Bremsstrahlung from Electron-Ion Encounters in a Magnetic Field*

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The emission by a free electron undergoing Coulomb interactions in the presence of a strong external magnetic field has been computed making use of the quantum-mechanical formalism especially designed for continuous radiation. The result holds for all frequencies except the ones at the cyclotron resonance proper. It is shown that the general results coincide with the known bremsstrahlung formula for frequencies much above the cyclotron resonance, and that it moreover goes into our previous results in the neighborhood of the resonance.

1. INTRODUCTION

IN two previous papers,[1,2] henceforth called I and II, the radiation emitted by free electrons undergoing Coulomb interactions in the presence of a magnetic field has been computed subject to the restriction that the frequencies observed were close to the cyclotron resonance. The results were therefore interpreted as the contour of the cyclotron spectral line.

In order to present a complete analysis of the radiation from charged particles in a magnetic field, an extension of these results into the frequency domains at greater distance from the resonance is necessary. Although it may be possible with a great deal of care to extend the approach of Papers I and II to spectral regions far from the resonance, it was decided to carry out the actual calculation with the aid of a somewhat different formalism. This formalism[3] is the standard technique in the absence of resonance terms, and is conventionally used in the evaluation of bremsstrahlung emission. The major innovation of our calculation is the introduction of magnetic eigenfunctions instead of plane-wave functions in a manner analogous to the procedure of Papers I and II. The results consequently describe the over-all radiation spectrum satisfactorily, but fail to give accurate answers in the close neighborhood of the resonance. It will be shown in Sec. 4, however, that in the "wings" of the cyclotron resonance line, both methods yield essentially identical results, and that thus a combination of the two describes the spectrum completely within reasonable limits of accuracy.

The actual calculations were in many respects similar to the ones presented in detail in Papers I and II. Since, moreover, they do not contain basically new procedures, we felt that it would be permissible to leave some of the details out of this article, in particular, those of a purely computational nature.

The derivations are summarized in Sec. 2, and the general results stated in Sec. 3, including low- and high-frequency limits that are of practical interest. In Sec. 4, finally, a comparison with our previous calculations in the neighborhood of the resonance frequency is carried out.

2. COMPUTATION OF THE SPECTRUM

In the following, a system is considered that consists of a free electron subject to magnetic field and Coulomb interactions, and a photon. The transition probability per unit time from an initial state i of this system to a final state f reads[4]

$$W=(2\pi/\hbar)|\mathbf{K}_{fi}|^2\rho_f, \tag{1}$$

where ρ_f is the number of final states per unit energy interval.

Let us specify the initial state of the system by the electron's magnetic quantum number n, which is the quantity corresponding to the electron's kinetic energy in directions perpendicular to the magnetic field, the momentum vector k_i parallel to the magnetic field (which we assume to be along the z axis), and a quantum number s that reflects the electron's location relative to the fixed scattering center. Initially, no photon is present. The final state in this notation is specified by a magnetic quantum number $n+m$, a momentum vector $\mathbf{k}$, a positional quantum number $s+m+\delta_i$ ($\delta_i=0, \pm 1$), and the momentum $\hbar c k_\nu$ of the photon.

We have to consider, in addition, two intermediate states 1 and 2, corresponding to the situation in which first the photon is emitted (state 1) with subsequent scattering, and the situation where first the scattering takes place (state 2), followed by the photon emission. The quantum numbers and wave numbers of state 1 of the system electron plus photon are thus ($n+\delta_i$, $k_1=k_i-\mathbf{k}_\nu\cdot\hat{z}$, s; k_ν), whereas for state 2 we have ($n+m-\delta_i$, $k_2=k_f+\mathbf{k}_\nu\cdot\hat{z}$, $s+m+\delta_i$; K_ν).

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[1] R. Goldman and L. Oster, Phys. Rev. **129**, 1469 (1963).

[2] R. Goldman, Phys. Rev. **133**, A647 (1964).

[3] W. Heitler, *The Quantum Theory of Radiation* (Cambridge University Press, New York, 1955), 2nd ed.

[4] Reference 3, Sec. 25, Eq. (10).

Physically, the introduction of these intermediate states accounts for the fact that the Coulomb scattering, aside from the photon emission, results in a redistribution of the kinetic energy of the electron among the directions parallel and perpendicular to the magnetic field. Similarly, ρ_f accounts for the fact that from any initial state many, and in this respect different, final states can be reached.

