In addition we conclude from (16) that

$$\|A_k(W)V_{k,k-1}\|^2_{\mathrm{HS}} \to 0 \quad \text{for} \quad \mathrm{Re}W \to -\infty\,. \quad (18)$$

The product AB of a HS operator A with a bounded operator B is again a HS operator whose HS norm satisfies $\|AB\|_{\mathrm{HS}} \leqslant \|A\|_{\mathrm{HS}}\|B\|$, $\|B\|$ being the operator norm of B. Therefore

$$\|A_{k-1}(W)\|_{\mathrm{HS}} \leqslant \|A_k(W)V_{k,k-1}\|_{\mathrm{HS}}\|G_{k-1}(W)\|\,. \quad (19)$$

But $G_{k-1}(W)$ is the resolvent of the self-adjoint[2] operator H_{k-1} on $\mathcal{H}_{k-1}$ which is bounded from below.[1] Therefore $G_{k-1}(W)$ is bounded whenever W is not on the spectrum of H_{k-1}, and $\|G_{k-1}(W)\| \leqslant \mathrm{const}\,|\mathrm{Re}W|^{-1}$ for $\mathrm{Re}W \to -\infty$. Since the spectrum of H_{k-1} contains the set $\arg(W-W_0)=0$ (in its continuous part), we conclude from (17), (18), and (19) that $A_{k-1}(W)$ has indeed the properties (8a), (8b), so that our induction proof is completed.

[2] T. Kato, Trans. Am. Math. Soc. **70**, 195 (1951).

For $k=2$, theorem (8) and (17) yield the final result: $I(W)$ is a Hilbert-Schmidt operator on the space $\mathcal{H}_1$ for all W not on the continuous spectrum of the full Hamiltonian, if (and only if) the potentials, which describe the pair interactions, are square integrable.

This is identical with Weinberg's conjecture, since the HS norm for $\mathcal{H}_1$ is the same as Weinberg's "center-of-mass HS norm."

ACKNOWLEDGMENT

The author gratefully acknowledges the partial financial support received from the Swiss National Fund.

PHYSICAL REVIEW VOLUME 135, NUMBER 3B 10 AUGUST 1964

One-Nucleon Exchange in Pion-Nucleon Scattering*

ARTHUR W. MARTIN† AND JACK L. URETSKY

Argonne National Laboratory, Argonne, Illinois

(Received 26 March 1964)

We have studied the partial-wave amplitudes for pion-nucleon scattering in the approximation where the "driving force" comes from single-nucleon exchange. We start from the assumption that the amplitudes satisfy dispersion relations, but find that this is not enough to define a unique problem. Some restrictions upon the choice of amplitude to be inserted in the dispersion relations are found, but some ambiguities remain. Settling these in a "reasonable" way leads to the conclusion that the one-nucleon-exchange force is "too strong."

I. INTRODUCTION

IT has long been part of the folklore of physics that the resonance observed in the $T=\frac{3}{2}$, $J=\frac{3}{2}^+$ scattering state of the pion-nucleon system is induced by the exchange of a single nucleon in the "crossed" pion-nucleon scattering channel. This belief very likely originated in the static-model calculations of Chew,[1] where it was found that a two-parameter (coupling constant and high-energy cutoff) fit could reproduce the position and width of the observed resonance. The belief was strengthened by the fact that the coupling constant determined by the two-parameter fit agreed with the one obtained subsequently from the forward-scattering dispersion relations.[2] In this way the single-nucleon-exchange diagram was understood to provide the primary driving force in the 3, 3 scattering amplitude.

One would like to adopt the viewpoint that the static model is an approximation to a more complete, relativistic theory which would permit calculations free of arbitrarily imposed cutoff parameters. The discovery of the Mandelstam representation,[3] from which partial-wave dispersion relations were deduced, appeared to provide a means of doing this. Frautschi and Walecka[4] investigated this approach and were able to show that the single-nucleon-exchange force is more than sufficiently attractive to account for the 3, 3 resonance.

Following the qualitative success of the Frautschi-Walecka calculation, it was possible to feel that one had a reasonable understanding of pion-nucleon scattering over a substantial range of energies. For example, if the expression for the single-nucleon-exchange force is reduced to nonrelativistic form, a Yukawa potential is obtained with the sign $(-1)^{l+J+T}$ (a positive sign corresponding to an attractive interaction). This is roughly in agreement with what one is led to deduce from the observed scattering.

The next step was to seek a sharper understanding by carrying through a more precise version of the

* Work performed under the auspices of the U. S. Atomic Energy Commission.

† Present address: Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California.

