Making the same consideration as done by other authors[15,19] and considering the nitrogen nucleons as equally probable photopion sources for high-energy photons, we have deduced the total cross section per equivalent quantum for the $\pi^{\pm}$ photoproduction from free nucleons. In the case of the reaction $\gamma+n\rightarrow\pi^{-}+p$, we have considered also the ratio π^{-}/π^{+} for free nucleons obtained by Pine and Bazin[20] from the photoproduction from deuterium. The total cross section calculated gives agreement within 15% with the experimental value $(388\pm65)\times10^{-30}$ cm^2.

The calculated values for the cross section per equivalent quantum and per nucleon for pion production gives the same agreement with the measured values of $(328\pm60)\times10^{-30}$ cm^2 for photostars with one charged pion and $(52\pm13)\times10^{-30}$ cm^2 for photostars with a pion pair.

These results confirm the considerable contribution of the light nuclei to the process of photodisintegration in nuclear emulsion, and confirm the hypothesis that at high energies, the photoproduction of real pions occurs on the individual nucleons also in the case of complex nuclei. The reabsorption of the real $\pi^{\pm}$ mesons photoproduced was found to be negligible with our experimental resolution.

[17] We have used for this purpose the value of the cross section versus E_γ given for these reactions by Komar *et al.* (Ref. 1).

[18] A. N. Gorbunov and V. M. Spiridonov, Zh. Eksperim. i Teor. Fiz. **33**, 21 (1957) [English transl.: Soviet Physics—JETP **6**, 16 (1958)].

[19] C. E. Roos and V. Z. Peterson, Phys. Rev. **124**, 1610 (1961).

[20] J. Pine and M. Bazin, Phys. Rev. **132**, 2735 (1963).

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Model Three-Body Problem*

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Solutions are obtained for a three-dimensional model three-body problem involving a spinless D particle and a spinless n particle with coupling $D\rightleftarrows n+n$. D-n scattering and D-n bound states are studied. The model is soluble in the sense that one obtains a linear, one-dimensional Fredholm equation for each partial wave in n-D scattering. We have solved the equations numerically on a high-speed computer for different values of the interaction strength and for different values of a size parameter used in the interaction form factor. In particular, we have studied the interaction-strength limit which corresponds to making the D a bound state of the n's. In this limit there are two three-body bound s states. The n-D scattering phase shifts obey a Levinson's theorem and also show the expected kink at the threshold for $n+D\rightarrow 3n$. The angular distribution for n-D scattering has considerable variation and shows the backward peak characteristic of an exchange mechanism. When parameters are chosen in the model to make the D fit the deuteron, the major features of nucleon-deuteron scattering are reproduced except at very low energies when the three-particle bound states dominate and our neglect of spin is important.

I. INTRODUCTION

THE theory of scattering beyond the two-body problem has recently been the subject of vigorous attack from many quarters. This is not surprising in view of the wide importance of the problem and the rudimentary state of the theory. Some of the recent efforts have been devoted to putting the formal situation in order for the full problem,[1] but these developments do not remove the essential difficulties associated with going beyond the two-body problem even in classical physics, namely the extra degrees of freedom. It may be that computers will soon enter a stage where the full three-body problem can be computed "exactly," but that stage has not yet arrived.

A more modest approach in which the three-body problem is simplified to the point where "exact" computation is possible has recently been introduced by one of us.[2] In this paper, we present calculations based

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[1] L. D. Faddev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]. C. A. Lovelace, Lecture Notes for the Edinburg Summer School, July 1963 (unpublished); S. Weinberg, Phys. Rev. **133**, B232 (1964). L. Rosenberg, *ibid.* **134**, B937 (1964). A model similar to our potential limit has been studied in a different context by A. N. Mitra. See A. N. Mitra, Nucl. Phys. **32**, 529 (1962) and A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1265 (1963).

[2] R. D. Amado, Phys. Rev. **132**, 485 (1963), hereafter referred to as A.

on that theory. These are presented not primarily as an approximation to some actual physical situation, but rather as a theoretical model in which the three-body aspects are exactly treated. Our main concern is to explore the exact consequences of allowing three bodies to interact and to study these as a function of the parameters of the model. To that end we take the simplest possible version of the theory. It is noteworthy that even in this version, in which two-body scattering is simple, the three-body amplitude is quite rich in structure.

The model deals with the world of a spinless particle, n, and another, D. The particles are named for the nucleon and deuteron, but resemble them in little else. Section IV is devoted to a comparison of our results with the three-nucleon system for orientation, but we stress again that we are not primarily concerned here with making a model of it. The interaction allows only the process $D \rightleftarrows n+n$. Both n and D are free to move and are assigned nonrelativistic energy-momentum relations. Thus n-n scattering occurs only in s states, since the interaction always forces the n-n systems into the D intermediate state. The scattering is characterized by the strength and form factor of the n-n-D coupling and by ϵ, the rest energy or binding energy of the D. This energy provides an energy scale to the problem.[3] The coupling strength may take on all values from zero up to a critical value. These correspond to variations in the wave-function renormalization Z of the D between 1 and 0. In the limiting case of maximal coupling, Z is zero and the model is identical to a potential model in which the n-n interaction is a separable potential and the D is a bound state in that potential. For $Z\neq 0$ the potential analog does not obtain, since one can then weaken the n-n coupling by varying Z, but keeping the position of the D fixed. It is clear that this cannot be done in a separable-potential theory. The form factor represents the spatial structure of the interaction; in the bound-state limit for the D, it is simply related to the bound-state wave function. In our computation, we take this to be of the Hulthén form[4] and hence introduce another parameter—the range of the Hulthén function. Thus our model contains two parameters, the range of the Hulthén form factor, and the strength of coupling, or, equivalently, the wave-function normalization of the D. Of course, in this model n-n scattering is trivially soluble. The point of the model is to turn it to n-D scattering, for which case one can derive an integral equation for the scattering amplitude.[2] This equation is not trivially soluble, but because of the restrictive nature of the n-n interaction turns out to be more complicated than the Lippmann-Schwinger equation[5] for potential scattering with a nonlocal energy-dependent potential. That is, the intermediate states, in the center of mass, are completely characterized by a single momentum vector, as are the intermediate states in the potential scattering equations. The equation can be solved on a high-speed computer by Fredholm methods after partial-wave analysis.