With these considerations in mind, we can write

$$|\mathbf{K}_{fi}|=\sum_{1,2}\left[\frac{V_{f1}H_{1i}}{E_i-E_1}+\frac{H_{f2}V_{2i}}{E_i-E_2}\right]. \quad (2)$$

Here, the letter V refers to the Coulomb operator in cylindrical coordinates,

$$V=-Ze^2/(r^2+z^2)^{1/2}, \quad (3)$$

so that for two states α and β with wave functions ψ_α and ψ_β,

$$V_{\alpha\beta}=\int\psi_\alpha V\psi_\beta d\tau, \quad (4)$$

with the integration extended over all space.

$H_{\alpha\beta}$ is the interaction Hamiltonian.[5] In the case of matrix elements involving the diıections x and y perpendicular to the magnetic field,

$$E_1-E_0=\hbar ck_\nu+\delta_i\hbar\omega_c+(\hbar^2/2m_e)(k_i-\mathbf{k}_\nu\cdot\hat{z})^2-(\hbar^2/2m_e)k_i^2. \quad (5)$$

Correct to order v/c, we find

$$E_1-E_i\approx E_f-E_2\approx E_i-E_2. \quad (6)$$

In Eq. (5), the gyrofrequency

$$\omega_c=eH/m_ec \quad (7)$$

has been used, with H being the magnetic field strength. Similarly, for the matrix elements involving the z axis parallel to H, we find

$$E_1-E_i=\hbar ck_\nu+(\hbar^2/2m_e)(k_i-\mathbf{k}_\nu\cdot\hat{z})^2-(\hbar^2/2m_e)k_i^2 \approx E_f-E_2\approx E_i-E_2. \quad (8)$$

Between the wave vectors $\mathbf{k}_i$ and $\mathbf{k}_f$ of initial and final state, finally, the relation,

$$(\hbar^2/2m_e)k_f^2+(n+m)\hbar\omega_c+\hbar k_\nu c=(\hbar^2/2m_e)k_i^2+n\hbar\omega_c \quad (9)$$

holds. For the total system (including the photon), E_f-E_i is of course zero.

In conventional notation $|n_l,k_l,s_l\rangle$ for any state l, the matrix elements read

$$\mathbf{K}_{fi}=\sum_{\delta_i}\sum_m\{\langle n+m,\ (2m_ek)^{1/2}/\hbar,\ s+m+\delta_i|V_{\alpha\beta}|n+\delta_i,\ k_i-\mathbf{k}_\nu\cdot\hat{z},\ s\rangle\langle n+\delta_i,\ k_i-\mathbf{k}_\nu\cdot\hat{z},\ s|\mathbf{H}_{\alpha\beta}|n,\ k_i,\ s\rangle$$
$$-\langle n+m,\ (2m_ek)^{1/2}/\hbar,\ s+m+\delta_i|\mathbf{H}_{\alpha\beta}|n+m+\delta_i,\ (2m_ek)^{1/2}/\hbar+\mathbf{k}_\nu\cdot\hat{z},\ s+m+\delta_i\rangle$$
$$\times\langle n+m-\delta_i,\ (2m_ek)^{1/2}/\hbar+\mathbf{k}_\nu\cdot\hat{z},\ s+m+\delta_i|V_{\alpha\beta}|n,\ k_i,\ s\rangle\}-\hbar ck_\nu-\delta_i\hbar\omega_c+\hbar^2k_i(\mathbf{k}_\nu\cdot\hat{z})/m_e. \quad (10)$$

In Eq. (10) the abbreviation

$$k^{1/2}\equiv(\hbar^2k_i^2/2m_e-m\hbar\omega_c-\hbar k_\nu c)^{1/2} \quad (11)$$

was used. The interaction Hamiltonian $H_{\alpha\beta}$, where α and β refer to initial and final state of the electron, and n_ν to the number of photons, reads

$$H_{\alpha,n_\nu+1|\beta,n_\nu}=-e(2\pi\hbar^2c^2/k\nu)^{1/2}[(n_\nu+1)/c]^{1/2}\left\{\frac{E_\alpha-E_\beta}{i\hbar}\int\psi_\alpha{}^*x\psi_\beta d\tau,\ \frac{E_\alpha-E_\beta}{i\hbar}\int\psi_\alpha{}^*y\psi_\beta d\tau,\ \int\psi_\alpha{}^*\left[-\frac{i\hbar}{m_e}\frac{\partial}{\partial z}\exp(-i\mathbf{k}_\nu\cdot\hat{z})\right]\psi_\beta d\tau\right\}. \quad (12)$$

Here,

$$E_\alpha=n\hbar\omega_c,\quad E_\beta=(n-1)\hbar\omega_c \quad (13)$$

are the electron's kinetic energies perpendicular to the magnetic field directions.

