the two zeros. This is equivalent to the statement that the phase shift is decreasing through $\frac{1}{2}\pi$ at the second zero, because the derivative of the phase shift is proportional to $(-\text{Re}D'/N)$. Thus, our second zero is not due to the Abers-Zachariasen mechanism, and no alternative solution of the Abers-Zachariasen type is present.

ACKNOWLEDGMENTS

The author would like to thank Professor G. Feldman, Professor T. Fulton, and Dr. R. Silbar for useful conversations and for reading the manuscript; Professor R. Schult and Professor M. Grisaru for instructive correspondence and relevant conversation.

PHYSICAL REVIEW VOLUME 135, NUMBER 3B 10 AUGUST 1964

Possible Connection Between Gravitation and Fundamental Length*

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(Received 11 June 1959; revised manuscript received 5 August 1963)

An analysis of the effect of gravitation on hypothetical experiments indicates that it is impossible to measure the position of a particle with error less than $\Delta x \gtrsim \sqrt{G} = 1.6\times 10^{-33}$ cm, where G is the gravitational constant in natural units. A similar limitation applies to the precise synchronization of clocks. It is possible that this result may aid in the solution of the divergence problems of field theory. An equivalence is established between the postulate of a fundamental length and a postulate about gravitational field fluctuations, and it is suggested that the formulation of a fundamental length theory which does not involve gravitational effects in an important way may be impossible.

I. INTRODUCTION

THE presence of divergences in quantum field theory leads one to consider the possibility of modifying the formalism by introducing a fundamental length into the theory. Although the proof by Källén[1] has recently been questioned,[2,3] it still seems not unlikely that the renormalization constants of quantum electrodynamics and other field theories are indeed infinite. Although the renormalization theory permits one to get finite results for physically observable quantities in any order of perturbation theory, the existence of the infinite quantities makes one feel somewhat uneasy about the theory. Moreover, in the model proposed by Lee,[4] it has been shown[5] that the infinite coupling constant renormalization leads in an exact solution to the existence of physically unacceptable "ghost" states, which destroy the unitarity of the S matrix; and it may be[6] that similar difficulties are contained in the more realistic field theories as well.

It is often stated that the divergences arise from the concept of a point particle. This is true, but in a somewhat indirect sense. In the present theory, due to the possibility of pair creation, a single particle cannot be localized more closely than its Compton wave length without losing its identity as a single particle; i.e., if the mass of the particle is m, its position will be uncertain by $\Delta x \gtrsim 1/m$. (Throughout this paper we use natural units: $\hbar = c = 1$.) Therefore, it might be more accurate to say that the divergences arise from the assumption that field quantities (such as electric field strength, charge density, etc.) averaged over arbitrarily small space-time regions are observable in principle, thus making it physically meaningful to make use of local interactions in the theory. The work of Bohr and Rosenfeld[7–9] tells us how these quantities can be measured in the case of quantum electrodynamics using test bodies equipped with springs, etc. However, these authors assume that test bodies having any desired properties can exist in principle. It is clear that the average of a field quantity in a volume V cannot be measured by a test body unless the test body itself is known to be located in the volume under study. It is therefore possible to state that the divergences in a field theory arise, not from the assumption that the particles being studied in the theory are point particles, but from the assumption that point (or arbitrarily

* This research was begun under the auspices of the U. S. Atomic Energy Commission, while the author was a research associate at Brookhaven National Laboratory, Upton, New York.

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[1] G. Källén, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. V, Part I, Chap. VII.

[2] K. A. Johnson, Phys. Rev. **112**, 1367 (1958).

[3] S. G. Gasiorowics, D. R. Yennie, and H. Suura, Phys. Rev. Letters **2**, 513 (1959).

[4] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[5] G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).

[6] L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon Press, Inc., New York, 1955), pp. 52–69.

[7] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **12**, No. 8 (1933).

[8] N. Bohr and L. Rosenfeld, Phys. Rev. **78**, 794 (1950).

[9] L. Rosenfeld, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (Pergamon Press, Inc., New York, 1955), pp. 70–95.

small) particles can exist "in principle," and can be used as test bodies for measuring the various field quantities. For instance, in the case of quantum electrodynamics, in which the particles being studied are electrons and photons, the electron cannot be localized more closely than $\Delta x \gtrsim 1/m$ without losing its identity as a single particle. Nevertheless, the theory contains divergences because the interactions are local, and this depends for its physical interpretation on the assumption that test particles exist in principle which can be localized with unlimited precision.

Therefore, if there should exist a fundamental length[10] ℓ such that no particle can be localized with greater precision than $\Delta x \gtrsim \ell$, it would appear from the above discussion that this would remove the physical cause of the divergences, and incorporation of the fundamental length into the formalism should lead to a natural cutoff for the divergent integrals. Moreover, the existence of such a fundamental limitation on the possibilities of measurement would be of some interest in itself, independently of its effect on the divergence problem.

The above discussion suggests at once a possible physical postulate which would lead to a fundamental length. If we postulate that no elementary particles can exist with mass greater than M, then, as mentioned above, an elementary particle will always have $\Delta x \gtrsim 1/M$, and composite particles will presumably have "radius" $\gtrsim 1/M$ also. Therefore, the best possible test particle can only be used to measure field quantities averaged over regions of dimensions of order $1/M$ in each direction, so that the length $1/M$ satisfies our definition of a fundamental length. If we assume that M is of the order of the masses of the heaviest baryons known at present, we obtain $\ell \sim 10^{-13}$ cm.

Another possibility, which has been speculated on by several authors,[6,11,12] is that the fundamental length might arise in some way from the consideration of gravitational effects. In this case, we would have $\ell \sim \sqrt{G} = 1.6 \times 10^{-33}$ cm, where G is the gravitational constant in natural units.[13] Although the first possibility would probably be favored by most physicists at the present time, this latter idea cannot be ruled out completely, and it is our intention in this paper to examine it from the point of view of a few thought experiments.

Before starting, a few remarks on the meaning of a fundamental length in terms of experimental results might be in order. In the first place, it seems clear that a *single* measurement can always be read to whatever accuracy one pleases. For instance, one could focus light from the body being measured through a pinhole on a photographic plate at a great distance, so that a small change in the position of the body would produce a large change in the position of the spot produced on the plate by a single photon used in the measurement. The uncertainty manifests itself in the nonreproducibility of the results. That is, successive measurements show fluctuations. Another point that should be mentioned is that, for reasons of covariance, if a fundamental length exists we would expect a similar limitation to apply to the reading of a clock. Thus, the physical content of a fundamental length postulate is that successive measurements of the position of a body, or of the reading of a clock, will show fluctuations at least of the order of ℓ.

This paper will consist of two independent parts. The first part (Secs. II–V) deals with the question of whether present physical ideas about gravitation, together with the uncertainty principle, are sufficient to lead to a fundamental length. In these sections, it is initially assumed that one can set up a well-defined Lorentzian coordinate system, and that the position of a particle is a well-defined quantity in this frame of reference. It is then shown that the gravitational effect of the act of measurement on the particle being measured produces an uncontrollable change in its position, such that the result of an immediately subsequent measurement of the same kind cannot be forecast with greater accuracy than $\Delta x \sim \sqrt{G}$. In detail, Sec. II shows that a particle cannot be bound within a region of radius smaller than $R \sim \sqrt{G}$; Sec. III deals with position measurements on free particles, and Sec. IV with clocks; Sec. V shows that the results hold also for measurements on macroscopic bodies. In these sections we make no *a priori* restrictions on the possible properties of elementary particles. Thus it is always assumed, for instance, that the bodies being measured are sufficiently heavy to avoid complications due to pair creation, and the clocks satisfy the relevant criteria of Salecker and Wigner.[14] Hence the possibility is not ruled out of a larger fundamental length due, perhaps, to a maximum elementary particle mass as discussed above. Our policy is always to make the most optimistic assumptions about the realizability of particles having desired properties, so that the resulting limitation on the accuracy of measurement is as general as possible. We also assume that the gravitational interaction has approximately its classical form, at least on the average. The results of these sections cannot be considered as rigorously established, because they deal with only a finite number of thought experiments, and the possibility always exists that someone will be clever enough to design a hypothetical method of measurement which avoids these results. Nevertheless, it is believed that the results are reasonable, and the arguments for them fairly convincing. This is reinforced by the fact that more formal considerations[11,12] lead to similar results;

[10] From now on, whenever the term "fundamental length" is used in this paper, it refers to a length having the physical interpretation discussed here, that is, a limitation on the possibility of measurement.

