrix of determinant 1. This decomposition (i.e., $U_3=U_1\times SU_3$) of the group U_3 can be obtained by forming two traceless linear combinations of B_{11}, B_{22}, B_{33}.

Under a Lorentz transformation of the coordinates we can obtain the corresponding transformations of the $B_{\mu\nu}$. Thus if L is a Lorentz matrix the corresponding transformations are

$$|x'\rangle=L|x\rangle,\quad |x\rangle=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \tag{II.19}$$

for the coordinates, and

$$|\chi'\rangle=R|\chi\rangle \tag{II.20}$$

for $|\chi\rangle$, where R belongs to the group of complex orthogonal transformations.[5,6] Thus there exists a correspondence between L and R transformations such that

$$R^\dagger B_{\mu\nu}R=L_\mu{}^\alpha L_\nu{}^\beta B_{\alpha\beta}. \tag{II.21}$$

The relations (II.21) do not represent a connection between Lorentz and unitary groups since both R and L correspond to the different representations of the same group. Furthermore, translations of the coordinates do not appear in (II.21). However, all that (II.21) suggests refers to the fact that the Lie algebra of U_3 can, in connection with the transformation properties of the electromagnetic field, appear in Lorentz-covariant statements like (II.21). This rather suggestive representation cannot be established if one adheres to the usual representation where the generators of SU_3 are involved in the elementary particle interactions as a linear array of 8 matrices rather than as the components of tensor operator $B_{\mu\nu}$ as displayed by (II.20) and (II.21). The same B matrices also appear in the definition (II.1) of the photon angular momentum operator.

The integral of (II.2) for the quantized $|\eta\rangle$ will consist of a sum of terms of the form

$$S_{\mu\nu}=\tfrac{1}{2}\langle a|B_{\mu\nu}|a\rangle,\quad |a\rangle=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}, \tag{II.22}$$

where the three-dimensional oscillator operators a_i satisfy the commutation relations

$$[a_i,a_j{}^+]=\delta_{ij}. \tag{II.23}$$

The operators $S_{\mu\nu}$ as defined by (II.22) satisfy the commutation relations (II.9), (II.11), and (II.12). Hence we see that the Fock representation (see Chap. VII in MQT) of the three-dimensional harmonic oscillator provides, via the definition (II.22), a representation of the three-dimensional unitary group U_3.

In order to exhibit the Lie algebra of SU_3 in a convenient form we can begin with the following representation of the Lie algebra of SU_2. Consider the four Hermitian operators

$$I_\mu=\tfrac{1}{2}\langle N|\tau_\mu|N\rangle, \tag{II.24}$$

where the τ_i $(i=1,2,3)$ are the usual Pauli matrices and τ_4 is the 2×2 unit matrix. The symbol $|N\rangle$ refers to the coordinates of a two-dimensional oscillator

$$|N\rangle=\begin{bmatrix}a_p\\a_n\end{bmatrix}.$$

The elements of the Lie algebra of SU_2 can be represented in the form of a 2×2 matrix, where only the usual ladder operators appear. We form a traceless operator by putting

$$S_2=\tfrac{1}{2}\,\mathrm{trace}(\tau^\mu I_\mu)-\tau^\mu I_\mu,$$

where

$$\tau^\mu I_\mu=\tau_4I_4-\tau_1I_1-\tau_2I_2-\tau_3I_3.$$

Hence, using (II.24), we obtain

$$S_2=\begin{bmatrix}I_0&I_-\\I_+&-I_0\end{bmatrix}, \tag{II.25}$$

where $I_+=a_p{}^+a_n$, $I_-=a_n{}^+a_p$, $I_0=\frac{1}{2}(a_p{}^+a_p-a_n{}^+a_n)=\frac{1}{2}(N_p-N_n)$. The operators I_0 and I^2 form a complete commuting set, where I is the eigenvalue of

$$I^2=\tfrac{1}{2}\operatorname{trace}(S_2)^2=I_0{}^2+\tfrac{1}{2}(I_+I_-+I_-I_+)=I(I+1)\,,$$
$$I=\tfrac{1}{2}(a_p{}^+a_p+a_n{}^+a_n)=\tfrac{1}{2}(N_p+N_n)\,.$$