[1] G. F. Chew, Phys. Rev. **94**, 1748 (1954); **95**, 1669 (1954); G. F. Chew and F. E. Low, *ibid.* **101**, 1570 (1956).

[2] U. Haber-Schaim, Phys. Rev. **104**, 1113 (1956).

[3] S. Mandelstam, Phys. Rev. **112**, 1344 (1958); **115**, 1741 and 1752 (1959).

[4] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960), referred to hereafter as F–W.

Frautschi-Walecka calculation. This was important because there were two somewhat contradictory indications of the strength of the single-nucleon-exchange force in the 3, 3 amplitude. The estimate of F–W put the resonance at too low an energy, suggesting that the force was too strong to be consistent with experiment. On the other hand, a different technique of estimation by Baker,[5] confirmed by Martin and Wali,[6] put the resonance at too high an energy and suggested that the one-nucleon force was, in fact, too weak. In this situation an optimist might have been led to hope that a more careful calculation would give a result lying between the two extremes.

A recent calculation by Abers and Zemach[7] (see also Ball and Wong[8]) suggested, however, that the F–W result was closer to the truth. In fact, F–W underestimated the strength of the one-nucleon-exchange force, which was found by Abers and Zemach to be strong enough to give a 3, 3 bound state. There are some features of the A–Z calculation that require further investigation, to be provided here, but the qualitative conclusion is confirmed by our own work. One purpose of this work was to do a careful version of the A–Z calculation in order to investigate the strength of the one-nucleon-exchange force.

It is not difficult to state the problem we set out to solve: to find partial-wave pion-nucleon scattering amplitudes that satisfy elastic unitarity on the physical cuts, that have the correct threshold behavior, and that have the dynamic singularities of the single-nucleon-exchange force. One might hope that this problem is well defined. Certainly, if one hopes to carry through a "bootstrap" program[9] for meson-baryon systems, this type of problem must be confronted and solved.

In setting up the calculation, however, we found ourselves afflicted by a plague of ambiguities. This paper is devoted mainly to the analysis of these ambiguities and an attempt to resolve them. The fact that some of these difficulties exist has been recognized by a number of authors,[10] but the specific question we propose to answer has not, to our knowledge, been satisfactorily dealt with in the literature. It is somewhat as an afterthought that we are able to report upon the original subject of investigation, the strength of the one-nucleon-exchange force.

The next section of the paper is devoted to the technical details of the analysis. In it we consider the implications of requiring partial-wave scattering amplitudes to have "correct" threshold behavior. The requirement turns out to bear upon the asymptotic growth of the amplitudes and some rather severe restrictions can be read out of a standard mathematical theorem that is reproduced in Appendix II. Section III qualitatively describes the results of our numerical calculations. The detailed numerical results are not considered to be of sufficient physical interest to warrant their inclusion. Section IV is devoted to our conclusions and contains suggestions for further work. Our notation and some of the relevant formulas are consigned to Appendix I. We remark that a previous paper[11] by one of us contains erroneous equations that are correctly given in the Appendix.

II. ANALYSIS

We now focus our attention upon the partial-wave amplitudes as analytic functions of the total center-of-mass energy W. As a convenient point of reference we choose the familiar partial-wave amplitudes[4,12] $f_{l\pm}(W)$ (see Appendix I for a summary of the notation) in terms of which other amplitudes will be defined. The amplitudes $f_{l\pm}(W)$ satisfy the "elastic" unitarity condition $f(W)=q^{-1}\times\sin\delta e^{i\delta}$ (δ real) in the physical region and are related through the MacDowell symmetry[13] $f_{l+}(-W)=-f_{l+1,-}(W)$. It is sufficient, therefore, to work with the amplitudes $f_{l+}(W)$ alone; we denote them by $f_l(W)$.

Let us now formulate the mathematical problem. The amplitude $f_l(W)$ is a real analytic function in the cut W plane[13,14]; it satisfies the elastic unitarity condition on both the left- and right-hand physical branch cuts and has a specified discontinuity across certain dynamic branch cuts (as well as possible specified poles). Further, we suppose that the only dynamic singularities are those that arise from the single-nucleon-exchange forces. Finally, we demand that the amplitude $f_l(W)$ exhibit the "correct"[15] threshold zeros at the two physical thresholds, namely, $f_l(W)\propto q^{2l}$ for $W\approx M+\mu$ and $f_l(W)\propto q^{2l+2}$ for $W\approx -(M+\mu)$.