[11] J. A. Wheeler, Ann. Phys. (N. Y.) **2**, 604 (1957).

[12] S. Deser, Rev. Mod. Phys. **29**, 417 (1957).

[13] In cgs units we would have $\ell \sim (G\hbar/c^3)^{1/2}$.

[14] H. Salecker and E. P. Wigner, Phys. Rev. **109**, 571 (1958).

in this connection, an Appendix to this paper contains a derivation of the fundamental length limitation on clock synchronization which does not depend on the particular method of measurement used.

The second part of the paper (Sec. VI) deals with the relation between fundamental length and gravitational field fluctuations; it is quite independent of the preceding sections. The idea is roughly as follows: Suppose there exists a fundamental length ℓ. Since a space-time coordinate system, to be physically meaningful, must be referred to physical bodies, it follows that no Lorentzian coordinate system can be set up capable of specifying the coordinates of a space-time event more precisely than $\Delta x \gtrsim \ell$. Conversely, if the limitation on the coordinate system holds, the limitation on the localizability of particles follows immediately. Thus, the fundamental length postulate may be equivalently stated as a postulate of a limitation on realizable coordinate systems. Now a coordinate system may be pictured as a distribution of bodies and clocks throughout space; the clocks are synchronized by means of light signals, and the distances between the bodies are known and held constant, also by means of light signals. In this picture, the fundamental length appears as a limitation on the accuracy of the synchronization of the clocks, and of the knowledge of the distances between the bodies. In terms of the light signal experiments this means, for instance, that the time required for a light signal to propagate from body A to body B and back (as measured by a clock at A) is subject to uncontrollable fluctuations. However, from the point of view of general relativity, it is completely equivalent to define the coordinates associated with each body and clock reading by some arbitrary convention, and to regard the light signal experiments as yielding information about the space-time metric associated with the coordinated system so defined. From this point of view, fluctuations in the results of the light-signal experiments are to be regarded as indicating fluctuations in the metric, i.e., in the gravitational field. Thus, it seems qualitatively plausible that a fundamental length postulate is equivalent to a postulate about gravitational field fluctuations. In Sec. VI, this is taken up in more detail, and it is shown, again by means of thought experiments, that such an equivalence does exist. If there is a fundamental length, it leads to uncertainties in the measurement of gravitational fields; on the other hand, if the gravitational field is uncertain, its unknown effect on the motion of a particle between measurements is such as to lead to a fundamental length. The question of the measurability of gravitational fields has been discussed before,[15,16] but not from this point of view.

The results are discussed briefly in Sec. VII.

[15] E. P. Wigner, Revs. Mod. Phys. **29**, 255 (1957); Phys. Rev. **120**, 643 (1960).

[16] Wright Air Development Center Technical Report 57–216; Armed Services Technical Information Agency Document No. AD 118180, 1957 (unpublished).

II. BOUND PARTICLES

Consider a particle of rest mass m bound by some force field in such a way that it is confined in a spherical region of radius R. According to Heisenberg's principle, the uncertainty in the momentum of the particle is given by $\Delta p \gtrsim 1/R$. The gravitating mass M of the particle is equal to its average energy, and is of the order

$$M=\langle(m^2+p^2)^{1/2}\rangle \gtrsim \langle|p|\rangle \gtrsim \Delta p \gtrsim 1/R. \tag{1}$$

The distribution of energy in this case is spherically symmetrical, at least on the average, so we may use the Schwarzschild exterior solution for the gravitational potential outside the region in which the particle is confined.[17] We have, for the potential at a distance $r \gtrsim R$ from the center of the region:

$$\phi=-GM/r \lesssim -G/Rr. \tag{2}$$

Just outside the region in which the particle is confined, (2) reduces to

$$\phi=-GM/R \lesssim -G/R^2. \tag{3}$$

However, the physical interpretation of the general theory of relativity requires[17] that

$$g_{00}=1+2\phi \geq 0. \tag{4}$$

Combining (3) and (4), we obtain

$$R^2 \gtrsim 2G, \quad R \gtrsim (2G)^{1/2}. \tag{5}$$

Equation (5) is the result we were seeking in this section. It tells us that a particle cannot be bound in a region whose radius is less in order of magnitude than $\sqrt{G}$. This means that any particle can be considered more or less free as far as motion over distances of order of magnitude $\sqrt{G}$ or less is concerned. Therefore, to show in general that a particle cannot be localized with less uncertainty than $\Delta x \gtrsim \sqrt{G}$, it will suffice to demonstrate this for free particles, which will be done in the next section.

The distance R used in this section is defined by the equation $\sigma=2\pi R$, where σ is the circumference of a circle drawn around the region in question. The radius of the region as measured directly by ideal measuring rods is somewhat greater[17] than this R, so the inequality (5) holds for this radius as well.

There may be some question as to whether (4) really needs to be required, since, it may be said, it is at least conceivable that some physical interpretation may be given the Schwarzschild solution in the region in which (4) is not satisfied. However, it is easy to show that a signal can never penetrate into this region from outside, so that the region in which (4) fails to hold is certainly

[17] L. D. Landau and E. Lifshitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Press, Inc., Cambridge, Mass., 1951), Chap. 11.

inaccessible to observation. Thus, if a bound particle is to be observable at all, the region in which it is bound must satisfy (5).

III. FREE PARTICLES

A. Microscope Experiment: Nonrelativistic Treatment

The simplest method for measuring the position of a free particle is by means of a microscope. In this experiment, a photon of frequency ν is scattered by the particle into the aperture of a microscope, where it is focused and observed. The direction of the trajectory of the photon from the particle into the microscope is not known exactly but is spread over an angle ϵ. The photon interacts strongly with the particle over a region of radius r. Now if it is desired to measure the x coordinate of the particle by this method, there are several sources of uncertainty. In the first place, due to the limited resolving power of the microscope, we have

$$\Delta x \gtrsim \frac{1}{\nu \sin\epsilon} \gtrsim \frac{1}{\nu}. \tag{6}$$

Also, the photon may be scattered from any point in the region of radius r surrounding the particle. Therefore,

$$\Delta x \gtrsim r. \tag{7}$$

Presumably, $r \sim 1/\nu$, but this will not be needed.

The photon cannot be focused while it is still interacting strongly with the particle; therefore, the time τ which must elapse between the scattering event and the recording of the result must in general be of the order of the time required for the photon to move a distance r away from the particle. Thus, at least $\tau \gtrsim r$.