The generalization of (II.24) and (II.25) to SU_3 is based on the definition

$$S_3=\tfrac{1}{2}[B^{\mu\nu}S_{\mu\nu}-\tfrac{1}{3}\operatorname{trace}(B^{\mu\nu}S_{\mu\nu})]. \tag{II.26}$$

Hence, using (II.22), we get the useful representation

$$S_3=\begin{bmatrix}\frac{1}{2}Y+I_0 & I_+ & s_-\\ I_- & \frac{1}{2}Y-I_0 & J_+\\ s_+ & J_- & -Y\end{bmatrix}, \tag{II.27}$$

where the off-diagonal elements $I_+=a_p{}^+a_n$, $I_-=a_n{}^+a_p$, $s_-=a_p{}^+a_\Lambda$, $s_+=a_\Lambda{}^+a_p$, $J_+=a_n{}^+a_\Lambda$, $J_-=a_\Lambda{}^+a_n$ are the ladder operators of the associated Lie algebra of SU_3. The suffixes 1, 2, and 3 of a_i have been replaced by (p,n,Λ), respectively. The operators Y and I_0 are the only mutually commuting generators in the Lie algebra. Thus the group SU_3 is of rank two. The operators Y and I_0 are given by

$$Y=N_p+N_n-\tfrac{2}{3}B=\tfrac{1}{3}B+S\,, \tag{II.28}$$
$$I_0=\tfrac{1}{2}(N_p-N_n)\,, \tag{II.29}$$

where $B=N_p+N_n+N_\Lambda$, $S=-N_\Lambda$ and where $N_p=a_p{}^+a_p$, $N_n=a_n{}^+a_n$, $N_\Lambda=a_\Lambda{}^+a_\Lambda$ are number operators. In this case the operators I_0, Y and $\mathbf{F}^2$ form a complete commuting set, where

$$\mathbf{F}^2=\tfrac{1}{6}\operatorname{trace}(S_3)^2=\tfrac{1}{3}B(\tfrac{1}{3}B+1)\,.$$

As in the case of SU_2 the trace operation here must be understood as the sum of diagonal elements in the three-dimensional matrix $S_3{}^2$ whose elements themselves are operators. It is clearly seen from the above discussion of SU_3 that it has double-valued representations.

In the limit of full unitary symmetry (which may be expected to hold rigorously at very high energies), or in the absence of symmetry breaking perturbations one can obtain a manifestly Lorentz and unitary invariant theory in which a pseudoscalar octet of mesons is coupled to nucleons and the Λ particle This can most conveniently be achieved by using the analogy with the SU_2 invariant theory

We introduce the 4-vector Φ_μ ($\mu=1, 2,3, 4$) in a linear combination

$$\Omega=\tfrac{1}{2}\operatorname{trace}(\Phi^\mu\tau_\mu)-\Phi^\mu\tau_\mu$$

and exhibit the coupling of π mesons to nucleon as

$$H_I=iG\langle\bar{\psi}|\gamma_5\Omega|\psi\rangle\,, \tag{II.30}$$

where

$$|\psi\rangle=\begin{bmatrix}\psi_p\\ \psi_n\end{bmatrix},$$

and the Ω operator has the form

$$\Omega=\begin{bmatrix}\Phi_3 & \Phi_-\\ \Phi_+ & -\Phi_3\end{bmatrix}=\begin{bmatrix}\pi_0 & \pi_-\\ \pi_+ & -\pi_0\end{bmatrix},\quad \Phi_\pm=\Phi_1\pm i\Phi_2,\quad \phi_3=\pi_0.$$

The generalization of (II.30) to the octet model follows if we define the traceless operator ζ in the form

$$\zeta=\tfrac{1}{2}[\tfrac{1}{3}\operatorname{trace}(\psi^{\mu\nu}B_{\mu\nu})-\psi^{\mu\nu}B_{\mu\nu}]\,, \tag{II.31}$$

where $\psi_{\mu\nu}$ is a real symmetric tensor in four-dimensional space. The operator ζ can be written as

$$\zeta=\begin{bmatrix}(1/\sqrt{6})\chi_0+(1/\sqrt{2})\pi_0 & \pi^- & K^-\\ \pi^+ & (1/\sqrt{6})\chi_0-(1/\sqrt{2})\pi_0 & K^0\\ K^+ & \bar{K}^0 & -(\sqrt{\frac{2}{3}})\chi_0\end{bmatrix}, \tag{II.32}$$