For the wave functions we take the same expressions as in I and II.[6] Hence, we find[7] for any two states (n,k,s) and $(n+n',\ k',\ s+n')$

$$V_{n,k,s|n+n',k',s+n'}=\int\psi_{n+n',k',s+n'}{}^*[-Ze^2/(r^2+z^2)^{1/2}]\psi_{n,k,s}d\tau=\frac{Ze^2}{i\hbar v_z}K_{n'-n}(qs^{1/2})I_{n'-n}(qn^{1/2}),\quad n<s,$$
$$=\frac{Ze^2}{i\hbar v_z}K_{n'-n}(qn^{1/2})I_{n'-n}(qs^{1/2}),\quad n>s. \quad (14)$$

The velocity v in Eq. (14) represents the average of the electron velocity component parallel to the magnetic field, taken between the initial and final state. The parameters

$$q=(k-k')\gamma^{-1/2},\quad \gamma=eH/2c\hbar. \quad (15)$$

I and K are, as usual, the modified Bessel functions.

[5] Reference 3, Sec. 25, Eq. (2).
[6] Paper I, Eq. (4).
[7] Paper II, Eq. (39).

The evaluation of the matrix elements from Eq. (10) is completely analogous to the procedure employed in paper II. We note in particular that [cf. Eqs. (10) and (11), paper II]

$$\langle \psi_{n-1,k,s}{}^* x \psi_{n,k,s} \rangle = \tfrac{1}{2}(n/\gamma)^{1/2} \qquad (16a)$$

and

$$\langle \psi_{n-1,k,s}{}^* y \psi_{n,k,s} \rangle = \tfrac{1}{2} i (n/\gamma)^{1/2}. \qquad (16b)$$

Furthermore,

$$H_{1i} = -e(2\pi\hbar^2 c^2/k_\nu)^{1/2}(n_\nu+1)^{1/2} \times (\hbar/2m_c)(2k_i - \mathbf{k}_\nu \cdot \hat{z}) \qquad (17a)$$

and

$$H_{f2} = -e(2\pi\hbar^2 c^2/k_\nu)^{1/2}(n_\nu+1)^{1/2} \times (\hbar/2m_e)(2k_f + \mathbf{k}_\nu \cdot \hat{z}). \qquad (17b)$$

Finally, we rewrite Eq. (1) in a more convenient form. To this end, we first normalize to unit final-state density $\rho_f = 1$. Second, we consider the total transition probability *per unit time* for a beam of electrons of density N_e and velocity components $v_\perp = (v_x^2 + v_y^2)^{1/2}$ and v_z. For this purpose we recall the meaning of the quantum number s which represents the relative location of the guiding center of the beam with respect to the scatterers, i.e., in classical language, the collision parameter. The total transition probability per unit time is then given by an average over the (random) location of scatterers, or now in quantum language, over the degenerate quantum numbers s, such that

$$w ds = (2\pi/\hbar)|\mathbf{K}_{fi}|^2 N_e dA_s, \qquad (18)$$

where

$$dA_s = 2\pi s ds \qquad (19)$$

is the area in the plane perpendicular to the magnetic field corresponding to the quantum number s. Hence, for unit density $N_e = 1$,

$$W = \langle w \rangle = \frac{2\pi}{\hbar} \int_{s=0}^{\infty} |\mathbf{K}_{fi}|^2 \frac{dA_s}{ds} ds. \qquad (20)$$

3. THE RESULTS

The transition probability per interaction[8] for the emission of a photon with energy $\hbar\omega$ (on neglecting Doppler effects) is then obtained with the aid of the formulas of Paper II, in particular, Appendices B–D.