In this subsection, we treat the problem from a nonrelativistic point of view, in order to give a simple physical picture. The final result is then obtained relativistically in the following subsection. The gravitating mass of the photon is ν, so that during the time when the photon is proceeding from the particle toward the microscope, the particle experiences a gravitational acceleration in the direction of the photon given by

$$a \sim G\nu / r^2. \tag{8}$$

If the particle does not attain relativistic velocities, the time required for the photon to escape from the region of strong interaction is of the order of r, so from (8), if the particle is originally at rest, it acquires a velocity in the direction of the photon given by

$$v \sim G\nu / r. \tag{9}$$

The average velocity of the particle during the process is of the same order of magnitude as the final velocity, so that in the time r it moves a distance

$$L \sim G\nu. \tag{10}$$

This motion is in the direction taken by the photon which, however, is unknown. The projection of the gravitational motion on the x axis, therefore, is uncertain by approximately $L \sin\epsilon$, which gives with (10) a further uncertainty in our knowledge of the x coordinate of the particle at the end of the experiment.

$$\Delta x \gtrsim G\nu \sin\epsilon. \tag{11}$$

Combining (6) and (11), we obtain

$$\Delta x \gtrsim \sqrt{G}. \tag{12}$$

The above derivation is simple, but not entirely correct even from a nonrelativistic point of view, since it does not take proper account of the conservation of momentum in assuming that the particle is at rest at the time the photon starts toward the microscope. Another derivation which exhibits the momentum conservation explicitly, is the following:

Due to the gravitational attraction of the particle, the photon acquires an increased energy and momentum while it is in the vicinity of the particle. If the momentum of the photon when it is far from the particle (such as when it is focused in the microscope) is ν, its momentum during the scattering process is of the order

$$k \sim \nu + mG\nu / r. \tag{13}$$

The direction of this momentum is unknown, however, since the direction taken by the photon is spread over the angle ϵ. This leads to an uncertainty in the x component of the momentum of the particle during the scattering process given by

$$\Delta p_x \sim k \sin\epsilon \sim (\nu + mG\nu / r) \sin\epsilon, \tag{14}$$

with a corresponding velocity uncertainty

$$\Delta v_x \sim \left(\frac{\nu}{m} + \frac{G\nu}{r}\right) \sin\epsilon. \tag{15}$$

The extra velocity due to the gravitational interaction will be removed by the gravitational pull of the photon as it moves away, but will persist while the photon is in the vicinity of the particle, that is, at least for a time $\sim r$. This leads to an uncertainty in the final position of the particle of the order

$$\Delta x \gtrsim r \Delta v_x \sim \left(\frac{\nu r}{m} + G\nu\right) \sin\epsilon \gtrsim G\nu \sin\epsilon. \tag{16}$$

(16) is identical with (11), and can again be combined with (6) to give the desired final result (12).

The basic idea in both of these derivations is very simple: In order to reduce the uncertainty (6), it is necessary to use photons of very high energy; but a high-energy photon carries with it a strong gravitational field, which tends to move the particle. Moreover, the effect of a gravitational field on the motion of the particle is independent of the nature of the particle, a property possessed by no other force. Thus, while the position uncertainty arising from any other force (e.g., electromagnetic) could in principle be made arbitrarily small simply by measuring the position of a very heavy particle, the result (12) is independent of the mass of the particle being measured and is therefore a fundamental limitation on the possibility of localizing any conceivable particle.

The discussion in this subsection suffers from a failure to consider relativistic effects, and indeed it is easy to see that the nonrelativistic approximation breaks down just at the point where the gravitational interaction becomes the dominant source of error. The uncertainty due to (11) becomes comparable with that due to (6) and (7) at about the same values of ν and r at which the velocity given by (9) becomes of the order of unity. Moreover, the Newtonian law of attraction (8) is not valid for the case of a gravitational field which is changing rapidly with time, such as that of a photon. Nevertheless, the result (12) is still correct, as will be shown in the next subsection.

B. Microscope Experiment: General Relativistic Treatment

In this subsection we wish to establish that a result similar to (10) holds in the general relativistic treatment of the gravitational interaction. Since in a general discussion it is necessary to consider the possibility of particles other than photons being used to locate the particle, and since this causes no special difficulties, we will proceed to calculate approximately the gravitational field of a test particle of rest mass μ and momentum k. The case of the photon is obtained by letting $\mu\rightarrow 0$. Its energy ν and velocity v are then given by

$$\nu=(\mu^2+k^2)^{1/2},$$
$$v=k(\mu^2+k^2)^{-1/2}. \tag{17}$$

In calculating the detailed motion of the particle being measured under the gravitational action of the test particle, the time dependence of the gravitational field is quite important, and this, of course, involves retardation effects in an important way.[18] However, we do not need to calculate the detailed motion, and for our purposes it will suffice to know the average value of the gravitational potential while the two particles are in interaction, without inquiring as to how it came about. To this end, we can transform the test particle to rest by a Lorentz transformation, use the Schwarzschild solution for its gravitational field, and then transform back to the laboratory coordinate system. Since the Schwarzschild solution is a static solution, this neglects the fact that the state of motion of the test particle changes during the experiment. However, its gravitational field should in a rather short time attain (at least in order of magnitude) the value it would have if the motion of the test particle had been uniform in the direction taken after the scattering. We therefore proceed as follows.

We choose a coordinate system with x^1 axis in the direction taken by the test particle after the scattering; the origin is a point which we take to be the beginning of the motion of the test particle in the x^1 direction. The world line of the test particle is thus $x^1=vx^0$, for the portion of its motion that we are considering. The measured particle is somewhere "behind" the test particle, that is, its x^1 coordinate is less than vx^0. We now transform the test particle to rest by a Lorentz transformation, denoting the rest frame by primes. In the rest frame, the measured particle moves along the negative $(x')^1$ axis. Along this axis, the metric tensor is given by[17]

$$(g')_{11}=\frac{-1}{1+2\phi'};\quad (g')_{00}=1+2\phi';$$
$$(g')_{22}=(g')_{33}=-1;\quad (g')_{ij}=0,\quad i\neq j, \tag{18}$$

where

$$\phi'=-\frac{G\mu}{|(x')^1|}=\frac{G\mu}{(x')^1}. \tag{19}$$

The primed and unprimed coordinates are related by the transformation

$$\begin{aligned}
&(x')^2=x^2;\quad (x')^3=x^3,\\
&(x')^1=(x^1-vx^0)(1-v^2)^{-1/2};\\
&(x')^0=(x^0-vx^1)(1-v^2)^{-1/2},\\
&\quad x^1=[(x')^1+v(x')^0](1-v^2)^{-1/2};\\
&\quad x^0=[(x')^0+v(x')^1](1-v^2)^{-1/2}.
\end{aligned} \tag{20}$$

To transform the metric tensor given by (18) to the unprimed system, we use the general transformation law for a second-order covariant tensor

$$g_{ij}=\frac{\partial(x')^k}{\partial x^i}\frac{\partial(x')^l}{\partial x^j}(g')_{kl}, \tag{21}$$

where we use the usual Einstein summation convention. To find the components of the unprimed metric tensor in the region of interest to us, we use (18), (19), (20),

[18] The two-body problem in general relativity has been treated systematically by B. Bertotti, Nuovo Cimento **12**, 226 (1954); **4**, 898 (1956).

and (21), and easily obtain:

$$g_{22}=g_{33}=-1;\quad g_{20}=g_{30}=g_{21}=g_{31}=g_{23}=0,$$

$$g_{00}=\frac{1+2\phi}{1+2\phi(1-v^2)}+2\phi;$$

$$g_{11}=\frac{-1+2\phi v^2}{1+2\phi(1-v^2)}+2v^2\phi;$$

$$g_{10}=g_{01}=\frac{-2v\phi}{1+2\phi(1-v^2)}-2v\phi, \tag{22}$$

where

$$\phi=\frac{\phi'}{1-v^2}=-\frac{G\nu}{r}, \tag{23}$$

and $r=vx^0-x^1$ is again the mean "distance" between test particle and measured particle during the scattering process.