where we have used the notation

$$\tfrac{1}{3}(\psi_{11}+\psi_{22}-2\psi_{33})=(\sqrt{\tfrac{2}{3}})\chi_0\,,\quad \psi_{11}-\psi_{22}=(1/\sqrt{2})\pi_0\,,$$

and

$$\psi_{12}\pm i\psi_{34}=\pi^\pm\,,\quad \psi_{31}\pm i\psi_{24}=K^\pm\,,\quad \psi_{23}+i\psi_{14}=K^0\,,$$
$$\psi_{23}-i\psi_{14}=\bar{K}^0.$$

The normalization is such that the squares of the diagonal terms sum to $\pi_0{}^2+\chi_0{}^2$. The effective coupling of the octet to the $(pn\Lambda)$ system is

$$H_I=iG\langle\bar{\chi}|\gamma_5\zeta|\chi\rangle\,, \tag{II.33}$$

where

$$|\chi\rangle=\begin{bmatrix}\psi_p\\ \psi_n\\ \psi_\Lambda\end{bmatrix}.$$

This is a known unitary spin-invariant theory replacing isotopic spin-invariant theory.

III. THE NEW GROUP (ILU_4)

So far we have discussed the construction of the group SU_3 from the Lie algebra of the three-dimensional rotation group. Bilinear combinations of the generators K_1, K_2, K_3 provided a complete set of generators of infinitesimal unitary transformations in three dimensions. Any connection found with Lorentz group was merely a formal manipulation without any extra physical consequences. However, the method has provided us with a rather novel technique for constructing new groups out of known physically relevant groups. It is, of course, sufficient to find only one set of generators for a group, since all other representations of the group can be found from the commutation relations of these generators.

It has been shown by Wigner[9] that the Lorentz group

[9] E. Wigner, Ann. Math. 40, 149 (1939).

has no finite dimensional unitary representations. Therefore it is not possible to establish a connection between the finite dimensional unitary invariance of the elementary particle interactions and their Lorentz invariance properties. Only a group larger than the Lorentz group can be expected to possess both finite and infinite dimensional unitary representations. Therefore the wave function describing a physical state can not be completely given by just specifying a Lorentz frame of reference. With every Lorentz frame of reference we must associate additional properties pertaining to a finite unitary covariance of the state. An example of a symmetry group with the above-mentioned properties can be constructed by generalizing the techniques used in the previous section. Instead of bilinear combinations of the generators of the three-dimensional rotation group (which is a subgroup of the homogeneous Lorentz group) we shall consider bilinear combinations of the homogeneous Lorentz group generators. In the spirit of the theory discussed in the previous section we again take a symmetrical tensor operator which is formally the same as the electromagnetic field energy momentum tensor. Thus, we write the traceless matrices

$$\Gamma_{\mu\nu}=\tfrac{1}{4}g_{\mu\nu}M_{\alpha\beta}M^{\alpha\beta}-\tfrac{1}{2}(M_{\mu\cdot}{}^{\rho}M_{\nu\rho}+M_{\nu\cdot}{}^{\rho}M_{\mu\rho}), \tag{III.1}$$

where

$$g^{\mu\nu}\Gamma_{\mu\nu}=0, \tag{III.2}$$

and where the $M_{\mu\nu}=-M_{\nu\mu}$ are a set of six generators of infinitesimal homogeneous Lorentz transformations defined as (see MQT, p. 50)

$$M_1=M_{23}=\begin{bmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{bmatrix},\quad M_2=M_{31}=\begin{bmatrix}0&0&i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0\end{bmatrix},\quad M_3=M_{12}=\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$

$$N_1=M_{14}=\begin{bmatrix}0&0&0&-i\\0&0&0&0\\0&0&0&0\\-i&0&0&0\end{bmatrix},\quad N_2=M_{24}=\begin{bmatrix}0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&-i&0&0\end{bmatrix},\quad N_3=M_{34}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&-i&0\end{bmatrix}. \tag{III.3}$$