The effective three-body "potential" as represented by the Born approximation, involves the exchange of an n from incoming D to incoming n to form the outgoing D; it is an "exchange potential" and is attractive in even partial waves and repulsive in odd. We look for three-particle s-wave bound states in this "potential" by finding the zeros of the Fredholm determinant. In the case of maximal coupling ($Z=0$ for D), there are two s-wave three-particle bound states for all values of the Hulthén range searched. One is weakly bound and the other strongly bound; it is, in fact, much too strongly bound to represent the triton if the parameters of the D are fitted to the deuteron. If the coupling is weakened slightly, the weaker bound state disappears; but the other stays for a wide range of coupling. There are no p-wave bound states since the p-wave "potential" is repulsive, and there are none in higher partial waves. The effect of keeping only the two-particle intermediate states on the positions of the bound states in s wave is investigated. It is found that even though the inelastic threshold may be far from the three-particle binding energy, leaving out the three-particle states makes a qualitative difference and is therefore a poor approximation.

Since the second bound state is so near the scattering threshold for strong coupling, the s-wave scattering amplitude is very large at low energies. This is even true when the state is virtual but with opposite sign for the scattering length. For the s-wave n-D scattering, there is a kink at the threshold for $n+D\rightarrow 3n$, and then a sharp minimum. This can be understood in terms of a generalized Levinson's theorem.[6] Since for strong coupling there are two bound states, we can take the s-wave phase shift to begin at 2π, and we would then expect it to fall through $3\pi/2$ (antiresonance) π and $\pi/2$ (antiresonance), arriving at zero for infinite energies. The point at which it passes π will be a zero of the real and imaginary parts of the amplitude in potential scattering. Since for strong coupling this point comes above the breakup threshold, the cross section is not exactly zero, but the closeness to threshold makes the energy variation rapid. If the coupling is weakened so that there is only one bound state, the phase shift starts at π, but rises and then falls back through π below the breakup threshold, giving a real zero. In both cases, the other partial waves mask any effect of this on the total cross section.

The angular distribution shows the backward peak-

[3] We restrict ourselves to the use of a stable D. The case of $3n\rightarrow 3n$ scattering for unstable D is very interesting, and we hope to deal with it later.

[4] L. Hulthén and M. Sugawara, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

[5] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[6] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **9**, 25 (1949).

ing associated with an exchange type of reaction, but has considerably more structure than just the Born approximation (as well as being generally much smaller). Typically, the angular distribution is of the sort expected for a direct reaction, with variation over two orders of magnitude. We have made no attempt to fit these with some of the modern approximation procedures, but such attempts would surely be interesting.

At very high energies the amplitude tends to the first Born approximation. This will be demonstrated in a subsequent paper. This is true even if $Z=0$, for which case the Born series does not converge for any energy.[7] In view of this, we have calculated only to energies sufficiently high that we are approaching the Born answer.

In Sec. II we present a summary of the model and the major points of our calculational method. In Sec. III the results for the bound state and scattering data are presented for various strengths of coupling and various Hulthén ranges. In Sec. IV the results are compared with the three-nucleon system, and in Sec. V there is a discussion of the results and future programs.

II. CALCULATIONAL METHOD

The integral equation for the n-D scattering amplitude in the center-of-mass system given in A is ($\hbar=2m=1$):

$$(\mathbf{k}'|T(E)|\mathbf{k}) = (\mathbf{k}'|B(E)|\mathbf{k}) + \frac{1}{(2\pi)^3}\int d^3k''(\mathbf{k}'|B(E)|\mathbf{k}'') \times \frac{S(E-\frac{3}{2}k''^2+\epsilon+i\eta)}{E-\frac{3}{2}k''^2+\epsilon+i\eta}(\mathbf{k}''|T(E)|\mathbf{k}), \quad (1)$$

where the Born function is

$$(\mathbf{k}'|B(E)|\mathbf{k}) = \Gamma^2 \frac{f[(\mathbf{k}+\frac{1}{2}\mathbf{k}')^2]f[(\mathbf{k}'+\frac{1}{2}\mathbf{k})^2]}{E-[k^2+k'^2+(\mathbf{k}+\mathbf{k}')^2]+i\eta}. \quad (2)$$

It represents the basic n-D interaction involving the exchange of an n. The function

$$[S(x)]^{-1} = 1 - \frac{\Gamma^2}{2(2\pi)^3}x\int d^3n \frac{f^2(n^2)}{(2n^2+\epsilon)^2(x-\epsilon-2n^2)} \quad (3)$$

represents the sum of "bubbles" for D in intermediate states. The renormalization coupling constant Γ is related to the D wave-function renormalization Z by

$$Z = 1 - \frac{\Gamma^2}{2(2\pi)^3}\int d^3n \frac{f^2(n^2)}{(2n^2+\epsilon)^2}; \quad (4)$$

$f(q^2)$ is the form factor for the n-n-D vertex; k, k' are the initial and final n momenta. E is the total energy variable, and ϵ is the D binding energy or rest energy.

To solve the integral equation, we first make a partial-wave analysis:

$$(\mathbf{k}'|T(E)|\mathbf{k}) = \sum_0^\infty (2l+1)P_l(\cos\theta)(k'|T_l(E)|k),$$

$$(\mathbf{k}'|B(E)|\mathbf{k}) = \sum_0^\infty (2l+1)P_l(\cos\theta)(k'|B_l(E)|k), \quad (5)$$

$$\cos\theta = \mathbf{k}'\cdot\mathbf{k}/k'k.$$

There is no partial-wave projection for

$$S(E-\tfrac{3}{2}k''^2+\epsilon+i\eta),$$

which is a function of k''^2 only. We obtain an uncoupled set of one-dimensional linear integral equations. For each partial wave we have

$$(k'|T_l(E)|k) = (k'|B_l(E)|k) + \int_0^\infty dk''(k'|K_l(E)|k'')(k''|T_l(E)|k), \quad (6)$$

where the kernel is

$$(k'|K_l(E)|k'') = \frac{k''^2}{2\pi^2}(k'|B_l(E)|k'')\frac{S(E-\frac{3}{2}k''^2+\epsilon+i\eta)}{(E-\frac{3}{2}k''^2+\epsilon+i\eta)}. \quad (7)$$

The equation can be cast formally into an inversion problem:

$$\int_0^\infty dk''[\delta(k'-k'')-(k'|K_l(E)|k'')]\times(k''|T_l(E)|k) = (k'|B_l(E)|k). \quad (8)$$

This is the starting point of our calculations. However, there are, in general, complications due to the complex nature of singular points in the kernel. So, we turn first to a discussion of the parameters and functions associated with the equation.