$$W(\omega)d\omega = \frac{8}{3\hbar\omega} \frac{e^6}{c^3 m_e^2} \frac{1}{v_z} \left\{ \sum_{i=1}^{3} \frac{\omega^2}{(\omega-\omega_c\delta_i)^2} \frac{1}{2} \frac{2E_b + 2|\delta_i|E_z - |\delta_i|E_b}{E} \sum_{\Delta m=-\infty}^{+\infty} (\Delta m)^2 (E/E_z)^{3/2} \times \tfrac{1}{2}(I(\epsilon),K(\epsilon))_{m',m;\Delta m=m'-m=p} + \sum_{i=1}^{2} \frac{\omega^2}{(\omega-\omega_c\delta_i)^2} \frac{1}{2} (E/E_z)^{1/2} \tfrac{1}{2} q^2 (I(q_i),K(q_i))_{n',n;n'-n=0} \right\} d\omega, \qquad (21)$$

where the bracket symbol is defined by the relation

$$(I,K)_{l',l} \equiv I_{l'-l}{}^2 K_{l'-l-1} K_{l'-l+1} - K_{l'-l}{}^2 I_{l'-l-1} I_{l'-l+1}. \qquad (22)$$

The argument ϵ is given by the expression

$$\epsilon \equiv p - \delta_i + (\omega/\omega_c)(E_b/E_z)^{1/2}. \qquad (23)$$

E_b and E_z are the electron's kinetic energy components corresponding to the velocity components $v_\perp$ and v_z so that

$$E = E_b + E_z, \qquad (24)$$

q_i is given by

$$q_i = (\delta_i + \omega/\omega_c)(E_b/E_z)^{1/2}, \quad \delta_i = 0, \pm 1. \qquad (25)$$

The general result of Eq. (21) cannot be reduced to a much simpler form: Particular cases can easily be handled numerically by an electronic computer. It is possible, however, to give somewhat more manageable expressions in certain limiting cases of interest.

In the "*low-frequency domain*" $\omega \ll \omega_c$ (which begins actually a few half-widths away from the resonance),

$$W(\omega)d\omega = \frac{8e^6}{3\hbar\omega c^3 m_e^2} \frac{1}{v_z} \frac{E_b}{E} \times \sum_{\Delta m=1}^{\infty} \left(\frac{E}{E_z}\right)^{3/2} (\Delta m)^2 (I,K)_{n',n} d\omega. \qquad (26)$$

In the limit $(E_b/E_z)^{1/2} \gg 1$ we have

$$W(\omega)d\omega = \frac{8e^6}{3\hbar\omega c^3 m_e^2} \frac{1}{v_z} \frac{E_b}{E} \frac{1}{2} \ln\left[\frac{\pi E_z^{1/2} E_b^{1/2}}{2\hbar\omega_c}\right] \qquad (27)$$

whereas, for $(E_b/E_z)^{1/2} \ll 1$,

$$W(\omega)d\omega = \frac{8}{3\hbar\omega c^3 m_e^2} \frac{1}{v_z} \frac{E_b}{E} \ln\left[\frac{\pi E_z}{2\hbar\omega_c}\right]. \qquad (28)$$

In the *high-frequency domain* $\omega \gg \omega_c$ (but formally still $\omega \lesssim \omega_p$ where ω_p is the plasma frequency),

$$W(\omega)d\omega = \frac{8e^6}{3\hbar\omega c^3 m_e^2} \frac{1}{v_z} \sum_{\max(\Delta m)}^{\min(\Delta m)} \frac{1}{\Delta m}. \qquad (29)$$

Equation (19) agrees with the low-frequency limit of ordinary bremsstrahlung,[9] provided that the summation over Δm is cut off at values that correspond to the customary cutoff parameters λ_D (Debye length, due to the plasma correlation) and b_0 (90°-deflection parameter, due to the "straight line approximation").[10]

The present calculation then extends the results of Papers I and II into the domain of low frequencies far from the resonance, and also into the domain of high

[8] Transition probabilities per unit time and per interaction differ by a factor of v.

[9] L. Oster, Phys. Fluids **7**, 263 (1964).

[10] L. Oster, Rev. Mod. Phys. **33**, 525 (1961).

frequencies (as compared with the cyclotron resonance) where they blend into the customary cross section for bremsstrahlung emission in the absence of a magnetic field. As one would expect, the presence of the magnetic field does not affect the emission at these higher frequencies. It is important to note, however, that the presence of the magnetic field changes the cross section at low frequencies, even at great distance from the cyclotron resonance.