The fifteen generators $J_{\mu\nu}$ (also $M_{\mu\nu}$) and $\Lambda_{\mu\nu}$ (also $\Gamma_{\mu\nu}$) of the group L_0U_4 satisfy the commutation relations

$$[J_{\mu\nu},J_{\alpha\beta}]=i(g_{\alpha\nu}J_{\mu\beta}+g_{\beta\nu}J_{\alpha\mu}-g_{\alpha\mu}J_{\nu\beta}-g_{\mu\beta}J_{\alpha\nu}), \tag{III.4}$$

$$[\Lambda_{\mu\nu},J_{\alpha\beta}]=i(g_{\alpha\nu}\Lambda_{\mu\beta}+g_{\alpha\mu}\Lambda_{\beta\nu}-g_{\beta\nu}\Lambda_{\alpha\mu}-g_{\mu\beta}\Lambda_{\alpha\nu}), \tag{III.5}$$

$$[\Lambda_{\mu\nu},\Lambda_{\alpha\beta}]=i(g_{\beta\nu}J_{\alpha\mu}+g_{\mu\beta}J_{\alpha\nu}-g_{\alpha\nu}J_{\mu\beta}-g_{\alpha\mu}J_{\nu\beta}), \tag{III.6}$$

where in the derivation of (III.5) and (III.6) analogy with (II.10) for $M_{\mu\nu}$ has been used and where $\Lambda_{\mu\nu}$ is an abstract set of generators of the group.

From (III.4), (III.5), and (III.6), we observe that the set of matrices $J_{\mu\nu}$ and $\Lambda_{\mu\nu}$ are closed under commutation. Thus the commutator of any two of the operators $J_{\mu\nu}$, $\Lambda_{\mu\nu}$ is a linear combination either of the $J_{\mu\nu}$ or the $\Lambda_{\mu\nu}$. These matrices constitute a set of generators for the fifteen parameter continuous transformation group L_0U_4 which is a subgroup of 16-parameter homogeneous group LU_4.

A set of matrices representing the $\Lambda_{\mu\nu}$ can be found by using (III.1) and (III.3)

$$\Gamma_{11}=\frac{1}{2}\begin{bmatrix}3&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\quad \Gamma_{22}=\frac{1}{2}\begin{bmatrix}-1&0&0&0\\0&3&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},$$

$$\Gamma_{33}=\frac{1}{2}\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&3&0\\0&0&0&-1\end{bmatrix},\quad \Gamma_{44}=\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-3\end{bmatrix}, \tag{III.7}$$

$$\Gamma_{23}=\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{bmatrix},\quad \Gamma_{31}=\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix},\quad \Gamma_{12}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \tag{III.8}$$

$$\Gamma_{14}=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{bmatrix},\quad \Gamma_{24}=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{bmatrix},\quad \Gamma_{34}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}. \tag{III.9}$$

The following properties of these matrices must be noted: (i) The matrices Γ_{ij} ($i, j=1, 2, 3$) are Hermitian and commute with F, (ii) The matrices Γ_{i4} ($i=1, 2, 3$) are anti-Hermitian and anticommute with F, where

$$F=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{III.10}$$

From (i) and (ii) it follows that the matrices $\Gamma_{\mu\nu}$ can be looked upon as generators of complex Lorentz transformations[10] which (a) leave the indefinite quadratic form

$$\langle z|F|z\rangle = |z_4|^2-|z_1|^2-|z_2|^2-|z_3|^2 \tag{III.11}$$

invariant, that is they satisfy the conditions

$$L^\dagger F L = F \tag{III.12}$$

(where the sign † denotes Hermitian conjugation and L is now a complex Lorentz matrix), and that (b) the determinant of L is 1. The above conditions are satisfied also by members of L_0U_4 generated by $M_{\mu\nu}$ defined by (III.3) which are also generators of the homogeneous Lorentz group (the latter is a subgroup of L_0U_4).