The interaction is characterized by ϵ, Γ, Z, and $f(q^2)$. Among these, ϵ is chosen as the energy scale and is set at $\epsilon=1.5$ for all calculations. Moreover, since Γ and Z are related through (4), only Z and $f(q^2)$ are adjustable. The choice of $f(q^2)\equiv 1$ has been discussed in A, where it is shown that a singular integral equation results when $Z=0$. In this paper, we choose a Hulthén form[4] for it:

$$f(q^2) = 1/(q^2+\beta^2), \quad (9)$$

where β is an adjustable parameter and is qualitatively the inverse range of the n-n potential in configuration space. Since $f(q^2)$ is related to the internal momentum distribution function $\phi(q^2)$ of D when $Z=0$ by

$$\Gamma^2 f(q^2) = (2q^2+\epsilon)\phi(q^2), \quad (10)$$

[7] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).

we restrict β to values where $\beta^2 > \frac{1}{2}\epsilon$. Note that we are keeping the D binding energy ϵ constant when we vary β.

For each value of l, the solution of the integral equation is a T-matrix element, $(k'|T_l(E)|k)$, which is a function of 3 variables, k', k, and E. The physical n-D scattering amplitude is a particular one of these solutions corresponding to $k'=k$ and $E=3k^2/2-\epsilon$. There is no need to find such general off-the-energy-shell amplitudes. For n-D scattering, we fix the energy variable on the incident momentum via the relation $E=3k^2/2-\epsilon$. Now, the integral equation involves only two variables, k' and k. However, for the bound-state problem associated with the n-D system, we do not fix E on k but treat it as a free variable with a range $E<-\epsilon$. The bound states, if any, of the n-D system are, of course, not explicitly exhibited in the integral equation because a complete set of eigenstates of the free Hamiltonian is used in the intermediate states. Rather, they emerge as dynamical consequences of the interaction parametrized by Z and β.

With the vertex function given explicitly, we can evaluate all integrals involved in Γ^2, $(k'|\beta_l(E)|k)$, and $S(E-\frac{3}{2}k''^2+\epsilon+i\eta)$ and, with the specification of the energy variable in mind, discuss any complications which the functions may introduce into the problem. First, let us define for convenience some symbols:

$$\begin{aligned}
\alpha &\equiv (\tfrac{1}{2}\epsilon)^{1/2}, \\
\sigma &\equiv E-\tfrac{3}{2}k''^2+i\eta, \\
a &\equiv a(k',k;E) \equiv k^2+k'^2-\tfrac{1}{2}E-i\eta, \\
b &\equiv b(k',k;\beta) \equiv \tfrac{1}{4}k'^2+k^2+\beta^2, \\
c &\equiv c(k',k;\beta) \equiv \tfrac{1}{4}k^2+k'^2+\beta^2.
\end{aligned} \tag{11}$$

Then we obtain the following results:

(I)

$$\Gamma^2 = 64\pi\alpha\beta(\alpha+\beta)^3(1-Z). \tag{12}$$

(II.a) for $k'\neq 0$ and $k\neq 0$,

$$(k'|B_l(E)|k) = \left(\frac{\Gamma^2}{2}\right)\frac{(-1)^{l+1}}{k'k}\left[\frac{1}{b-a}\frac{1}{c-a}Q_l\left(\frac{a}{k'k}\right) + \frac{1}{c-b}\frac{1}{a-b}Q_l\left(\frac{b}{k'k}\right) + \frac{1}{a-c}\frac{1}{b-c}Q_l\left(\frac{c}{k'k}\right)\right]. \tag{13a}$$

(II.b) for $k'=0$ or $k=0$,

$$(k'|B_l(E)|k) = -(\Gamma^2/2)(1/abc)\delta_{l0}, \tag{13b}$$

where $Q_l(z)$ is the Legendre function of the second kind.

(III) We define

$$[S(E-\tfrac{3}{2}k''^2+\epsilon+i\eta)]^{-1} \equiv Z+(\Gamma^2/32\pi)\zeta, \tag{14}$$

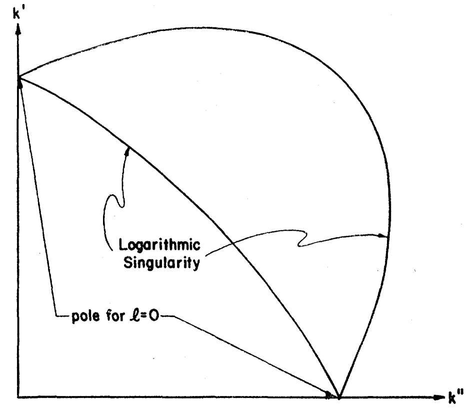

FIG. 1. Section of intersecting ellipses locating singularities of the Born function.

then, (a) for $\sigma\leq 0$, ζ is real and is

$$\zeta = \frac{1}{2\beta(\alpha+\beta)[\alpha+(-\frac{1}{2}\sigma)^{1/2}][\beta+(-\frac{1}{2}\sigma)^{1/2}]} \times\left[\frac{1}{\alpha+\beta}+\frac{1}{\beta+(-\frac{1}{2}\sigma)^{1/2}}\right]; \tag{15a}$$

(b) for $\sigma>0$, ζ is complex and

$$\begin{aligned}
\mathrm{Re}\zeta &= \frac{4}{(\alpha+\beta)(\epsilon+\sigma)(2\beta^2+\sigma)}\left[\frac{2\alpha\beta-\sigma}{2\beta^2+\sigma}-\frac{\epsilon+\sigma}{4\beta(\alpha+\beta)}\right], \\
\mathrm{Im}\zeta &= \frac{\sqrt{2}\sigma^{1/2}}{(\epsilon+\sigma)(\beta^2+\frac{1}{2}\sigma)^2}.
\end{aligned} \tag{15b}$$