[5] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).
[6] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).
[7] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963), referred to hereafter as A–Z.
[8] J. S. Ball and D. Y. Wong, Phys. Rev. **133**, B179 (1964).
[9] See, for example, F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[10] We gather that the authors of Ref. 4, 7, and 8, and also G. F. Chew and C. E. Jones [University of California Lawrence Radiation Laboratory (Berkeley) Report UCRL-10992, August 1963 (unpublished)] and undoubtedly many others who have worked in this field, are aware of the ambiguities to which we refer.

[11] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961). Dr. Kotani has kindly pointed out to one of us that several of the formulas of that paper are in error. The correct forms are contained in Appendix I of the present paper. As remarked in the text, the results of the two analyses are qualitatively the same.
[12] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).
[13] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).
[14] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).
[15] From a purely logical standpoint, this requirement is not at all straightforward. The threshold behavior that we specify is well known to be a rigorous consequence of the assumption that the scattering amplitude satisfies a Mandelstam representation. The demonstration requires, however, that the "dynamic branch cuts" be considerably more complicated than those arising from one-nucleon exchange. Thus, it becomes a matter of taste as to which consequences of the Mandelstam representation should be kept in the formulation of an approximate problem. One of us (J. L. U.) thanks Professor Charles Zemach for some interesting discussions of this point.

We are almost ready to incorporate all of these statements into a Cauchy integral representation and thereby gain for ourselves a nonlinear "dispersive" integral equation for the amplitude $f_l(W)$. Before doing this it is necessary to discuss the asymptotic behavior of the amplitude. Let us first suppose that the amplitude $f_l(W)$ vanishes at infinity and so may be expressed by an unsubtracted dispersion integral. Let the contour of integration be such that it borders all branch cuts on both sides and is closed on the arcs at infinity. It is trivial to show[11] that the integration along the contours bordering the dynamic branch cuts reproduces the Born approximation one-nucleon-exchange amplitude $f_l^B(W)$. The possibility of this integration giving an additional entire function is eliminated by the assumed asymptotic behavior of $f_l(W)$ and the known asymptotic behavior of $f_l^B(W)$. Then, using the elastic unitarity condition on the two physical cuts, we may write

$$f_l(W)=f_l^B(W)+\frac{1}{\pi}\int_{M+\mu}^{\infty}dW'\frac{q(W')|f_l(W')|^2}{W'-W}-\frac{1}{\pi}\int_{M+\mu}^{\infty}dW'\frac{q(W')|f_l(-W')|^2}{W'+W}. \quad (1)$$

While this amplitude clearly satisfies unitarity and possesses the desired dynamic singularities, it will not ordinarily possess the desired threshold behavior. The reason for this failure is that, for single-particle exchanges, the Born approximation itself exhibits the "correct" threshold behavior and, as a result, cannot cancel the finite threshold contributions of the integrals over the physical cuts. While it is possible that the two positive-definite integrals (for W at one of the thresholds) might cancel each other, this cannot happen simultaneously at both thresholds and, therefore, is not the answer to the difficulty. Instead, the answer is found in the approximation of taking only single-particle-exchange contributions to the "driving force" represented by $f_l^B(W)$. Higher order contributions, those that provide the double spectral functions to the Mandelstam representation, do not, in general, vanish at the thresholds and are required to achieve the "correct" threshold behavior[16] of the f_l's.

[16] Although we have shown that the approximate f_l^B's coming from single-particle exchanges *cannot* give the correct threshold behavior to the f_l's in Eq. (1), it is certainly not apparent that this trouble would be completely alleviated if the exact dynamical cuts (f_l^B) were specified. The point is that although the Mandelstam double-dispersion relation implies a certain threshold behavior for the partial-wave amplitudes, the Mandelstam representation does not contain a unitarity requirement. On the other hand, the right-hand side of Eq. (1) contains the unitarity requirement and, in consequence, need no longer contain the threshold behavior of the Mandelstam representation. The difficulty probably stems from the fact that solutions of "dispersive" integral equations are generally nonunique. Thus, it is likely that if special properties, such as threshold behavior, are to be required of the solution, then they must be imposed at the outset.