The fact that the group SU_3 is a subgroup of L_0U_4 is clear from the quadratic form (III.11). However, it is more convincing to demonstrate this fact in terms of the Lie algebra of L_0U_4. We shall write the commutation relations (III.4), (III.5), and (III.6) of the L_0U_4 Lie algebra in three-dimensional notation putting

$$J_{ij}=\epsilon_{ijl}J_l, \quad J_{i4}=D_i;$$

we obtain

$$[J_i,J_j]=i\epsilon_{ijl}J_l, \tag{III.13}$$

$$[D_i,D_j]=-i\epsilon_{ijl}J_l, \tag{III.14}$$

$$[J_i,D_j]=i\epsilon_{ijl}D_l, \tag{III.15}$$

$$[J_i,\Lambda_{jl}]=i(\epsilon_{ijs}\Lambda_{sl}+\epsilon_{ils}\Lambda_{sj}), \tag{III.16}$$

$$[D_i,\Lambda_{jl}]=i(\delta_{il}\Lambda_{j4}+\delta_{ij}\Lambda_{4l}), \tag{III.17}$$

$$[J_i,\Lambda_{j4}]=i\epsilon_{ijl}\Lambda_{l4}, \tag{III.18}$$

$$[J_i,\Lambda_{44}]=0, \tag{III.19}$$

$$[D_i,\Lambda_{44}]=2i\Lambda_{i4}, \tag{III.20}$$

$$[D_i,\Lambda_{j4}]=i(\delta_{ij}\Lambda_{44}+\Lambda_{ij}), \tag{III.21}$$

$$[\Lambda_{ij},\Lambda_{ls}]=i(\delta_{is}\epsilon_{jlk}+\delta_{il}\epsilon_{jsk}+\delta_{js}\epsilon_{ilk}+\delta_{jl}\epsilon_{isk})J_k, \tag{III.22}$$

$$[\Lambda_{ij},\Lambda_{4l}]=i(\delta_{jl}D_i+\delta_{il}D_j), \tag{III.23}$$

$$[\Lambda_{i4},\Lambda_{j4}]=-i\epsilon_{ijl}J_l, \tag{III.24}$$

$$[\Lambda_{i4},\Lambda_{44}]=-2iD_i, \tag{III.25}$$

$$[\Lambda_{44},\Lambda_{ij}]=0. \tag{III.26}$$

Now, a comparison of (III.13), (III.16), and (III.22) with (II.9), (II.11), and (II.12), respectively, shows that SU_3 is a subgroup of L_0U_4. At this point it may be tempting to see whether the above commutation relations contain an inhomogeneous character in them. We shall, therefore, compare them with the commutation relations of the usuali nhomogeneous Lorentz group. In addition to the commutation relations (III.13), (III.14) and (III.15), the inhomogeneous Lorentz group comprises the commutation relations

$$[J_{\mu\nu},p_\rho]=i(g_{\rho\nu}p_\mu-g_{\mu\rho}p_\nu), \tag{III.27}$$

$$[p_\mu,p_\nu]=0, \tag{III.28}$$

between angular momentum and linear momentum operators, or in three-dimensional notation we write

$$[J_i,p_j]=i\epsilon_{ijl}p_l, \tag{III.29}$$

$$[D_i,p_j]=i\delta_{ij}p_4, \tag{III.30}$$

$$[J_i,p_4]=0 \tag{III.31}$$

$$[D_i,p_4]=ip_i. \tag{III.32}$$

Hence we see that if we assume the correspondence $\frac{1}{2}\Lambda_{44}\rightarrow p_4$ and $\Lambda_{i4}\rightarrow p_i$, the commutation relations of L_0U_4 corresponding to (III.27) are given by (III.18), (III.21), (III.19), and (III.20), respectively. With the exception of (III.21) the remaining rules are of the same form as (III.29), (III.31), and (III.32). Furthermore, the (III.24) show that $\Lambda_{i4}(\rightarrow p_i)$, do not commute, nor does $\frac{1}{2}\Lambda_{44}(\rightarrow p_4)$ with the Λ_{i4}, as shown by (III.25). Thus the commutation relations (III.28) are completely modified.[11]

In this connection, an interesting speculation is to think of translations in microphysics as discrete operations. However, this seems to conflict with the well-established concept of translation in quantum mechanics. It would also violate the usual commutation laws for small momenta, becoming less and less important at very large momenta. Furthermore, to be compatible with the aim stated at the beginning of this section we must seek a more natural way to introduce translations where one can accommodate both finite and infinite dimensional unitary representations in the same transformation group. In this case spin and unitary spin may be recognized in the "little groups" within the infinite dimensional unitary representations of the new group.

A simple way is to observe that the commutation relations (III.4), (III.5), and (III.6) of L_0U_4 are also

[10] The possibility for a complex Lorentz group was stipulated earlier in connection with two-valued representations of Lorentz group [see p. 240, Eq. (VIII.5.55) in MQT].