The analytic properties of the functions $(k'|B_l(E)|k)$ and $S(E-\frac{3}{2}k''^2+\epsilon+i\eta)$ are now obtained from these relations. $S(E-\frac{3}{2}k''^2+\epsilon+i\eta)$ has a cut in the complex E plane from 0 to ∞. That it, in general, becomes complex for $E>0$ is due to the "breakup" of the "bubbles" for D, that is, to the production of three real particles in the intermediate states ($n+D\rightarrow 3n$), the threshold for this being $E=0$. From the properties of $Q_l(z)$ and the fact that for $k'\geq 0$ and $k\geq 0$, the two terms $b/k'k$ and $c/k'k$ are both greater than 1, we see that the Born function will be complex and will possess logarithmic and even pole (at $k'=0$ or $k=0$) singularities when $|a/k'k|\leq 1$. This occurs when $E>0$. However, the inhomogeneous Born function will be real for all values of E if we fix E on k via the relation $E=3k^2/2-\epsilon$. Physically, that $|a/k'k|\leq 1$ for some values of E is a reflection of the fact that at this energy, a real n is being exchanged in the intermediate state. This again produces an absorptive effect on the elastic n-D scattering. Thus we see that the product $(k'|B_l(E)|k'')S(E-\frac{3}{2}k''^2+\epsilon+i\eta)$ plays the role of an energy-dependent "optical potential" in the kernel

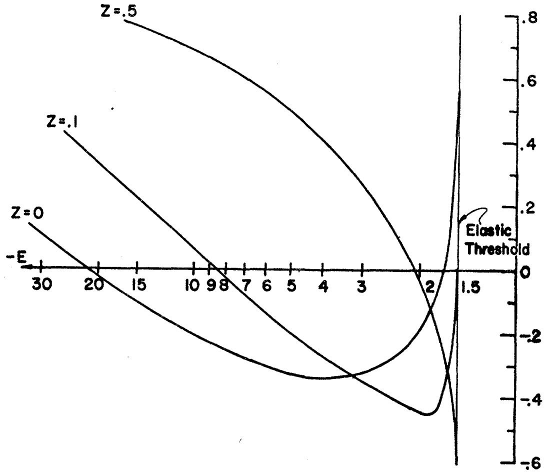

FIG. 2. s-wave Fredholm determinant as a function of energy below scattering threshold for $\beta=7$ and various Z. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

with the inelastic threshold at $E=0$. (The elastic n-D threshold is at $E=-\epsilon$.)

In the actual computation, our treatment of the singular nature of the Born function may be summarized by a diagram in k'-k'' space. (See Fig. 1.) Two quarter ellipses are defined by the equations

$$k'^2+k''^2\pm k'k''=\tfrac{1}{2}E,\quad E>0. \tag{16}$$

Within the area bounded by the two elliptic arcs, $(k'|B_l(E)|k'')$ is complex and is evaluated through a complex $Q_l(a/k'k'')$. On the edges, it has, in general, a logarithmic singularity, which we "smooth out" by averaging $Q_l(a/k'k'')$ over a thin strip about the edge. At the two intercepts on the k' and k'' axes, the logarithmic singularity develops into poles. There, for $l\neq 0$, the Born function is zero from (13b). For $l=0$, the Born function is set equal to zero at the pole on the k' axis but is evaluated as a genuine pole on the k'' axis. The asymmetry in treatment arises from phase-space factor k''^2 for the intermediate state in the kernel.

Having discussed the singular points encountered in the evaluation of the kernel, we return to the inversion problem. With the choice of the vertex function, $f(q^2)$, it is easy to see by power counting that the kernel is sufficiently convergent for the Fredholm method of solution to apply. For any fixed value of l and k, we approximate the integral of (8) by a finite sum using Simpson's rule. The kernel is evaluated in k'-k'' space on a square mesh of $N\times N$ points. Equation (8) now becomes a matrix equation.

For the bound-state problem, k need not be specified. We treat E as a free variable, and search for zeros of the Fredholm determinant $D_l(E)$ below the elastic threshold $E=-\epsilon$. The kernel is real for $E<-\epsilon$, and the Fredholm determinant is

$$D_l(E)=\det|\delta_{ij}-K_{ij}(E)|\,, \tag{17}$$

where $K_{ij}(E)$ is proportional to the kernel evaluated at the mesh point (k'_i,k''_j). The position of the zeroes gives the bound-state energies, which depend on the parameters β and Z. We assume the Fredholm numerator does not vanish at the zero of D_l and hence that these zeros are actual bound states.

In n-D scattering, we set $E=3k^2/2-\epsilon$. The inhomogeneous Born function is real but the kernel is now complex. For $E<0$, there is only one pole—the n-D propagator pole—to integrate over, while for $E>0$, there is an additional pole when $l=0$ from the Born function at $k'=0$. The numerical integration over the pole is treated in the usual way. Now the kernel can be evaluated everywhere on the mesh points. The inversion[8] of the matrix equation yields for given value of k and l a complex vector which is a sequence of off-the-energy-shell amplitudes $(k'_i|T_l(E)|k)$. The physical amplitude is, of course, the one with $k'_i=k$.

III. RESULTS

A. Bound States

We here examine the position and number of three-body bound states for our model as functions of β, the Hulthén range parameter, and Z, the wave-function renormalization constant of the D. We do this by methods discussed in the previous section; namely, by searching for zeros of the Fredholm determinant $D_l(E)$ of Eq. (17).

Inherent inaccuracies in our programs and technical difficulties associated with the computer prevent us from examining certain values of our parameters and variables. For example, β much greater than 15.0–20.0 requires too large a range of integration to be practical. For values of β less than 2.0–1.5 and/or the energy E too close to -1.5 (the elastic n-D threshold), the kernel of the integral equation becomes very peaked near the origin, and the matrix inversion routine fails to give satisfactory results. Nevertheless, we quote results in these unattainable limits on the basis of trends established by the machine. We feel this is justified because of the empirically established smooth behavior of the quantities in question as a function of the parameters, and also because an examination of the analytic structure of our equations indicates the unlikelihood of any spectacular effects in the limits mentioned.