4. CROSS SECTION IN THE NEIGHBORHOOD OF THE RESONANCE

It remains to be shown that the general solution (21) predicts the emission correctly "in the neighborhood" of the resonance (corresponding to the wings of the cyclotron line). We do not expect Eq. (21) to be accurate at the resonance proper, due to the fact that the formalism employed in this paper is basically one tailored for continuous emission problems. But since we had derived in Papers I and II accurate cross sections for the resonance line itself, we can be satisfied if these cross sections blend into the expressions predicted by Eq. (21). To be precise, we expect Eq. (21) to yield for $\omega \to \omega_c$ the same results as our previous calculations for $\omega > \omega_c$ or $\omega < \omega_c$.

In order to verify this statement, we first note that for $\omega \approx \omega_c$

$$\sum_{\Delta m=1}^{\infty} (I,K)_{\Delta m} = \tfrac{1}{2}(E_z/E)^{3/2} \ln\left[\frac{\pi E_z^{1/2} E_b^{1/2}}{2\hbar\omega_c}\right], \quad (E_b/E_z)^{1/2} \gg 1, \tag{30a}$$

$$= \tfrac{1}{2}(E_z/E)^{3/2} \ln\left[\frac{\pi E_z}{2\hbar\omega_c}\right], \quad (E_b/E_z)^{1/2} \ll 1. \tag{30b}$$

Inserting (30a) and (30b) into Eq. (21), we find after some straightforward manipulations

$$W(\omega)d\omega = \frac{8e^6}{3\hbar\omega c^3 m_e^2}\left[\frac{\omega^2}{(\omega-\omega_c)^2} + \frac{\omega^2}{(\omega+\omega_c)^2}\right]\left\{\frac{1}{4}\left(1+\frac{E_z}{E}\right)\ln\left[\frac{\pi E_z^{1/2} E_b^{1/2}}{2\hbar\omega_c}\right] + \frac{1}{4}\left(\frac{E}{E_z}\right)^{1/2}\right\}, \quad (E_b/E_z)^{1/2} \gg 1 \tag{31a}$$

and

$$W(\omega)d\omega = \frac{8e^6}{3\hbar\omega c^3 m_e^2}\left[\frac{\omega^2}{(\omega-\omega_c)^2} + \frac{\omega^2}{(\omega+\omega_c)^2}\right]\left\{\frac{1}{4}\left(1+\frac{E_z}{E}\right)\ln\left[\frac{\pi E_z}{2\hbar\omega_c}\right] + \frac{1}{4}\left(\frac{E}{E_z}\right)^{1/2}\right\}, \quad (E_b/E_z)^{1/2} \ll 1. \tag{31b}$$

Equations (31a) and (31b) have to be compared with the corresponding expressions in paper II, namely, Eqs. (II, 122) and (II, 123). We first note that the "collision frequency" ν_a given in Paper II corresponds to the parts of Eqs. (31a) and (31b) which are inside of the curly brackets except for a factor $E^{3/2}$ and a combination of constants which are of no concern to our present discussion.

If $E_b^{1/2} \gg E_z^{1/2}$, Eq. (II, 123) to dominant order becomes

$$-\nu_a \sim \tfrac{1}{4}(E_b/E_z)^{1/2} \gg 1. \tag{32}$$

Inspection of Eq. (31a) shows that the results of the present calculation are identical.

Extension of Eqs. (II, 123) and (31b) into the domain where $E_b \approx E_z = \frac{1}{2}E$ leads to an expression

$$-\nu_a \sim \ln(E/\hbar\omega_c) + O(1), \tag{33}$$

which is the same in both cases.

The only discrepancy occurs when Eq. (II, 123) and Eq. (31b) are compared ($E_z^{1/2} \gg E_b^{1/2}$). From the practical point of view, this discrepancy is of no consequence, since a beam whose energy initially is concentrated along the magnetic field will after a few Coulomb interactions have an energy distribution function which is essentially isotropic. For this brief transition period, our treatment is not valid due to the implicit assumption of a time-constant distribution function. This "adiabatic hypothesis," incidentally, is common in calculations of radiative interactions.

The compatibility of the present formulas and the formulas derived in Paper II can be corroborated by considering the angular average, i.e., by integrating the spectrum over all velocities with an isotropic distribution. It is legitimate in this context to treat the logarithmic terms as slowly varying functions. Then, a straightforward calculation which we dispense with here, shows complete agreement between the various formulations.

This completes the quantum mechanical calculation of the emission of electrons in a magnetic field while undergoing Coulomb interactions.