We must therefore attempt to supply the correct threshold zeros as an added requirement upon the partial-wave amplitudes. The conventional scheme for accomplishing this is to "disperse" in amplitudes with the zeros divided out. Thus (with intentional sloppiness in our treatment of the "left-hand" threshold in order to keep the argument simple), we consider the new amplitudes

$$h_l(W)=(W^k/q^{2l})f_l(W)\equiv\rho_l(W)f_l(W) \quad (2)$$

and remark that the factor $\rho_l(W)$ has the behavior

$$\rho_l(W)\propto W^{-(2l-k)} \quad \text{for large } W, \quad (3a)$$

$$\rho_l(W)\propto W^{2l+k} \quad \text{for vanishing } W. \quad (3b)$$

The integral exponent k is not specified except that it is obviously required to be less than or equal to $2l$. It is supposed that $h_l(W)$ satisfies the unsubtracted dispersion relation

$$h_l(W)=h_l^B(W)+\frac{1}{\pi}\int_{M+\mu}^{\infty}dW'q(W')\rho_l^{-1}(W')\times\left[\frac{|h_l(W')|^2}{W'-W}-(-1)^k\frac{|h_l(-W')|^2}{W'+W}\right], \quad (4)$$

where $h_l^B(W)=\rho_l(W)f_l^B(W)$.

Let us now consider some of the implications of Eq. (4). First, we recognize that along the real axis a solution of the equation will satisfy the elastic unitarity condition. In particular, $h_l(W)$ must fall off at least as fast as $W^{-(2l+1-k)}$ for large, real W. Since $h_l^B(W)$ can be shown to have this behavior (along every ray in the W plane), it follows that[17]

$$\lim \text{l.u.b.}|W[f_l(W)-f_l^B(W)]|<\infty, \quad W\to\pm\infty. \quad (5)$$

Along the imaginary axis, on the other hand, Eq. (4) becomes

$$\hat{h}_l(iy\equiv h_l(iy)-h_l^B(iy)=\frac{1}{\pi}\int_{M+\mu}^{\infty}dW'\frac{q(W')\rho_l^{-1}(W')}{W'^2+y^2}[(W'+iy)|h_l(W')|^2-(-1)^k(W'-iy)|h_l(-W')|^2]. \quad (6)$$

From this it follows that for k even (odd), the imaginary (real) part of $\hat{h}_l(iy)$ cannot fall off any faster than $y^{-1}(y^{-2})$. The conclusion is then immediate that $f_l-f_l^B$ will grow at least as fast as

$$|W[f_l(W)-f_l^B(W)]|\gtrsim\left\{\begin{matrix}|W|^{2l-k} & k\text{ even}\\ |W|^{2l-k-1} & k\text{ odd}\end{matrix}\right\}W\to\pm i\infty. \quad (7)$$

[17] It is necessary to call upon the least upper bound (l.u.b.) because $\lim|Wf_l(W)|$ need not exist; the function $Wf_l(W)$ may oscillate indefinitely. We would disagree vigorously with analyses that assume, without justification, that the limit exists, e.g., those of M. Sugawara and A. Kanazawa, Phys. Rev. **126**, 2251 (1962) and A. P. Balachandran, J. Math. Phys. **5**, 614 (1964).

At this point it would be handy to have a theorem to invoke, and it turns out that one of the Phragmen-Lindelöf theorems (Appendix II) is exactly what is needed. It allows us to conclude that if k is not large enough, then an $f_l(W)$ that satisfies both conditions (5) and (7) must be of exponential growth and, therefore, cannot satisfy the dispersion relation (4). The precise condition on k is $k \geqslant 2l$ for k even or $k \geqslant 2l-1$ for k odd. Of these two, the natural choice is $k=2l$. [Note that the estimate for k odd, Eq. (7), is generous in that it depends on the cancellation of two positive definite integrals—a circumstance not likely to arise in actual calculations.] Thus, we conclude that (still ignoring the correct treatment of the left-hand physical threshold) we should disperse with the amplitudes

$$h_l(W)=(W^2/q^2)^l f_l(W). \tag{8}$$

This conclusion leads us to another observation that we think is of considerable interest. Suppose that $h_l(W)$ [Eq. (4)] is nonvanishing at the origin. It follows from Eq. (8) that $f_l(W)$ will have a pole there of order $4l$. But why select $W=0$? All of the preceding argument would have gone through unchanged if we had replaced W^k by a kth-order polynomial in W, thus providing $f_l(W)$ with a k-fold collection of poles (and k arbitrary constants) in the finite W plane to accompany the pole of order $2l$ that occurs at the origin as a result of the factor q^{2l}. One has a strong suspicion that these necessary additional poles should collect at the origin, but we have not been clever enough to prove this.

There is yet another difficulty to be settled before we can proceed to the solving of equations. The quantity q^2 vanishes not only at the MacDowell-symmetric thresholds $W=\pm(M+\mu)$, but also at the thresholds $W=\pm(M-\mu)$ of the "crossed" pion-nucleon channels. The $f_l(W)$ defined by Eq. (8) therefore has l-fold zeros also at the "crossed" thresholds. Do we in fact want these?