[11] It is interesting to note that replacement of D_i and Λ_{i4} by iD_i and $i\Lambda_{i4}$, respectively, in the commutation relations for L_0U_4 yields the Lie algebra of the group SU_4. The latter contains O_4 (the four-dimensional orthogonal group) and SU_3 as subgroups. Thus the relation of the new group to SU_4 is similar to the relation of O_4 to the homogeneous Lorentz group (see p. 255 of MQT). Therefore, representations of L_0U_4 can be constructed from the representations of SU_4.

satisfied by the operator representations

$$-J_{\mu\nu}=x_\mu p_\nu - x_\nu p_\mu\,, \quad \text{(III.33)}$$

$$\Lambda_{\mu\nu}=x_\mu p_\nu + x_\nu p_\mu - \tfrac{1}{2}g_{\mu\nu}x_\rho p^\rho\,, \quad \text{(III.34)}$$

where the coordinates x_μ and momenta p_μ satisfy the usual commutation relations

$$[x_\mu,p_\nu]=ig_{\mu\nu}\,,\quad [p_\mu,p_\nu]=0\,. \quad \text{(III.35)}$$

We may, now, easily establish the commutation relations

$$[J_{\mu\nu},p_\alpha]=-i(g_{\alpha\mu}p_\nu - g_{\alpha\nu}p_\mu)\,, \quad \text{(III.36)}$$

$$[\Lambda_{\mu\nu},p_\alpha]=i(g_{\alpha\mu}p_\nu + g_{\alpha\nu}p_\mu - \tfrac{1}{2}g_{\mu\nu}p_\alpha)\,. \quad \text{(III.37)}$$

In this case we have, together with (III.4), (III.5), (III.6), the Lie algebra of a 20-parameter inhomogeneous group. The group includes, of course, transformations of the form $\exp(i\phi)$. For this group $p_\mu p^\mu$ is no longer a group invariant, except for the inhomogeneous Lorentz group which is now a subgroup of the generalized inhomogeneous Lorentz group ILU_4. This seems to be a nontrivial way of combining both space time and internal symmetries of dynamical systems into a single group, since SU_3 also is a subgroup of ILU_4. Unlike the inhomogeneous Lorentz group, the rest mass does not, by itself alone, commute with the new group.

The next important step in this direction is to discover the implied relevant dynamics by ILU_4. If the latter turns out to be too ambitious we must at least find a physical parameter whereby one can discuss some kind of "contraction" of ILU_4 so as to yield inhomogeneous Lorentz group in some limit of this parameter.

A further relevant, but not too pressing at present, remark is to look upon ILU_4 transformations valid only locally and not applicable over an extended space time region. In this latter sense the group ILU_4 may be envisaged as a subgroup of an infinite continuous group in the elements of which arbitrary functions occur. In this case, gauge group, and coordinate transformations in general relativity may be studied with a broader view.

IV. THE BASIC QUANTUM NUMBERS OF LU_4

Following up the method used in Sec. II for the discussion of SU_3 we introduce the operators a_μ ($\mu=1, 2, 3, 4$) which obey the commutation relations

$$[a_\mu,a_\nu^\dagger]=g_{\mu\nu}\,. \quad \text{(IV.1)}$$

The sixteen linear operators $Q_{\mu\nu}=a_\mu^\dagger a_\nu$ satisfy the commutation relations

$$[Q_{\mu\nu},Q_{\alpha\beta}]=g_{\alpha\nu}Q_{\mu\beta}-g_{\mu\beta}Q_{\alpha\nu} \quad \text{(IV.2)}$$

and can be used to construct generators of infinitesimal LU_4 transformations by putting

$$\Lambda_{\mu\nu}=\mathfrak{M}g_{\mu\nu}-(a_\mu^\dagger a_\nu + a_\nu^\dagger a_\mu)=\langle a|F\Gamma_{\mu\nu}|a\rangle\,, \quad \text{(IV.3)}$$

$$J_{\mu\nu}=i(a_\mu^\dagger a_\nu - a_\nu^\dagger a_\mu)=\langle a|FM_{\mu\nu}|a\rangle\,, \quad \text{(IV.4)}$$

[12] See MQT, p. 254, Eq. (VIII.8.3).