We now proceed to discuss the sources of error, and inquire whether it is possible to have $\Delta x\ll\sqrt{G}$. First, to avoid a breakdown of space-time structure in the rest frame, we must have [cf. Eq. (4)]:

$$-2\phi'=2(G\nu/r)(1-v^2)<1. \tag{24}$$

It is clear that (6) and (7) continue to hold. Combining them with (24), we find

$$\Delta x^2\gtrsim r/\nu\gtrsim 2G(1-v^2). \tag{25}$$

Therefore, we see that we certainly cannot have $\Delta x^2\ll G$ unless

$$(1-v^2)\ll 1,\quad \text{and}\quad -\phi=G\nu/r\gg 1. \tag{26}$$

Thus, we need only consider the case where (26) is satisfied. Using (26), we can set $v=1$ in (22), which then becomes

$$g_{00}=\frac{1+2\phi(1+\alpha)}{\alpha}=\frac{1-2(G\nu/r)(1+\alpha)}{\alpha},$$

$$g_{11}=\frac{-1-2(G\nu/r)(1+\alpha)}{\alpha}, \tag{27}$$

$$g_{10}=g_{01}=\frac{2(G\nu/r)(1+\alpha)}{\alpha},$$

where

$$\alpha=1+2\phi(1-v^2). \tag{28}$$

Because of (24),

$$0<\alpha<1. \tag{29}$$

If the reader has any doubts as to the validity of (27) in the limit $\mu\rightarrow 0$, he may remove them by directly substituting (27), along with its extension into other regions of space, into the Einstein field equations, with the energy-momentum tensor of a massless particle moving with constant momentum.

The components of the metric tensor "seen" by the measured particle during the time we are concerned with are given approximately by (27). We do not need to calculate its motion in detail, but merely note that its world line must be time-like. That is, if u is the velocity of the measured particle in the x^1 direction, we must have[19]

$$ds^2=\{g_{00}+2g_{10}u+g_{11}u^2\}(dx^0)^2\geq 0. \tag{30}$$

(27) and (30) may be combined to give

$$u\geq(\eta-1)/(\eta+1), \tag{31}$$

where

$$\eta=(2G\nu/r)(1+\alpha). \tag{32}$$

The two particles remain in interaction until the test particle has moved a distance r away from the measured particle. Since the velocity of the test particle cannot be greater than 1, the time τ required for it to move a distance r from the measured particle is at least $\tau\gtrsim r/(1-u)$. The distance moved by the measured particle during this time is

$$L=u\tau\gtrsim\frac{ur}{1-u}\geq\frac{r(\eta-1)}{2}\sim\frac{r\eta}{2}, \tag{33}$$

since by (26) and (29) $\eta\gg 1$. From (29), (32), and (33) then,

$$L\gtrsim G\nu. \tag{34}$$

Equation (34) gives the distance moved by the measured particle in the direction taken by the test particle, and is identical with (10). As before, its projection on the x axis fixed in the laboratory is uncertain by $L\sin\epsilon$, leading to an uncertainty in the final x coordinate of the particle,

$$\Delta x\gtrsim G\nu\sin\epsilon. \tag{35}$$

(6) and (35) combine to give $\Delta x\gtrsim\sqrt{G}$, which is identical with (12).

It should be noted that the treatment given here does not contradict momentum conservation, although it might appear to at first glance. If the test particle is moving with velocity close to that of light, its gravitational force on the measured particle is repulsive for much of the motion. The particle is accelerated to a velocity satisfying (31) largely by retardation effects during the early part of the motion, and is then slowed

[19] We are neglecting any velocity the particle might have in the x^2 or x^3 directions. However, it is easy to show that for fields satisfying (26), such a velocity could not persist long enough to interfere with the final result; in other words, the particle could not achieve the "escape velocity" in this direction.

down, being left with a momentum such that total momentum is conserved for the whole process. This is similar to the problem of the gravitational action of a pulse of light, which has been worked out by Tolman.[20]

We have neglected the interaction of the test particle with the gravitational field of the measured particle, but it seems hardly likely that this will affect the result. Its main effect should be to increase the kinetic energy of the test particle while in the vicinity of the measured particle, thus increasing the gravitational field of the test particle, and increasing the result (34). So it seems that inclusion of this effect would strengthen, not weaken, the result.

We have also neglected the effect of quantum fluctuations in the gravitational field. However, these would be expected to provide an additional source of uncertainty, not remove those already present. Hence, inclusion of this effect would, if anything, strengthen the result.

The main defect of the treatment given in this subsection is the neglect of the acceleration of the test particle during the measurement. It seems clear on physical grounds, however, that these effects must be transient; that is, after a sufficiently long time, the gravitational field of the test particle should be nearly the same as if its motion had been uniform in the direction taken by it after the scattering process. What we assume is that the field assumes its asymptotic form, at least in order of magnitude, for an appreciable portion of the time when the two particles are still close together. Since we are only interested in orders of magnitude, and don't care if we are in error by, say, a factor of 10 or 20, this assumption seems reasonable. To invalidate the results, one would have to assume that Eq. (22) does not even give an order of magnitude approximation to the field for any appreciable fraction of the time of the experiment. This seems rather far-fetched, though perhaps not impossible.

Another point worth mentioning is that the significance of the coordinate x^1 during the measuring process is not clear, and it may be incorrect to identify it with a distance. However, at the end of the measurement, as at the beginning, the strong gravitational field has disappeared and x^1 has all the properties of a distance in the ordinary sense. This is sufficient for our purposes: The value of x^1 at the end of the experiment differs from that at the beginning by an unknown amount, which makes it impossible to predict precisely the result of a subsequent position measurement.

To summarize the argument of this subsection: It is first shown that, because of (6) and (7), the result (12) cannot be avoided unless (26) is satisfied. It is then shown that if (26) is satisfied, the result (12) still holds, so it follows that it holds in all cases. The result holds no matter what kind of particles are used as test particles. However, the microscope method is not the only conceivable way in which the position of a particle may be ascertained. In the next subsection we give a brief discussion of some other methods, and some possible refinements on the microscope method.

C. Other Methods and Refinements

We first consider some possible refinements of the microscope method for locating the particle, in order to ascertain whether they could be used to violate (12). First, one might hope to "follow" the gravitational acceleration of the particle by some means, perhaps indirect. However, it is clear that this would not help. If we ascertain the direction of the gravitational acceleration experienced by the particle, we can infer from this the direction taken by the photon toward the microscope; so all we gain is an effective reduction in the angle ϵ, which does not appear in the final result. This argument is independent of the means used to follow the gravitational motion, so we can conclude that the uncertainties (6) and (35) cannot be reduced simultaneously. However, suppose it could be arranged that the microscope records the position of the particle at the end of its contact with the photon. In this case, its position at the beginning of the experiment would be unknown, but we would know its position at the end. But even if this is possible (which the author doubts), the knowledge thus gained has no physical significance. We can only say that we have measured the position of a particle precisely if we can predict precisely the result of an immediately subsequent identical measurement. If a microscope experiment yields the position of the particle at the end of its interaction with the photon, a second measurement will give its position at the end of its interaction with the second photon, and this cannot be predicted from the result of the first experiment with greater precision than that given by (12). This argument also provides an additional refutation of the possibility of avoiding (12) by following the gravitational acceleration of the particle. Even if this could be done, we could not predict the outcome of a subsequent identical experiment, except within the limitation imposed by (12). Still another possibility is to attempt to compensate the gravitational force, e.g., by having the particle emit a photon. But this could never compensate the uncertainty, only shift the average motion.