The results of our computations are shown in Figs. 2–11. In Fig. 2, $D_l(E)$ for the s wave is shown as a function of E for various choices of Z and with $\beta=7.0$. D_l is one for very large negative E, and for attractive potentials curves toward zero as E increases. Each zero of D_l indicates the position of a bound state. Increasing Z away from zero represents weakening the effective n-D interaction, and the effect of this on D_l

[8] In our computation, we use an IBM matrix subroutine which employs the elimination method.

is clear in the figure. In particular, the bound states become less bound as Z increases. The cusp in D at the elastic-scattering threshold is due to a square-root singularity.

In the $Z=0$ limit, the s-wave n-D interaction produces two three-particle bound states for all values of the Hulthén range. That at least one should occur with a binding energy greater than that of D is not surprising, since $Z=0$ corresponds to the potential limit, and if the potential between pairs is strong enough to produce a two-body bound state, it should surely produce a three-body bound state, provided the Pauli principle does not operate, as it does not in our case. In higher partial waves, there are no three-particle bound states, since the "exchange potential" is repulsive in p waves and not strong enough in d waves.

The effect of varying Z on the s-wave bound states is shown in Fig. 3 for $\beta=5$. One bound state quickly disappears as the coupling is weakened (Z becomes greater than zero), and the second moves toward the elastic scattering threshold and finally disappears for $Z\approx 0.7$. For $Z\approx 1$, the nn-D coupling is very weak, perturbation theory presumably holds, and there is no three-particle bound state. We have not studied this featureless limit.

Since these are s-wave bound states, the zero of $D_0(E)$ corresponding to them moves through the square-root branch cut at the elastic-scattering threshold and back along the negative E axis on the second sheet, now corresponding to a virtual state, as the coupling is turned down. That is, they *do not* become resonances. We have not searched carefully to preclude the possibility that they return for even weaker coupling to resonant positions, but this seems unlikely. In fact, anticipating, we can say we have found no choice of parameters which gives a three-particle resonance in any partial wave. As the bound states disappear, the n-D scattering length goes through infinity and changes sign.

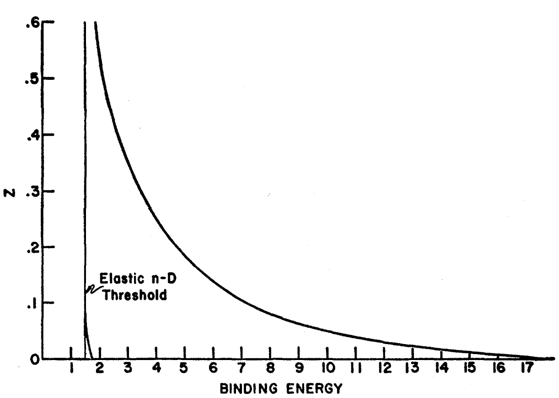

FIG. 3. Three-particle binding energy as a function of Z for $\beta=5$. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

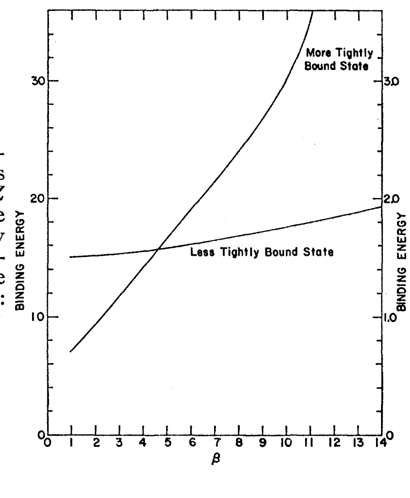

FIG. 4. Three-particle binding energy as a function of β for $Z=0$. Right-hand scale refers to less tightly bound state and left-hand scale to more tightly bound. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

The effect of varying β and keeping Z zero on the s-wave bound states is shown in Fig. 4. It is clear that increasing β corresponds to increasing the effective n-D interaction strength. Since the position of the D binding energy is kept fixed at the same time, the strengthening is being achieved by shortening the range ($\propto 1/\beta$) of the two-body force but increasing its strength. Since the three-particle states are much more compact than the two-particle states, they are much more sensitive to this. In particular, one would expect the more tightly bound three-particle state to respond quite strongly to this short-range potential, and it does. All indications are that its binding energy goes to minus infinity as β goes to infinity. This limit on β corresponds to making the form factor in (9) identically one, since as we increase β but keep the D binding fixed and $Z=0$, the coupling constant will also grow at just the proper rate. We should expect just this, since for $f=1$ the equation cannot be solved by Fredholm methods, and the existence of an infinite-energy "bound state" means that the homogeneous integral equation has solutions for large energy which are not determined by the Born approximation. These are probably diffraction solutions, but we have not yet established this.

It would be of great interest to check these exact results against various currently popular approximation schemes. We have not done this in general, but one simple check we have carried out is to do the calculation with $S\equiv 1$ in (3). This corresponds to neglecting three-particle intermediate states, while two-particle states are still treated correctly, including two-particle unitarity. According to the principle of dominance of closest singularities, this neglect should not be very serious for bound states, so long as the elastic-branch cut is correctly put in, as it is. The effect of doing the calculation with $S\equiv 1$ is compared with the correct result for D_0 with $\beta=5$ and $Z=0$ in Fig. 5. We see that the actual system has two bound states, one quite tightly bound, whereas the approximate one has only one relatively weak bound state. Hence, neglecting

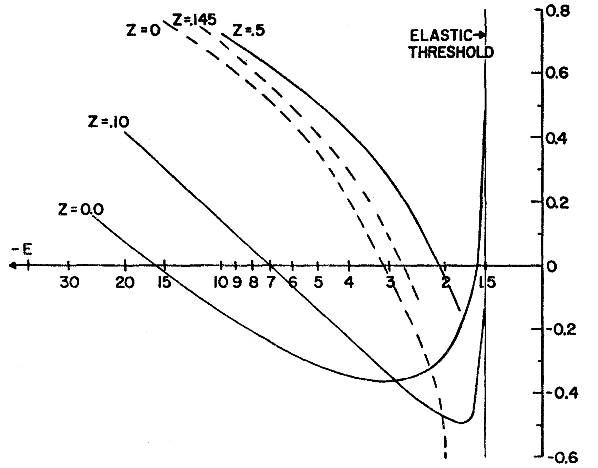

FIG. 5. Effect of neglect of three-particle states on s-wave Fredholm determinant for $\beta=5$, and various Z. Full curves are exact and dotted curves with no three-particle states, $S=1$. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

the three-particle intermediate states weakens considerably the effective n-D potential.