We are of the opinion that the answer to this question follows from an analysis made by Frye and Warnock.[18] They show that, as a consequence of crossing symmetry, the exact partial-wave amplitudes $f_l(W)$ cannot have zeros at $\pm(M-\mu)$. Instead, these points are branch-point singularities of the amplitude. It is now possible to argue that our formalism should be one that would give the correct answer if the exact dynamical branch cuts and discontinuites were specified. Therefore, in this spirit, we must avoid forcing these extra zeros upon the amplitudes $f_l(W)$.

The ultimate result of the preceding considerations now becomes evident. The amplitude $f_l(W)$ has an l-fold zero at $(M+\mu)$, and an $(l+1)$-fold zero at $-(M+\mu)$. These must be divided out by a polynomial of degree $2l+1$. On the other hand, the analysis based upon the Phragmen-Lindelöf theorem indicates that we must choose a kinematic factor $\rho_l(W)$ [Eq. (2)] which approaches a nonvanishing constant as W tends to infinity. Thus we must introduce $2l+1$ simple poles into the amplitude $f_l(W)$ and we believe it appropriate to put them all at the origin of the W plane. In this way we are led to consideration of an amplitude

$$h_l(W)=W^{2l+1}\{(W+M+\mu)[W^2-(M+\mu)^2]^l\}^{-1}f_l(W) \equiv \rho_l(W)f_l(W), \tag{9}$$

which we assume has no poles [apart from those in the driving force $h_l^B(W)$] in the complex W plane and falls off as $|W|^{-1}$ for large W.

It is now clear that we may write a dispersion relation like Eq. (4) for the $h_l(W)$ of Eq. (9) and treat it by the conventional N/D method[19] to obtain an integral equation for N. Also, the kernel of the integral equation is readily shown to be of the Fredholm type so that the suitability of elementary numerical techniques is assured.

III. RESULTS

The integral equations for the different partial waves (see Appendix I) were approximated by linear algebraic equations, in the usual way, by use of trapezoidal-rule numerical integrations. These algebraic equations were inverted with the aid of Argonne's CDC-3600 computer, and the matrix size was varied from 40×40 to 80×80 to verify that truncation errors were not serious. The high-energy end points were at ±400 pion masses. Varying this had little effect upon the low-energy results.

The integral equations were solved for orbital angular momenta $0\leqslant l\leqslant 4(\frac{1}{2}\leqslant J\leqslant \frac{9}{2})$ for both isospin states in pion-nucleon scattering. The computer program was so constructed that the amplitude $h_l(W)$ obtained from the N/D equations was reinserted into the original nonlinear dispersion relation in order to verify that a solution had, in fact, been obtained. Having made the choice of amplitude [Eq. (9)], we have no arbitrary parameters in the calculation, the N/D representation (with a once-subtracted dispersion relation for D) being independent of the choice of subtraction point. In discussing the results we turn first to the 3, 3 amplitude.

The solution of the N/D equations for the $T=\frac{3}{2}$, $J=\frac{3}{2}$ amplitude was negative (repulsive-like) for positive W (p-wave) and, when inserted back into the dispersion relation, was found not to be a solution of that equation. It was presumed that this indicated that the one-nucleon-exchange force was sufficiently attractive to produce a bound 3, 3 state. We tested this notion by examining the behavior of the amplitude as a function of the coupling constant g^2, ($g^2/4\pi=14.5$ was assumed to be the "physical" value). We found that a p-wave resonance first occurred when $g^2/4\pi$ was

[18] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

[19] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

about 4.5, and that the resonance moved in and became a bound state at about $g^2/4\pi=6.2$. These results provide qualitative confirmation of the calculation of Abers and Zemach who also concluded that the one-nucleon-exchange force was strong enough to give a bound p-wave 3, 3 state.

Another aspect of the 3, 3 amplitude is thought worthy of comment. This is the behavior as a function of the coupling constant of the 3, 3 d wave, which, it will be recalled, is the MacDowell partner of the 3, 3 p wave. When g^2 was small, the d-wave phase shift was found to be attractive. This would naturally be inferred from the sign of the one-nucleon-exchange term, which is positive in both the $p_{3/2}$ and $d_{3/2}$ states[20] (for isospin $T=\frac{3}{2}$). However, as the coupling constant was increased and the p-wave phase shift approached a resonant behavior ($g^2/4\pi\gtrsim 4$), the d-wave phase shift became repulsive. In other words, the unitarity integral over the physical p-wave branch cut represents a repulsive driving force in the d-wave amplitude which dominates the one-nucleon-exchange term even though, with regard to location, the physical p-wave cut is much farther away. This was found to be a general characteristic of our solutions; an attractive driving force in an $f_{l+}(W)$ amplitude resulted in a dominant repulsion in the related $f_{l+1,-}(W)$ amplitude, contrary to expectations based on the sign of the one-nucleon-exchange term.