A related method for locating a particle is that in which the time for a light signal to propagate from some reference point to the particle and back is measured. If the time of emission and/or return of the light signal is in doubt by Δt, then we have for the uncertainty in the position

$$\Delta x \gtrsim \Delta t \gtrsim 1/\Delta\nu \gtrsim 1/\nu, \qquad (36)$$

where ν is the average frequency of the light signal, $\Delta\nu$

[20] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, Oxford, 1934), Sec. 114.

the spread in frequency, and we have made use of the Heisenberg principle

$$\Delta\nu\Delta t \gtrsim 1. \quad (37)$$

As before, the light signal (consisting of at least one photon) may be reflected from a region of radius r around the particle, so that

$$\Delta x \gtrsim r. \quad (38)$$

From (36) and (38), we have

$$\Delta x^2 \gtrsim r/\nu. \quad (39)$$

In order to have $\Delta x^2 \ll G$, therefore, we must have $G\nu/r \gg 1$. In this case, (26) is satisfied, so that from (34) the particle moves during the experiment a distance of the order $G\nu$. If ν is not known exactly, this leads to an uncertainty in the final position given by

$$\Delta x \gtrsim G\Delta\nu. \quad (40)$$

(36), (37), and (40) combine to give (12) again. Refinements on this experiment similar to those already discussed for the microscope experiment might be tried, but would fail for similar reasons.

Another method for localizing a particle is to cause it to pass through a very narrow slit. However, the slit must be made up of elementary particles, and the precision with which they can be held in fixed positions is limited by (5). Therefore, the edges of the slit will be "fuzzy" over a region at least of extent $\sqrt{G}$, with the result that the slit could not be used to locate a particle closely enough to violate (12). For a similar reason, it will not help to tie the particle to a long pointer. Because of (5), it could never be tied tightly enough for the pointer to be sensitive to movements of the particle over distances of order of magnitude $\sqrt{G}$ or less. Indeed, the pointer itself would be quite "limp" with respect to movements of its constituent elementary particles over distances of order $\sqrt{G}$ or less.

Although a completely exhaustive discussion of methods for locating a particle is obviously impossible without taking up an undue amount of space, it is felt that the examples discussed in this section are sufficiently typical so that there can be little doubt about the result.

IV. CLOCKS

The usual method by which a clock is synchronized with some standard clock is by passing light signals between them, with a light signal consisting of at least one photon.[21] If the time of emission and/or return of the light signal from the standard is uncertain by Δt, the reading T of the other clock is still in doubt at the end of the experiment by an amount

$$\Delta T \gtrsim \Delta t \gtrsim 1/\Delta\nu \gtrsim 1/\nu, \quad (41)$$

where we have again made use of (37), and ν is again the frequency of the photon. T is also in error by an amount of the order τ, the time during which it is strongly interacting with the photon; that is, the reading brought back by the photon might correspond to any time within the interval τ. Also, $\tau \gtrsim r$. So we write

$$\Delta T \gtrsim \tau \gtrsim r. \quad (42)$$

If the clock remains stationary during its interaction with the photon, the time recorded by it during the interaction is

$$\delta T = \tau\sqrt{g_{00}}. \quad (43)$$

Inserting $v=\alpha=1$ in (27), which corresponds to the test particle being a photon, we obtain

$$g_{00} = 1 - 4G\nu/r. \quad (44)$$

Combining (43) and (44), and noting that $\tau \sim r$ in this case since the clock remains stationary, we find

$$\delta T \sim (1-4G\nu/r)^{1/2}\tau = (1-4G\nu/r)^{1/2} r.$$

If the frequency is not known precisely, this leads to an uncertainty in the final clock reading given by

$$\Delta T \gtrsim 2G\Delta\nu/(1-4G\nu/r)^{1/2} \gtrsim 2G\Delta\nu. \quad (45)$$

(37), (41), and (45) combine to give

$$\Delta T \gtrsim \sqrt{G}. \quad (46)$$

The second inequality in (45) holds because we have assumed that the clock remains at rest throughout, and this is impossible unless $g_{00} \geqslant 0$.

A more general derivation is as follows: From (41) and (42), we cannot have $\Delta T \ll \sqrt{G}$ unless $G\nu/r \gg 1$. In this case we can use the results of the preceding section. The clock cannot remain stationary in this situation but must move with a velocity satisfying (30). As shown previously, it moves a distance of the order $G\nu$ during its interaction with the photon; since its velocity cannot be greater than unity, the time τ during which photon and clock are in interaction is also of the order $G\nu$. The time recorded by the clock during this time is the proper time, given by [cf. Eq. (30)]

$$\delta T = \int ds \sim (g_{00} + 2g_{01}u + g_{11}u^2)^{1/2}\tau. \quad (47)$$

The elements of the metric tensor are given by (27), with $\alpha=1$. Using (27) and (47), it is easy to show that

$$\delta T \lesssim \tau \bigg/ \left(1+4\frac{G\nu}{r}\right)^{1/2} \ll \tau. \quad (48)$$

[21] An alternate treatment of the clock problem, applicable to macroscopic as well as microscopic clocks, is given in the Appendix.

That is, the clock is essentially stopped during the interval $\tau \sim G\nu$, and this must be taken into account in making predictions of future readings. However, if ν is not known precisely, we will not know how great an allowance to make for this, and the error introduced is given by

$$\Delta T \gtrsim G\Delta\nu. \tag{49}$$

(37), (41), and (49) can be combined to give (46) again.

The considerations of this section show that the same limitations apply to the synchronization of clocks as were derived in the earlier sections for the localization of particles. This was to be expected on grounds of relativistic invariance. Refinements similar to those previously discussed for the localization experiments might be tried, but could be refuted by similar arguments.

V. MACROSCOPIC BODIES

The arguments of the preceding sections do not include any assumptions about the mass or size of the objects being tested, so it is clear that they apply equally well to direct position and time determinations on macroscopic bodies.[21] However, there is another kind of macroscopic measurement which deserves further attention. This consists of determining, e.g., the positions of a large number of microscopic bodies and taking the average. This average might be interpreted as the position of a macroscopic body, or in the case of time measurements, as the reading of a macroscopic clock. However, this method does not avoid the results of the preceding sections. We show this below for the case of the macroscopic clock; it is clear that analogous arguments hold for position measurements.

Suppose we have a "macroscopic clock" made up of N microscopic clocks spread out over a region of radius R. The reading of the macroscopic clock is defined as

$$\bar{T} = N^{-1} \sum T_i, \tag{50}$$

where the summation goes over all the microscopic clocks, whose readings are taken independently. Thus, N photons must be used, and for simplicity we assume the errors in the frequency and time of all the photons are the same, although the result still holds without this assumption. The average distance of one of the microscopic clocks from the standard with which we are trying to synchronize the macroscopic clock is at least of the order of R. All the T_i are subject to the limitation (41), but due to the independence of these errors, this leads to an error in $\bar{T}$ of only

$$\Delta\bar{T} \gtrsim N^{-1/2}\Delta t. \tag{51}$$

However, the errors due to the gravitational effect on the rate of the clocks are not reduced, but increased. Due to the long-range character of the gravitational force, an average gravitational potential of order $G\nu/R$ persisting for a time R is seen by the entire macroscopic clock during each microscopic measurement. Thus, as in Sec. IV, the error in $\bar{T}$ produced by one microscopic measurement is given in order of magnitude by (49), and since N microscopic measurements must be made, we have

$$\Delta\bar{T} \gtrsim N^{1/2} G\Delta\nu. \tag{52}$$

Combining (51) and (52), we obtain

$$\Delta\bar{T} \gtrsim G. \tag{53}$$

That is, the fundamental length result holds equally well for the reading of a macroscopic clock, and, for similar reasons, for the position of a macroscopic body as well.