where

$$\mathfrak{M}=\tfrac{1}{2}\langle a|F|a\rangle=\tfrac{1}{2}a_\mu^\dagger a^\mu$$

$$|a\rangle=\begin{bmatrix} a_1\\ a_2\\ a_3\\ a_4\end{bmatrix}\,. \quad \text{(IV.5)}$$

These expressions satisfy the commutation relations (III.4), (III.5), and (III.6). The operator $\mathfrak{M}$ commutes with $\Lambda_{\mu\nu}$ and $J_{\mu\nu}$. It will be related[13] to a new baryon number b by

$$b=-\tfrac{1}{2}\mathfrak{M}-1=\tfrac{1}{4}(N_1+N_2+N_3-N_4) \quad \text{(IV.6)}$$

The analog of the operator statement (II.26) in the present case is obtained with the Γ and M matrices defined by (III.3), (III.7), (III.8), (III.9), as

$$S_4=-\tfrac{1}{4}(\Gamma^{\mu\nu}\Lambda_{\mu\nu}+M^{\mu\nu}J_{\mu\nu})$$

$$=\begin{bmatrix} \tfrac{1}{2}(\mathfrak{L}+Y)+I_0 & I_+ & s_- & -u_-\\ I_- & \tfrac{1}{2}(\mathfrak{L}+Y)-I_0 & J_+ & -v_-\\ s_+ & J_- & -Y & -q_-\\ u_+ & v_+ & q_+ & -\mathfrak{L}\end{bmatrix} \quad \text{(IV.7)}$$

where

$$I_0=\tfrac{1}{2}(N_1-N_2)\,,\quad \mathfrak{L}=N_4+b\,,\quad Y=-N_3+b \quad \text{(IV.8)}$$

and $N_\alpha=a_\alpha a_\alpha^\dagger$ ($\alpha=1, 2, 3, 4$, not summed). The off-diagonal operator assignments, in this case, are

$$\begin{aligned} J_+&=a_2a_3^\dagger\,, & J_-&=a_3a_2^\dagger\\ s_+&=a_3a_1^\dagger\,, & s_-&=a_1a_3^\dagger\\ I_+&=a_1a_2^\dagger\,, & I_-&=a_2a_1^\dagger \end{aligned} \quad \text{(IV.9)}$$

and

$$\begin{aligned} u_+&=a_1a_4^\dagger\,, & u_-&=a_4a_1^\dagger\\ v_+&=a_2a_4^\dagger\,, & v_-&=a_4a_2^\dagger\\ q_+&=a_3a_4^\dagger\,, & q_-&=a_4a_3^\dagger\,. \end{aligned} \quad \text{(IV.10)}$$

The operators I_0, $\mathfrak{L}$, and Y are, besides b, the only mutually commuting members of the LU_4 Lie algebra. Hence, the group LU_4 is of rank four. The additive operators I_0, $\mathfrak{L}$, Y will be assumed to refer to isotopic spin projection, lepton number, and hypercharge, respectively.[14] (IV.8) shows that the hypercharge Y is related to the new baryon number by

$$Y=b+S\,, \quad \text{(IV.11)}$$

where $S=-N_3$ is the strangeness quantum number. The lepton and baryon numbers are related through N_4

[13] Thus LU_4 is decomposable into the product $LU_4=U_1\times L_0U_4$ where $U_1=\exp(i\theta\mathfrak{M})$ is an LU_4-invariant unitary operator and L_0U_4 is now the fifteen-parameter subgroup of LU_4 discussed in Sec. III.

[14] It may seem somewhat hasty to name $\mathfrak{L}$ as "lepton number" rather than allowing it to maintain its original place in the Latin alphabet. However, this operator assignment can be regarded as a suggestive speculation since, at present, one does not know how various elementary particle interactions will fit into the new group.