The characteristics of our solutions as a whole may be summarized as follows. When the one-nucleon-exchange driving force (with $g^2/4\pi$ set at 14.5) was repulsive in a given amplitude, the solution of the N/D equations failed to satisfy the original dispersion relation. We interpret this as evidence for "ghost" poles in the N/D "solution." In particular, the s-wave amplitudes, which are qualitatively the same as those reported in Ref. 11, apparently contain ghosts so close to the physical region that no reliance can be placed upon the results. This feature of the solutions was further examined by reducing the magnitude of the coupling constant. It was found that for small enough g^2 ($g^2/4\pi\approx 4$) the dispersion relations were reasonably well satisfied. This indicated that the ghost poles had receded in the complex W plane.

For those amplitudes in which the driving force was attractive, on the other hand, the N/D solutions satisfied the dispersion relation check quite well, even with the coupling constant set at its "physical" value. (This does not hold, of course, for the 3, 3 bound-state solution since the bound-state pole was not included in the dispersion relation.) The distinctive aspect of the "attractive" $f_{l+}(W)$ amplitudes was that they nearly always resonated. The only exception found was the $T=\frac{1}{2}$, $g_{9/2}$ amplitude, for which the phase shift reached a maximum value of 52°.

[20] The "exchange" property of having opposite signs for odd and even l is only true in the limit of very large nucleon mass.

Finally, we considered it of interest to study the practical consequences of our assertion that the extra zeros of q^2 should not be imposed upon the amplitudes $f_l(W)$. To this end the calculation was repeated with the kinematic factor

$$\rho_l(W)=W^{2l+2}\{[(W+M)^2-\mu^2]q^{2l}\}^{-1}. \qquad (10)$$

For the most part the results of the two calculations were strikingly similar, significant differences occurring in only a few amplitudes. The $T=\frac{1}{2}$, $p_{1/2}$ amplitude (the amplitude in which the direct nucleon pole term makes the dominant contribution) was found to have a much stronger repulsion with the kinematic factor of Eq. (10). Another distinction concerns the $f_{l+1,-}(W)$ amplitudes in which the driving force is attractive. With our initial choice of $\rho_l(W)$, it will be recalled, the solutions evidenced a dominant repulsion. With the $\rho_l(W)$ of Eq. (10), however, the phase shifts for these amplitudes are attractive at low energy and then swing over to become repulsive at higher energy.

A simple way to view the differences between the two choices of kinematic factor is the following observation. The extra zeros and poles imposed upon the amplitudes by the $\rho_l(W)$ of Eq. (10) tend to isolate the MacDowell symmetric amplitudes from each other. That is, the role of one "unitarity branch cut" as a driving force in the other amplitude is reduced. One consequence is the behavior of the $f_{l+1,-}(W)$ noted above. Another is the fact that the $f_{l+}(W)$ amplitudes in which the driving force is attractive resonate at lower energies. The last point to be made is that both choices of kinematic factor led to the same "ghost" difficulties.

IV. CONCLUSIONS

We should now like to make some remarks concerning the sufficiency of the single-nucleon-exchange force for predicting features of pion-nucleon elastic scattering. We must, however, admit to a certain feeling of uneasiness in trying to discuss these matters in the context of the "dispersive" or "N/D" calculations that have been used. The high-order pole that we have arbitrarily chosen to put at the origin of the W plane seems to have very little to do with physics and almost certainly arises from an inadequate definition of the scattering "potential." We remind the reader that the same difficulty is present whenever the "potential" (or left-hand cut) is derived from single-particle exchanges and occurs in other treatments of the pion-nucleon scattering problem.[7,8]

Nevertheless, if we suppress our doubts and proceed to draw conclusions, then there is one that seems inescapable. The one-nucleon-exchange force, *as we have applied it*, is much too strong to be held accountable for the 3, 3 resonance. Of course, given an excessively strong force it is always possible to weaken it by the imposition of a cutoff as was done by Abers and Zemach[7] and by Ball and Wong,[8] and it is not surprising that they were able to find a 3, 3 resonance by such a procedure. Whether this demonstrated that we now have a deep understanding of low-energy pion-nucleon scattering is another question. Let us recall that the one-nucleon-exchange force is attractive in roughly half of the partial-wave states. The probability that it can be used to predict any given low-energy resonance is not hard to calculate. At any rate, it would appear that we have not progressed markedly beyond Chew's static model. The 3,3 resonance is still obtained from a two-parameter calculation just as it was in earlier days.