VI. EQUIVALENCE OF FUNDAMENTAL LENGTH WITH GRAVITATIONAL FIELD FLUCTUATIONS

A. Effect of Fundamental Length on Gravitational Field Measurements

In this subsection, we postulate the existence of a fundamental length and inquire as to its effect on gravitational field measurements. The fundamental length postulate still allows us some freedom of choice as to what assumption we make regarding errors in macroscopic measurements. For instance, as was shown in the preceding section, in the case of a fundamental length arising from gravitational effects we have

$$\Delta x \gtrsim \ell,$$

for any measurement, microscopic or macroscopic. Another possibility would be to postulate that the errors in position measurements on elementary particles are independent, but that the maximum number of such independent measurements that can be made in a space-time volume R^4 is given by $N=(R/\ell)^4$. In this case we would have, for the error in a macroscopic measurement spread out over a space-time region of four-volume R^4,

$$\Delta x \gtrsim \ell/\sqrt{N} \sim \ell(\ell/R)^2.$$

We can include both these possibilities, and many more, if we simply require

$$\Delta x \gtrsim \ell\beta(R/\ell), \tag{54}$$

for the error in a coordinate measurement on a body (macroscopic or microscopic) spread over a cubic space-time region of volume R^4. Here $\beta(y)$ is a non-increasing function which is approximately unity when y is unity and need not be defined for $y \ll 1$. Thus, for the two special cases mentioned above, we have $\beta=1$ and $\beta=y^{-2}$, respectively. We choose a coordinate system which is quasi-Lorentzian on the average and for the moment assume that the gravitational field is not too strong.

Now, assuming that (54) holds, suppose we wish to measure a component of the gravitational field F averaged over a space-time region of volume R^4. To do this, we must measure the acceleration of a body confined in the given region. We have

$$F=(v_2-v_1)/T\,,$$

where v_1 and v_2 are the results of two velocity measurements separated by a time T. We see that F is uncertain by at least

$$\Delta F \gtrsim \Delta v/T \gtrsim \Delta v/R\,, \tag{55}$$

where Δv is the error in either one of the velocity measurements, and we note that T cannot be greater than R. A velocity measurement, in turn, consists of two successive position measurements separated by a time $t\lesssim R$. We find

$$v=(x_2-x_1)/t\,;$$

and, with the aid of (54)

$$\Delta v \gtrsim \Delta x/t \gtrsim \Delta x/R \gtrsim \ell\beta(R/\ell)/R\,. \tag{56}$$

Combining (55) and (56), we obtain

$$\Delta F \gtrsim \ell\beta(R/\ell)/R^2\,. \tag{57}$$

From (57), we see that the average gravitational potential (or component of the metric tensor) in the region is uncertain by

$$\Delta g \gtrsim \ell\beta(R/\ell)/R\,. \tag{58}$$

Equation (58) should not be thought of as applying independently to all components of the metric tensor, but to those components, or linear combinations of components, which are taken as independent. In general, the different components may be related by some subsidiary ("gauge") condition which is used to fix the coordinate system. Equation (57) presumably applies to any Christoffel symbol $\Gamma^i{}_{jk}$, in the given coordinate system.

Another example, more closely related to the situation discussed in the introduction, is as follows: Let there be two bodies, A and B, each of radius R, separated by a distance $L\gtrsim R$ along the x axis. Each body is equipped with a clock, and the two clocks are kept synchronized as closely as possible; however, if (54) holds for clock readings as well as for position measurements (as it must), an exact synchronization will be impossible. In particular, if a light signal leaves A at a time t (measured on A's clock), and arrives at B at a time $t+\tau$, as measured on B's clock, then the time of propagation τ will be subject to fluctuations given by

$$\Delta\tau \gtrsim \ell\beta(R/\ell)\,. \tag{59}$$

On the other hand, we are permitted to regard the two bodies and their clocks as defining part of our coordinate system. Thus let the measured position of A at any time be assigned the space coordinates $x=y=z=0$, and that of B the coordinates $x=L$, $y=z=0$. The time of an event taking place at A is given by the reading of A's clock, and that of an event at B by B's clock. If we define our coordinate system in this way (which is as good a way as any), there can be no question of fluctuations in the coordinates of A and B, since these are fixed by convention; for the same reason the time of each single event can be known exactly. In this case, the light signal experiments must be regarded as yielding information about the metric tensor associated with the given coordinate system in the space-time region swept out by the world-line of the light signal. This world line must be a "null geodesic," i.e.

$$ds^2=g_{00}dt^2+2g_{01}dxdt+g_{11}dx^2=0\,.$$

We find for the time of propagation:

$$\tau=\left[\frac{-g_{01}+(g_{01}{}^2-g_{11}g_{00})^{1/2}}{g_{00}}\right]L\,, \tag{60}$$

in which the g's represent suitable averages over the region through which the light signal passes. Thus, a measurement of τ gives us some information about the metric tensor. If τ is subject to fluctuations given by (59), we see from (60) that the average value of one or more of the components of g must be uncertain by

$$\Delta g \gtrsim \ell\beta(R/\ell)/L\,. \tag{61}$$

Equation (61) agrees with (58) for the special case $L\sim R$. For the gravitational force, which is made up of derivatives of the g's, we have

$$\Delta F \gtrsim \ell\beta(R/\ell)/L^2\,, \tag{62}$$

which agrees with (57) if $L\sim R$.

We see from the above examples that the fundamental length postulate, as expressed by (54) and (59), has as a consequence the existence of certain gravitational field fluctuations, given by (57), (58), (61), and (62).

B. Effect of Gravitational Field Fluctuations on Coordinate Measurements

In this subsection, we initially postulate the existence of gravitational field fluctuations given by (57), (58), (61), and (62), and study the consequences of this postulate with respect to coordinate measurements.

First, suppose we wish to measure the position of a particle with radius R. The time required to carry out such a measurement is at least of the order of R, so that the time between two successive measurements is of the order of R. During this time, the body is acted on by the gravitational field, whose magnitude is not exactly known. We have for the uncertainty of the acceleration of the particle, from (57):

$$\Delta a \gtrsim \Delta F \gtrsim \ell\beta(R/\ell)/R^2. \quad (63)$$

During the time R, this unknown acceleration will lead to an uncertainty in the coordinate given by

$$\Delta x \gtrsim \ell\beta(R/\ell), \quad (64)$$

which is the same as (54). This means that, if (57) holds, the result of one position measurement cannot be used to predict the result of a subsequent one more precisely than is permitted by (64) or (54).

Furthermore, a glance at (60) is sufficient to show that (59) is a consequence of (61) for the experiment in which light propagates between two bodies.

As a further example, we consider the measurement of the x coordinate of a body of radius R by means of a microscope. We imagine that the microscope is placed above the body to be measured, at a distance $L \gtrsim R$, in the z direction. Now if F_x is the average x component of the gravitational field in the region traversed by a light signal between the body and the microscope, the light signal will be deflected in the x direction from its original trajectory, such that its final direction of propagation makes an angle θ with its original direction, where

$$\sin\theta \sim LF_x. \quad (65)$$

Thus, when focused in the microscope, the light signal will appear to have come, not from the point x at which the particle is "actually" located, but from a point

$$x_{\text{obs}} = x - L\sin\theta \sim x - L^2F_x. \quad (66)$$

We must correct for this effect in deducing the position of the body from the result of the measurement.

$$x \sim x_{\text{obs}} + L^2F_x. \quad (67)$$

If F_x is not known exactly, then x is uncertain by

$$\Delta x \gtrsim L^2\Delta F_x. \quad (68)$$

Equation (68), in conjunction with (62), gives

$$\Delta x \gtrsim \ell\beta(R/\ell), \quad (69)$$

which again is identical with (54).