B. Scattering

The scattering amplitudes for each partial wave have been computed according to the procedure outlined in Sec. II, and the results are partially summarized in Fig. 6 where the real part of the phase shifts are presented for the first three partial waves and for various choices of Z and β. Below the production threshold, $E=0$, the phase shift is real; above production the real part is defined by[9]

$$\tan[\mathrm{Re}\delta_l(k)]=\frac{\mathrm{Re}T_l}{\mathrm{Im}T_l}-\frac{3\pi}{k\,\mathrm{Re}T_l}\times\left\{1-\left[1+\frac{2k}{3\pi}\left(\mathrm{Im}T_l+\frac{k}{6\pi}|T_l|^2\right)\right]^{1/2}\right\},\quad(18)$$

$$T_l\equiv(k|T_l(E=\tfrac{3}{2}k^2-\epsilon)|k),$$

where the appropriate branch of the arctan is taken to give a continuous curve. The effect of breakup is noticeable in the kink in the s-wave phase shift at threshold. We have normalized the phase shifts to zero at infinite energies, and for $Z=0$ this leads to an s-wave phase shift of 2π at zero energy. This is what we would expect naively from Levinson's theorem,[6] since there are two s-wave bound states for $Z=0$. Even though Levinson's theorem has not been shown to be valid for three-particle scattering, some form of it almost certainly is. The s-wave phase shift of Fig. 6 for $Z=0$ is about as simple a one as one can imagine consistent with these conditions: 2π at zero energy, a kink at the inelastic threshold, and zero at infinite energies. All that can change is its rate of fall. We have not investigated this in detail, but, from the points we have in the figure and from the discussion of the previous section, it is clear that larger β, corresponding to a stronger but shorter range two-particle potential, makes the phase shift fall more slowly, and smaller β makes it fall more rapidly.

For $Z=0.145$, $\beta=5$, there is only one three-particle bound state and the s-wave phase shift begins at π. Now, however, the scattering is dominated by a nearby virtual state at low energy and the phase shift starts with positive slope. At higher energy it turns over and falls to zero slowly. For even larger Z, the virtual state would move out and the other bound state come closer to threshold, and the phase shift would begin at π with negative slope. For Z near 1, the phase shift would start and end at zero. Presumably the effect of varying β on all these cases would be as in the case of $Z=0$; i.e., increasing β slows the energy variation of the phase shift.

For p waves, the phase shift is negative, since we have an effective n-D "exchange potential." Since there are no p-wave bound states, nothing very dramatic happens when we vary Z. Although the results are not shown, there are also no startling results of varying β. In d waves the force is attractive, but not very strong, and this is reflected in the relatively small phase shifts and small sensitivity to Z.

Above the breakup threshold, the cross sections cannot be computed from the real part of the phase shift. However, both the elastic-scattering cross section and the total cross section can be determined from the

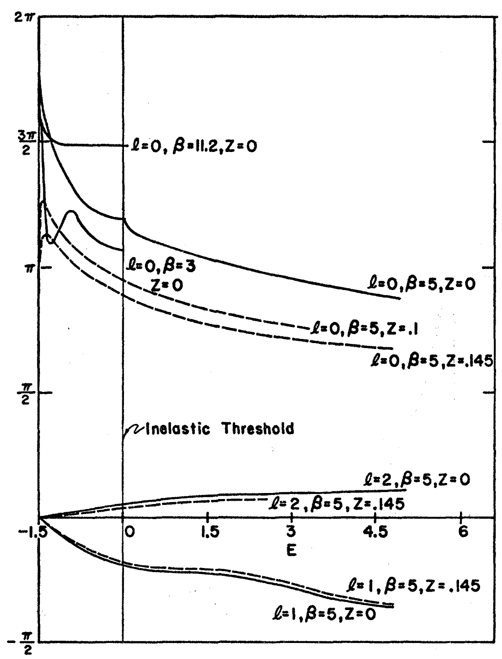

FIG. 6. Real part of the n-D phase shift as a function of energy for various l, β, and Z. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

[9] The elastic unitary relation is $\mathrm{Im}T_l=-(k/6\pi)|T_l|^2$.

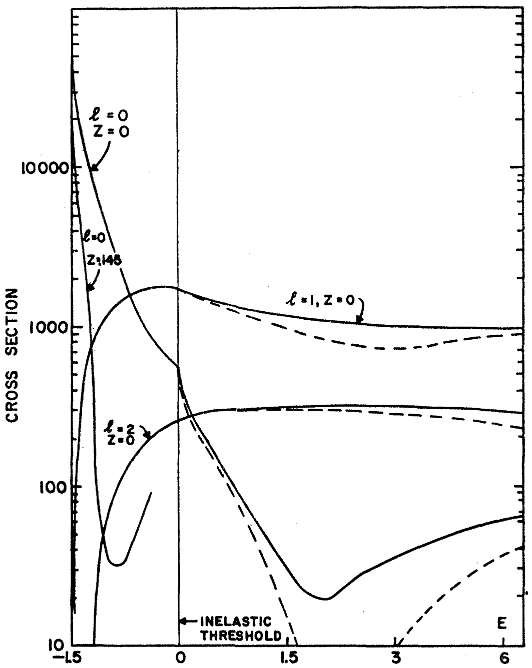

FIG. 7. Partial-wave cross sections versus energy. The solid curves are the total cross section, the dotted the elastic only. For $l=0$, $Z=0.145$ the cross section falls under the $l=2$, $Z=0$ above the inelastic threshold, and therefore we do not show it. All curves are for $\beta=5$. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

scattering amplitude and unitarity, and since there is only one inelastic channel, the breakup cross section can be obtained from these. The elastic-scattering and total cross sections for each partial wave are shown in Fig. 7 up to $l=2$ for $\beta=5$. The kink in the s wave at threshold is clearly visible. The effect of varying Z is shown only for the s wave, where it is dramatic. The very small cross section for $l=0$, $Z=0.145$ at low energies is due to the way in which the phase shift, for this set of parameters, stays near π. For $Z=0$, the s-wave phase shift crosses π above breakup threshold, and therefore there is a minimum, but no true zero in the elastic scattering at this point. In the other partial waves, increasing Z just reduces the cross section.