The excessive strength of the one-nucleon-exchange force is not, as we have already noted, confined to the 3,3 state. All of the $f_{l+}(W)$ amplitudes (up to $J=\frac{9}{2}$) in which the driving force is attractive are found to have low-energy resonances that seem to have little to do with physics. Clearly, this case does not share the good fortune of, say, nucleon-nucleon scattering for which the one-particle-exchange terms adequately reproduce the low-energy behavior of the higher partial waves.

What, then, is the interpretation of the experimental data if the one-nucleon-exchange force is so strong? The most obvious answer is that "short range" forces that we have not taken into account are effective in damping the one-nucleon contribution. Hence, one finds the need for a cutoff. Ball and Wong[8] obtained reasonable fits to the data for a number of partial waves using the same cutoff for each partial wave. The trouble here is that the required cutoff is at such a low energy (about 2.5 nucleon masses). It seems very unlikely that the ignored forces could conspire to give the same effective *low-energy* cutoff in all angular momentum and isospin states.

A more compelling guess is that we have, in fact, oversimplified the one-nucleon contribution by treating the nucleon as an elementary particle. From this viewpoint, the imposition of a cutoff compensates for the fact that the nucleon "really" is a point on a Regge trajectory. The cutoff then provides a simple, approximate way to Regge-ize the nucleon-exchange contribution.

We would feel more comfortable with the second explanation if we had more confidence in the ability of a "dispersive" formalism to make unique predictions. For example, we have solved dispersion relations for amplitudes so defined that the one-nucleon-exchange force is not strong enough to make the 3,3 state resonate for any finite cutoff. Such solutions, which are meaningful in a cutoff theory, would violate the Phragmen-Lindelöf theorem of Sec. II in the absence of a cutoff [let, for example, the exponent k in Eq. (2) be zero].

Despite the objections of the last paragraph, we do admit to a prejudice in favor of the "Regge-ized nucleon" explanation. It is not hard to visualize how such a concept can be given quantitative meaning in a "bootstrap" calculation. Start with the exchange of an un-Regge-ized nucleon (in the u channel) and no cutoff. The force resulting from this exchange gives rise to a 3,3 trajectory which, when added to the "crossed" channel, will give rise to the nucleon trajectory. In this way one obtains driving forces with built-in effective cutoffs. We anticipate that the iterative process will be a convergent one.

ACKNOWLEDGMENT

We are indebted to Professor Calvin Wilcox for suggesting the applicability of the Phragmen-Lindelöf theorems.

APPENDIX I: INTEGRAL EQUATION FORMULATION

The kinematics of pion-nucleon scattering have been discussed in great detail[4,12] and will not be repeated here. We employ the standard notation in which W is the total energy in the barycentric system, $q(W)$ is the magnitude of the three-momentum in that system, and M and μ are the nucleon and pion masses, respectively. Further, s, t, and μ are the familiar Mandelstam variables, and we use the natural units $\hbar=c=1$. Our basic partial-wave amplitudes are those of Refs. 4 and 12, namely,

$$\begin{aligned}f_{l\pm}(W)=(1/32\pi W^2)\{&[(W+M)^2-\mu^2]\\&\times[A_l+(W-M)B_l]+[(W-M)^2-\mu^2]\\&\times[-A_{l\pm1}+(W+M)B_{l\pm1}]\},\end{aligned}\quad\text{(A1)}$$

$$A_l(W)=\int_{-1}^{1}P_l(x)A(s,t,u)dx,\quad\text{etc.}\quad\text{(A2)}$$

which satisfy the elastic unitarity condition $f_{l\pm}(W)=\sin\delta_{l\pm}\exp(i\delta_{l\pm})/q$ in the physical region for both total-isotopic-spin states. The one-nucleon-exchange driving force $f_{l\pm}{}^B(W)$ for our calculation is obtained from Eqs. (A1) and (A2) by using the invariant amplitudes

$$\begin{aligned}&A^T(s,t,u)=0,\\&B^T(s,t,u)=\alpha_T g^2/(M^2-s)-\beta_T g^2/(M^2-u),\end{aligned}\quad\text{(A3)}$$

where the familiar isotopic-spin dependence is $\alpha_T=3(0)$ and $\beta_T=-1(2)$ for total isospin $T=\frac{1}{2}(\frac{3}{2})$. From Eq. (A2) it follows that