which, presumably, is related to muon number[15] S_l. Thus from (IV.8) we write

$$\mathcal{L}-b=S_l. \tag{IV.12}$$

Some of the commutation relations in terms of the ladder operators appearing in (IV.10) are

$$\begin{aligned}
&[\mathcal{M},I_\pm]=[\mathcal{M},s_\pm]=[\mathcal{M},J_\pm]=[\mathcal{M},u_\pm]\\
&\qquad\qquad=[\mathcal{M},v_\pm]=[\mathcal{M},q_\pm]=0\,,\\
&[\mathcal{L},I_\pm]=[\mathcal{L},I_\pm]=[\mathcal{L},s_\pm]=0\,,\\
&[\mathcal{L},u_\pm]=\pm u_\pm\,,\quad [\mathcal{L},v_\pm]=\pm v_\pm\,,\\
&\qquad\qquad [\mathcal{L},q_\pm]=\pm q_\pm\,,\\
&[Y,I_\pm]=0\,,\quad [Y,s_\pm]=\mp s_\pm\,,\quad [Y,J_\pm]=\pm J_\pm\,,\\
&[Y,u_\pm]=0\,,\quad [Y,v_\pm]=0\,,\quad [Y,q_\pm]=\mp q_\pm\,.
\end{aligned} \tag{IV.13}$$

We now give the commutation relations which reveal the fact that the isotopic spin group in addition to the SU_3 subgroup of LU_4 contains other subgroups (SU_2's) in its structure:

$$[I_0,I_\pm]=\pm I_\pm\,,\quad [I_+,I_-]=2I_0\,, \tag{IV.14}$$

$$[J_0,J_\pm]=\pm J_\pm\,,\quad [J_+,J_-]\ \ 2J_0\,, \tag{IV.15}$$

$$[s_0,s_\pm]=\pm s_\pm\,,\quad [s_+,s_-]=2s_0\,. \tag{IV.16}$$

Here

$$\begin{gathered}
J_0=\tfrac{1}{2}(N_2-N_3)\,,\quad s_0=\tfrac{1}{2}(N_3-N_1)\,,\\
J_0+s_0+I_0=0\,.
\end{gathered} \tag{IV.17}$$

Therefore the new symmetry group also contains the spin degree of freedom of elementary particles as a unitary content, as well as the isotopic spin. We must add, immediately, that a real detection of spin in ILU_4 must follow from a discussion similar to one used in the usual Lorentz group. Such a discussion will be deferred to the next paper.

An invariant of the group LU_4 is given by

$$\begin{aligned}
\mathfrak{b}^2&=(1/48)(\Lambda_{\mu\nu}\Lambda^{\mu\nu}+J_{\mu\nu}J^{\mu\nu})\\
&=\tfrac{1}{12}\,\mathrm{trace}(S_4)^2=b(b+1)\,,
\end{aligned} \tag{IV.18}$$

putting $2I=N_1+N_2$ and $b=I-\frac{1}{2}(Y+\mathcal{L})$ it can be written as

$$\mathfrak{b}^2=[I(I+1)+\tfrac{1}{2}(\mathcal{L}+Y)^2-(2I+1)\tfrac{1}{2}(\mathcal{L}+Y)]\,. \tag{IV.19}$$

The operators $\mathcal{L}$, Y, I_0, and $\mathfrak{b}^2$ form a complete commuting set.

We may use the eigen-values of $\mathfrak{b}^2$ to classify elementary particles just as the eigenvalues of I^2 are used to label various isotopic multiplets. We may therefore consider two general cases:

(i) $\mathfrak{b}^2>0$, for $b>0$

(ii) $\mathfrak{b}^2=0$, for $b=0$.

The eigenvalues of the occupation number operators $N_i=a_ia_i^\dagger$ ($i=1,2,3$) and $N_4=a_4a_4^\dagger$ range over $(0,1,2,\cdots)$ and $(\cdots-3,-2,-1,0)$, respectively. This can easily be established by using the four equations

$$a_\mu^\dagger|0\rangle=0 \tag{IV.20}$$

for the vacuum state.[16] Hence the eigenvalues of

$$b=\tfrac{1}{4}(N_1+N_2+N_3-N_4) \tag{IV.21}$$

are non-negative.

Detailed discussion of these points will be made the subject matter of the next paper on this symmetry group.

Note added in proof. The well-known properties of the group SU_3 and the discussion in this paper demonstrate, beyond any shadow of doubt, that the group LU_4 has double-valued representations. However, it does not seem to be possible to identify the various operators of the present group according to a conventional scheme. The latter may not be a necessity.

ACKNOWLEDGMENT

A critical reading of this paper by Professor N. Kemmer and many discussions with him have helped greatly in clarifying some of the ideas introduced here.

[15] The remarks of the previous footnote apply here also.

[16] See MQT, p. 194.