In general the breakup cross section—the difference between the total and elastic-scattering cross sections—is relatively small in the region examined. At higher energy they will both go to zero, the inelastic more rapidly than the elastic, since at very high energy the Born approximation dominates, and it is pure elastic.

The scattering data can also be reassembled into angular distributions. One such for an energy $E=-0.914$ (below breakup) and $\beta=5$ is shown in Fig. 8. From Fig. 6, it is clear that for $Z=0$ at this energy we have mostly s and p waves, and with opposite sign. This accounts for the backward peaking (exchange potential). For orientation, the Born approximation is also shown. It is relatively flat, and a factor of 15 larger than the correct answer. In plotting the exact curve in Fig. 8, the Born approximation has been used for the partial waves above $l=3$. The addition of these waves has little effect. For $Z=0.145$, the major change in Fig. 8 is to put the s-wave phase shift near π. Since the d wave is small at this energy, the angular distribution now dips to essentially zero at 90°.

At higher energy, the imaginary part of the amplitude is relatively more important, and hence the forward peak grows since the imaginary parts are all of the same sign. This is seen in Fig. 9, which gives the angular distribution for $E=4.835$ and $\beta=5$. For $Z=0$ the near symmetry about 90° is due to the fact that the s-wave phase shift is near π and the p wave is left to dominate. For $Z=0.145$, only the s wave changes much, and the effect of this is to push the minimum to backward angles. It should be noted that at this energy the angular distribution varies over a factor of 10. The Born term is again shown for comparison.

IV. NUCLEONS, DEUTERONS, AND TRITONS

Our model is, if not based on, at least inspired by the actual three-nucleon system, and it is therefore illuminating to try our results against those of that system. Since nucleons are not spinless bosons, we should not expect close agreement with experiment; but to the extent that nucleon exchange is the main source of the force between nucleon and deuteron—and it presumably is, because of the diffuse nature of the deuteron—the major trends of the nucleon-deuteron system should be reproduced.

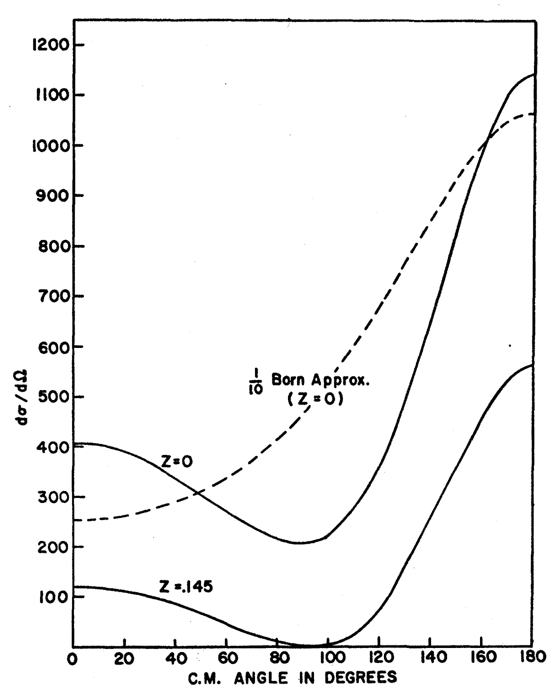

FIG. 8. Angular distribution at $E=-0.914$ and $\beta=5$. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

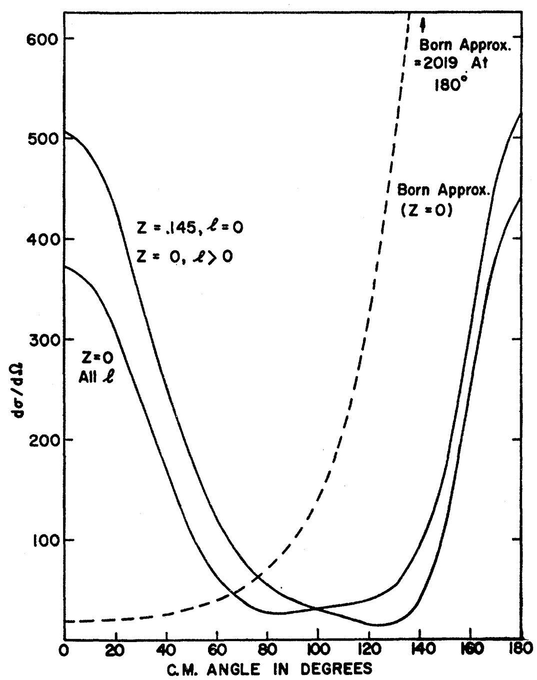

FIG. 9. Angular distribution $E=4.835$ and $\beta=5$. The mixed choice of Z for the one curve is motivated in Sec. IV. (Units: $\hbar=2m=1$, $\epsilon=1.5$.)

The correct characterization of the two-particle coupling is now given, of course, by $Z=0$—the deuteron is a pure bound state. Furthermore, we take $\beta=5$. This choice of parameters, plus putting $\hbar$, m, etc., back in, gives a good fit to the low-energy *two-body* data in the *triplet* state. However, it gives two three-particle s-wave bound states, of which one is much more tightly bound than the triton. This is partly a result of the neglect of spin. It is probably not a bad approximation to say that the spatial wave function of the triton is symmetric, but all pairs do not interact in triplet states, and the singlet force is weaker than the triplet. Furthermore, our characterization of the force does not include any saturating parts such as hard cores. These are presumably more important in the three-body system than in the two-body system since the triton is more tightly bound and the pairs spend more of the time close together in it. That is, while nucleon exchange might be the principal mechanism for n-D scattering at moderate energies, in the relatively compact triton, more complex interactions are also important. We are presently setting the problem up with spin and with the singlet-triplet difference, and are able to report on the importance of at least this sophistication. Hard cores can also be included, but it is more difficult.