$$\begin{aligned}A_l{}^T(W)=0,\quad B_l{}^T(W)=&2\delta_{l0}\alpha_T g^2/(M^2-W^2)\\&+(\beta_T g^2/q^2)Q_l(\epsilon),\end{aligned}\quad\text{(A4)}$$

for the one-nucleon-exchange force, where $Q_l(\epsilon)$ is the Legendre function of the second kind with argument

$$\epsilon=1-(W^2-M^2-2\mu^2)/2q^2.\quad\text{(A5)}$$

Using the partial-wave amplitudes $h_l(W)$ which are finite at both thresholds, we carry out the N/D separation in the usual way and obtain

$$h_l(W)=N_l(W)/D_l(W)\,, \tag{A6}$$

$$N_l(W)=\frac{1}{2\pi i}\int_{\text{d.c.}}\frac{h_l(W')D_l(W')dW'}{W'-W}, \tag{A7}$$

$$D_l(W)=1-\frac{(W-W_0)}{\pi}\times\int_{\text{p.c.}}\frac{q(W')N_l(W')dW'}{\rho_l(W')(W'-W_0)(W'-W)}, \tag{A8}$$

where the integral over the physical cuts (p.c.) is

$$\int_{\text{p.c.}}=\int_{-\infty}^{-(M+\mu)}+\int_{(M+\mu)}^{\infty} \tag{A9}$$

and we use the convention $q(-W)=q(W)$. The integration over the dynamic cuts (d.c.) in Eq. (A7) is along a contour bordering these cuts on both sides. Inserting (A8) into (A7) and using

$$\frac{1}{2\pi i}\int_{\text{d.c.}}\frac{h_l(W')dW'}{W'-W}=h_l^B(W)\,, \tag{A10}$$

we obtain the linear inhomogeneous integral equation[11] for $N_l(W)$, namely,

$$N_l(W)=h_l^B(W)+\frac{1}{\pi}\int_{\text{p.c.}}\frac{dW'q(W')N_l(W')}{\rho_l(W')(W'-W)}\times\left[h_l^B(W')-\frac{(W-W_0)}{(W'-W_0)}h_l^B(W)\right]. \tag{A11}$$

It is a trivial matter to prove that the solution of (A11) along with (A8) leads to an amplitude $h_l(W)$ which is independent of the subtraction point W_0.

APPENDIX II: PHRAGMEN-LINDELÖF THEOREM

We have made use of the following theorem (with a trivial modification) stated in Ref. 21.

Theorem (Phragmen-Lindelöf): Let $f(z)$ be regular in the half-plane $y>0$, continuous in the closed half-plane $y\geqslant 0$, bounded on the real axis ($|f(x)|\leqslant M$), and $f(z)=\mathcal{O}(e^{r^\beta})$, $\beta<1$, uniformly in θ, for a sequence $r=r_n\to\infty$. Then $|f(z)|\leqslant M$ for $y\geqslant 0$. Here $f(z)=\mathcal{O}[g(z)]$ means that $f(z)/g(z)$ is bounded on the specified values of $z=re^{i\theta}$.

The proof is so concise that we repeat it here. Consider

$$F(z)=f(z)\exp(-\epsilon z^\gamma e^{-i\gamma\pi/2})\,,$$

where $\beta<\gamma<1$ and $\epsilon>0$. Then

$$|F(z)|=|f(z)|\exp[-\epsilon r^\gamma\cos\gamma(\theta-\pi/2)]\,,$$

or

$$|F(z)|\leqslant|f(z)|\exp[-\epsilon r^\gamma\cos(\gamma\pi/2)]\,.$$

By construction $|F(z)|\to 0$ on the sequence of radii r_n for $0\leqslant\theta\leqslant\pi$. If r_n is large enough, $|F(z)|\leqslant M$ on $|z|=r_n$, $0\leqslant\theta\leqslant\pi$. Similarly, $|F(x)|\leqslant|f(x)|\leqslant M$ for $x\leqslant r_n$. That is, $|F(z)|\leqslant M$ everywhere on the boundary of a semicircular region. Since $F(z)$ is analytic inside this region, it must attain its maximum value on the boundary. It follows that $|F(z)|\leqslant M$ in the entire half-plane since r_n can be arbitrarily large. Hence, $|f(z)|\leqslant|F(z)|\exp(\epsilon r^\gamma)\leqslant M\exp(\epsilon r^\gamma)$ everywhere in the upper half-plane. Letting $\epsilon\to 0$ for each fixed z proves the theorem.

[21] R. P. Boas, Jr., *Entire Functions* (Academic Press Inc., New York, 1954), p. 3.