The conclusion of this subsection is that, if we postulate the existence of the gravitational field fluctuations (57), (58), (61), and (62), we can deduce the fundamental length limitations (54) and (59).

C. Remarks

The results of the preceding subsections show that the fundamental length postulate, as expressed in Eqs. (54) and (59) is equivalent to a postulate about gravitational field fluctuations, embodied in Eqs. (57), (58), (61), and (62), in the sense that either postulate may be deduced from the other. The derivations are not completely general, since they depend for their strict validity on the assumption of slowly moving bodies and of weak fields. That is, we must have $\Delta g \ll 1$, and $\Delta v \ll 1$. Comparing this with Eqs. (56) and (58), we see that the derivations are valid if $R \gg \ell$. As R becomes of the order of ℓ, but remains somewhat larger, the relativistic corrections become appreciable, but the results remain correct in order of magnitude. The derivation definitely breaks down for $R \lesssim \ell$. However, this case is inaccessible to observation anyway if (54) holds, since it refers to space-time regions of extent less than the fundamental length. It is also inaccessible if we postulate (58), since the fluctuations in the metric tensor then become of the order of unity, leading to a breakdown of the metric structure of space-time due to violation of Eq. (4). Hence, it seems that the qualitative results of the preceding subsections are probably more general than would appear at first glance.

It should also be emphasized that an attempt to carry out an argument similar to that of subsections A and B, but using some field other than the gravitational, would not lead to similar results without also making some postulates about the properties (e.g. mass, electric charge) of the bodies being measured. No field other than the gravitational has the property of imparting the same acceleration to all bodies, regardless of mass, charge, etc. Therefore, while the existence of the fundamental length would of course limit all field measurements to averages over regions of extent greater than ℓ, the equivalence between fundamental length and field fluctuations deduced above holds only for the gravitational field.

These results enable us to view the problem of formulating a fundamental length theory in a somewhat different light. Instead of asking how to formulate a theory in which (54) is obeyed, we can ask the completely equivalent question of how to formulate a theory in which (58) is obeyed. This question may be somewhat easier to answer. For example, if we restrict ourselves to the special case $\beta = 1$, we can answer it in a tentative way simply by expressing the metric tensor as

$$g_{ij} = g_{ij}(0) + h_{ij},$$

where $g_{ij}^{(0)}$ is the Lorentzian metric tensor, and inserting a term in the Lagrangian density:

$$\mathfrak{L}_g = (1/\ell^2)\partial_i h_{jk}\partial^i h^{jk}. \quad (70)$$

The coupling with the matter field is determined automatically up to first order in h, by covariance requirements; e.g., derivatives are replaced by covariant derivatives, etc. If we calculate the vacuum fluctuations of the metric tensor from the Lagrangian (70), we find, for the fluctuation in the average of a component of g over a cubic space-time region of volume R^4,

$$\Delta g \gtrsim \ell/R, \quad (71)$$

which is the same as (58) for the special case considered here.[22] However, this approach is just the quantization of the gravitational field in linear approximation with the role of the gravitational constant G taken by ℓ^2. Hence, such a theory would lead to gravitational effects contrary to experiment unless $\ell\sim\sqrt{G}$. Of course, it does not necessarily follow that *any* fundamental length theory would have to have $\ell\sim\sqrt{G}$; however, the equivalence between fundamental length and gravitational field fluctuations deduced in this section indicates that any fundamental length theory is likely to involve gravitational effects of some kind in an important way.

VII. DISCUSSION

We may summarize our conclusions briefly as follows: First, present-day physical ideas about gravitation, together with the uncertainty principle, imply the existence of a fundamental length of order $\sqrt{G}$. This fundamental length applies to macroscopic as well as microscope measurements; in the notation of Sec. VI, $\ell\sim\sqrt{G}$, $\beta=1$. This does not rule out the possibility of some larger fundamental length, but is to be thought of as a lower limit. However, a larger fundamental length probably could not be deduced from present-day concepts without making some assumptions about the properties of the particles being measured. Furthermore, it is shown in Sec. VI that a fundamental length postulate is equivalent to a postulate about gravitational field fluctuations, in which the fundamental length appears as the distance at which the metric structure of space-time breaks down due to these fluctuations. Hence, any fundamental length theory is likely to involve gravitational effects of some kind.

A word might be said about the correspondence between the results of Secs. II–V with those of Sec. VI. The fundamental length result derived in the earlier sections correspond to $\ell\sim\sqrt{G}$, $\beta=1$ in the notation of Sec. VI. According to Sec. VI, this is equivalent to postulating (58) which, according to (71), is just what one gets for the vacuum fluctuation of the gravitational field when one quantizes in linear approximation. It is also to be noted that our result for the gravitational field uncertainty is stronger than that of Peres and Rosen,[23] who derive essentially our Eq. (57) with $\ell\sim\sqrt{G}$, $\beta(y)=y^{-1}$. Discussions similar to theirs had been given previously by Anderson,[24] and by Regge.[25]

Superficially, it might appear from all this that the thing to do is to go ahead with the quantization of the gravitational field, sticking as close to the usual method of quantization as the peculiar properties of the gravitational field will allow. However, it should be remembered that the usual method treats field quantities averaged over arbitrarily small regions formally as observables which can be measured in principle with any desired degree of accuracy. But the existence of the fundamental length, through Eqs. (57) and (58), imposes definite limitations on the measurability of these quantities. The greater the desired accuracy, the larger must be the space-time region over which one averages. Hence, a theory in which the gravitational field is quantized in the usual way—at least if the physical interpretation is to be analogous to that of Bohr and Rosenfeld[7–9] for the electromagnetic field—can at best be an approximate theory, though it may be a very good approximation if one restricts oneself to space-time regions large compared with the fundamental length. The very careful analysis of DeWitt[26] leads to the same qualitative result. Thus, if one wishes to construct a theory which is applicable to regions of the order of the fundamental length in extent, fundamental changes in the quantization procedure would seem to be in order.

It should also be emphasized that, because of the equivalence established in Sec. VI, the remarks of the preceding paragraph apply regardless of what fundamental length postulate one makes. For example, if a fundamental length of the order of nuclear dimensions is postulated, it follows that gravitational field fluctuations become large in regions of the order of nuclear dimensions (and conversely). With such a fundamental length, according to (58), the fluctuations in the components of the metric tensor become of the order of unity in regions of the order of nuclear dimensions. The fluctuation in gravitational potential energy of a nucleon is therefore

$$\Delta E_g\sim M\,,$$

where M is the mass of a nucleon. Coulomb energies in this region are of the order

$$E_c\sim e^2/r=e^2M\,,$$

where e is the electronic charge, and r by hypothesis is about $1/M$. This means that, if one postulates a fundamental length of the order of nuclear dimensions, one must conclude that gravitational energy fluctuations in nuclei should be greater than Coulomb energies by a factor of about 137. Since no such effects are observed, this appears to be an argument *against* a fundamental length of the order of nuclear dimensions.

ACKNOWLEDGMENTS

When these ideas first occurred to the author, he discussed them with several colleagues. Among those who have contributed to the development of this work,

[22] The field Lagrangian (70) can only be used in conjunction with some subsidiary "gauge" condition. Therefore, Eq. (71) cannot be thought of as applying independently to all components of g, but only to those taken as independent dynamical variables. Cf. the discussion following Eq. (58).

[23] A. Peres and N. Rosen, Phys. Rev. **118**, 335 (1960).

[24] J. L. Anderson, Rev. Mex. Fis. **3**, 176 (1954).

[25] T. Regge, Nuovo Cimento **7**, 215 (1958).