To some extent, both the effect of the short-range repulsion and the singlet-triplet difference can be simulated by weakening the interaction in our model. It should be recalled that we can do this by increasing Z without changing the value of the deuteron energy. For $Z=0.145$, the weakly bound three-particle bound state disappears, and the other has the correct binding energy for the triton. Of course, changing Z from zero reduces the coupling constant at the $D\rightleftarrows n+p$ vertex, and therefore the nucleon exchange graph will not have the correct residue at its momentum-transfer pole. On the other hand, placing the triton correctly gives the correct position and residue to the pole in the energy corresponding to the triton. At low energies for n-D scattering this would be important, whereas at higher energies getting the exchange graph right should be more important. Since we have no spin, however, we cannot give separately the scattering in quartet and doublet states; nor are these separated experimentally. The triton comes only in the doublet state, and fitting its pole correctly is therefore not a big help in fitting experiments. This argument is to excuse the fact that we do not get even a qualitative fit to n-D scattering at low energies with $Z=0$ or $Z=0.145$. To some extent we can have both poles fit nearly correctly by putting $Z=0.145$ for the s wave and $Z=0$ for the other partial waves. From Fig. 6 it is clear that the difference between zero and 0.145 for Z is not very important beyond s waves. We use this hybrid theory to compare with experiment for the three-nucleon system. Strictly, our theory should have no free parameters, since β and Z can be fixed by the two-body data, but putting $Z=0.145$ for s waves corresponds to making one adjustment—fitting the triton.

The experimental results for the total n-D scattering cross section and our results are shown in Fig. 10.[10] The agreement is not startling, particularly at low energies, where the scattering lengths are incorrectly given, for the reasons already elaborated. However, at higher energies the trends and the order of magnitude are certainly reproduced. The angular distribution at

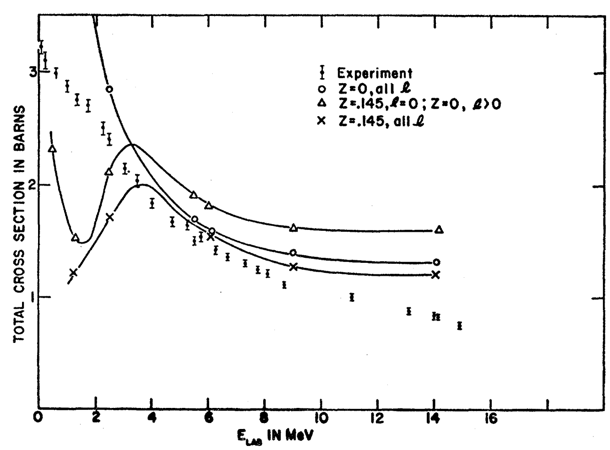

FIG. 10. Total cross section for neutron-deuteron scattering—experiment and theory. Only the points are calculated; the curves are drawn in for convenience.

[10] R. J. Howerton, University of California Radiation Laboratory Report, UCRL 5226 Rev., 1959 (unpublished).

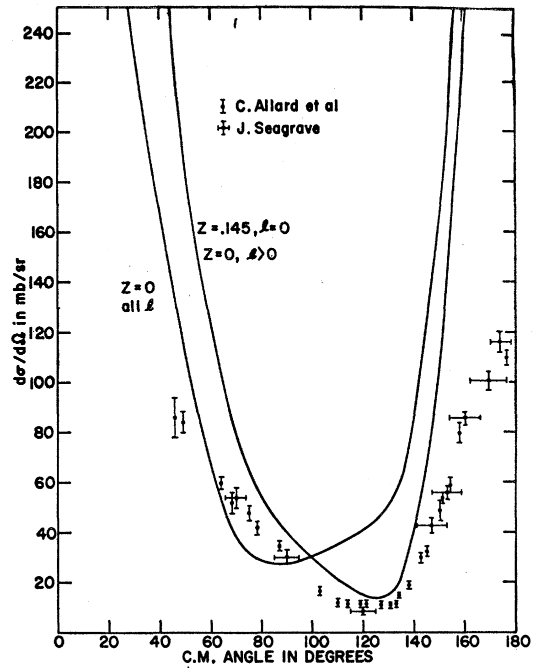

FIG. 11. Angular distribution for neutron-deuteron scattering at 14.1 MeV—experiment and theory.

14.1 MeV is as shown in Fig. 11.[11] The theoretical curve is the same as Fig. 9 with the units put in. Once again, we see that the major features of the data, as well as their order of magnitude, are reproduced. Both these agreements indicate that the qualitative features of the data are reproduced by the model and therefore that the principal mechanism is nucleon exchange. We are presently investigating the importance of adding spin. Hopefully this will improve things, particularly at lower energies. After that, however, refinements of the theory will presumably be much more difficult.

V. DISCUSSION

We have seen that the simple, soluble three-body model presented previously can be solved numerically and that the results are rich in the features of three-particle systems in spite of the simplicity of the model. In fact, the model has qualitatively many of the features of the three-particle system it most nearly resembles—the three-nucleon system. There are now essentially two sorts of next steps one can take. One can try to improve the model's resemblance to actual physical systems, or one can try to use the method as a probe into three-particle systems in general. One of the most appealing avenues in this direction is to use it to test currently popular approximation techniques. We have already shown, for example, that ignoring three-particle intermediate states in the integral equation gives a bad approximation for the bound states. One of us (RA) is currently involved in solving the equations using elastic "N over D" techniques[12] (these are approximate for the problem) in order to check them. Other such tests would be interesting.

Another program is connected with the question of unstable two-particle systems and their effect on the three-particle system. This can be investigated by making the D unstable or by introducing C–D–D poles in the two-particle amplitude.[13] Exact and approximate results of doing this would be interesting, as would be the occurrence in this way of three-particle resonances. We find none in our simple problem.

From a more general point of view it would be valuable to extend and improve the formalism to allow a richer representation of the two-particle force—for example, in order to allow one to treat hard cores. The inclusion of spin in the problem is straightforward and has been done by us. Whether the coupled equations which this leads to will be simple and manageable on the computer remains to be seen.

Having included spin, we can turn to a better model of the three-nucleon system. This is presently being done. Also the coupled-channel model of deuteron stripping presented in A is being computed. It will be interesting to compare the results of that calculation with the distorted-wave Born-approximation calculations. Going further afield, one might hope to investigate electron-atom scattering, although here the large number of bound states—a manifestation of the long range of the force—makes our method difficult to apply. Whether any actual problems in particle physics can be solved in this way, as opposed to being investigated formally as we outlined, remains to be seen.

ACKNOWLEDGMENTS

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[11] R. J. Howerton, University of California Radiation Laboratory Report, UCRL 5573, 1961 (unpublished).

[12] Cf. G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[13] L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956). This has been studied for V-θ scattering in the Lee model by P. K. Srivastava, *ibid.* **131**, 461 (1963).