[26] B. S. DeWitt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962).

either by encouragement and interest or by specific comments, are Dr. G. Feinberg, Dr. I. Goldberg, Dr. G. L. Miller, Dr. W. Rubinson, Dr. S. Smart, Dr. G. C. Wick, Dr. M. Wolfsberg, and Professor D. R. Yennie. Here and in the following acknowledgments, it should not be inferred that those mentioned are necessarily in agreement with the author on all points.

Conversations with Professor Henry Primakoff have been especially helpful and deserve separate mention. This work probably would never have been published had it not been for Professor Primakoff's continued encouragement during the long period of controversy with referees.

Thanks are due to Dr. R. W. Dodson and the entire chemistry department staff of Brookhaven National Laboratory for hospitality and kindness shown the author during his stay there, when this work was begun; also to the U. S. Atomic Energy Commission for financial support during this period.

The author is grateful to Miss L. Sanger (now Mrs. J. Sprackling), whose friendship, encouragement, and assistance were of the greatest importance to him during the conception and preparation of this work.

APPENDIX: ALTERNATE TREATMENT OF CLOCK PROBLEM

In this appendix we give a treatment of the clock problem which does not depend on the particular method of measurement being used, but only on commutation properties of operators. It is based on the concept of an "inner time" as discussed, for instance, by Aharonov and Bohm.[27]

Initially, we assume that all gravitational fields are weak, so that we can speak unambiguously of such things as "distance," "time," etc., with the gravitational effect on the rate of a clock being a small correction. We consider a physical system spread out over a region of radius R which is to be used as a clock. The energy operator of the system is $\mathcal{H}$. The "inner time" T (e.g. the position of a pointer on a scale marked off in time units) is a dynamical variable of the system whose time derivative is unity. Thus we have

$$i\dot{T}=[T,\mathcal{H}]=i. \tag{A1}$$

This refers, however, to the local time. The rate ρ of the clock relative to the "world time" (as measured by a clock at infinity) is

$$\rho=1+\phi=1-G\mathcal{H}/R. \tag{A2}$$

Here it is assumed that the gravitational potential is constant in the interior of the system, though it will be clear that the result does not really depend on this assumption. Combining (A1) and (A2), we find

$$[T,\rho]=-iG/R. \tag{A3}$$

[27] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).

The time required to make a measurement (e.g., synchronization by means of light signals with a nearby clock) is at least of the order of R, so the smallest possible change τ in the reading of the clock during the course of the measurement is

$$\tau=\rho R. \tag{A4}$$

Equations (A3) and (A4) may be combined to give

$$[T,\tau]=-iG. \tag{A5}$$

We therefore have the uncertainty relations

$$\Delta T\Delta\tau\gtrsim G, \tag{A6}$$

and

$$\Delta T\Delta(T+\tau)\gtrsim G. \tag{A7}$$

Since $T+\tau$ is just the reading of the clock at the end of the measurement (also the predicted result of an immediately subsequent measurement), the meaning of (A7) is that the clock cannot be kept synchronized with the "world time" with greater root-mean-square accuracy than $\sqrt{G}$. Successive synchronization measurements will therefore show unpredictable fluctuations, and all the consequences discussed previously will follow.

It is clear that the above considerations apply equally well to macroscopic clocks, since no assumptions were made about the specific nature of the dynamical variable T. In particular, it could be a macroscopic time such as that defined by Eq. (50).

It is also clear that the result applies equally well to the difference T_1-T_2 in the readings of two separate clocks (that is, the errors cannot be made to cancel). To see this, we simply note that

$$[(T_1-T_2),(\rho_1-\rho_2)]=-iG\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

The time for a synchronization measurement is at least R_1+R_2, so we have

$$\tau_1-\tau_2=(\rho_1-\rho_2)(R_1+R_2),$$

from which we find

$$i[(T_1-T_2),(\tau_1-\tau_2)]=G(R_1+R_2)^2/R_1R_2>G. \tag{A8}$$

It is also easy to verify that the assumption of weak fields is not necessary by doing a more accurate calculation. In what follows, we will use the words "distance" and "time" to refer to the radial and temporal coordinates, respectively, in the usual Schwarzschild coordinate system.[17] "Velocity" will mean "distance" per unit "time." It is understood that these coordinates do not have all the usual properties of distance and time, but it is nevertheless convenient to have some words to call them by. We also assume that R is large enough so that there is no singularity in the Schwarzschild solution.

With this understanding, we now have instead of (A2) the more exact relation

$$\rho=[1-2(G\mathcal{H}/R)]^{1/2}, \tag{A9}$$

and if T is to measure the local time we must have

$$[T,\mathcal{H}']=i, \tag{A10}$$

where

$$\mathcal{H}'=\mathcal{H}\rho^{-1} \tag{A11}$$

is the "local Hamiltonian." (A9) and (A11) may be solved to give

$$\mathcal{H}'=(R/2G)(\rho^{-1}-\rho). \tag{A12}$$

The velocity of light is now not unity but ρ^2, so the time for a measurement is R/ρ^2. It follows that

$$\tau=R\rho^{-1},$$
$$\rho=R/\tau. \tag{A13}$$

Combining (A12) and (A13), we find

$$\mathcal{H}'=(1/2G)(\tau-R^2/\tau);$$

$$d\mathcal{H}'/d\tau=(1/2G)(1+R^2/\tau^2); \tag{A14}$$

$$\frac{d\tau}{d\mathcal{H}'}=\frac{2G}{1+R^2/\tau^2}=\frac{2G}{1+\rho^2}\geqslant G. \tag{A15}$$

Combining (A10) and (A15), we finally obtain

$$-i[T,\tau]=-i[T,\mathcal{H}']\frac{d\tau}{d\mathcal{H}'}=2G/(1+\rho^2)\geqslant G. \tag{A16}$$

Thus, the more accurate treatment results in a change in sign of the commutator (the difference between $\mathcal{H}$ and $\mathcal{H}'$ and the effect on the velocity of light were not treated even in lowest order in the approximate treatment), but the resulting uncertainty relations are the same.

The results of this appendix permit a somewhat more precise interpretation than those of the body of the paper. The quantity T alone can apparently be measured with arbitrary accuracy, which means that one may say that two clocks are arbitrarily well synchronized at a particular time. However, one cannot prepare a pair of clocks which will remain reliably synchronized over a period of time. This situation is exactly the same as that discussed in the introduction: The result of a single measurement may be read with arbitrary accuracy, but successive measurements will show unpredictable fluctuations. Since the description of phenomena in terms of a Lorentzian coordinate system presupposes the physical possibility of setting up clocks at different points in space which can be relied on to remain synchronized, the conclusion is that such a coordinate system can only be set up with a mean error of the order of the fundamental length.

The considerations of this appendix, depending as they do only on the commutation properties of the operators, have some advantages over the methods used in the body of the paper. As stressed by Aharonov and Bohm,[27] when one considers a particular experiment, one runs the risk that the experiment chosen may not be sufficiently typical for the result to be generally valid. Hence, a result is to be believed only when it has been derived from the mathematical formalism. This point of view is unquestionably correct when one is concerned with interpreting a theory for which a definitive mathematical formalism exists. In the present case, however, as pointed out in the discussion section, the existence of the fundamental length may necessitate fundamental changes in the formalism, so that it is at least conceivable that the physical considerations of the body of the paper have a wider validity than the operator formalism. In this case, one must simply live with the risk of choosing atypical methods of measurements until a satisfactory formalism has been developed. This is the philosophy behind the relegation of these considerations to an appendix. In any case, it is highly satisfying that both approaches lead to identical results. Each reader may decide for himself the question of which approach is more convincing.