and baryons with possibly different spin and parity, since we have introduced two kinds of unitary triplets and from them composed meson and baryon supermultiplets. One of the pairs was identified as 0^- octet meson and $P_{1/2}$ octet baryons, respectively. The other pair has not been observed so far but the $D_{3/2}$ octet and the recently observed $\omega\pi$, $K\pi\pi,\cdots$, resonances might form another pair of octets.

As was noted earlier, we could have two nonets $(\bar{x}y)\pm(\bar{y}x)$ for the vector meson supermultiplets. One of them was identified as the observed vector meson supermultiplet. The other vector nonet, if they exist, necessarily obey the same mass sum rule but have opposite charge-conjugation parity compared to the usual nonet. The question of whether such a nonet exists in nature will be checked in future experiments.

Note added in proof. By eliminating the parameter from Eqs. (2.5) and (2.7), we have the following mass formula:

$$(M_\omega-M_\rho)(M_\phi-M_\rho)=\tfrac{4}{3}(M_{K^*}-M_\rho)(M_\omega+M_\phi-2M_{K^*})$$

from which the mass formula (2.9) is obtained by a procedure of linearization.

The author would like to thank Dr. M. Konuma for pointing out this relation.

ACKNOWLEDGMENTS

The author would like to express his sincere thanks to Professor Robert Oppenheimer for the hospitality at the Institute for Advanced Study. He also wishes to thank Professor Richard Capps for the hospitality at Northwestern University.

It is a pleasure to thank Dr. V. Gupta and Professor R. Capps for several interesting discussions and Dr. C. H. Albright for careful reading of the manuscript.

PHYSICAL REVIEW VOLUME 136, NUMBER 3B 9 NOVEMBER 1964

Spin Operators in the Kemmer Theory*

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(Received 4 June 1964)

A discussion of the various conserved spin operators in the Kemmer theory of spin-one particles is given. The algebraic properties and the interrelations of the various operators are also considered. The covariant forms of energy and spin projection operators are deduced and discussed.

INTRODUCTION

In recent years there has been an increasing interest in the study of vector mesons. It is partly due to the realization that a theoretical understanding of the various elementary-particle interactions seems to be closely connected with vector mesons. For example, a charged vector-meson field has been postulated as a possible intermediary field in the weak interactions.[1] Again in the strong interactions it is shown for some simple models that the mass of the nucleon is entirely due to boson-fermion interaction if a vector meson is introduced as an elementary system.[2] Furthermore, in the recent Regge pole hypothesis, a vector field (massive photon) is necessary if one wants to consider the nucleons as Regge poles as was shown by Gell-Mann and Goldberger.[3] Hence, in view of the appreciable role of vector meson in the recent works, a discussion of the various spin operators in the Kemmer theory which describes a vector meson in a manifestly covariant form, may be of interest as these operators may find applications in the study of polarization and scattering phenomena.

In this paper we shall give a set of spin operators which all commute with the Hamiltonian and hence they can be used to remove the spin degeneracy remaining after having specified the energy. First a three-vector conserved spin operator obeying the usual angular-momentum commutation relations is discussed. This is useful for the discussion of polarization involving plane-wave states. Then using the method employed by Bargmann, Michel and Telegdi,[4] Good,[5] and Fradkin and Good,[6] an axial four-vector spin operator analogous to Bargmann and Wigner's generators of the little group[7] and an antisymmetric tensor operator are deduced. The algebraic properties and the connection between these three operators are worked out. Finally, the covariant forms of the energy and spin projection

* Contribution No. 1523. Work was performed in the Ames Laboratory of the U. S. Atomic Energy Commission.

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operators are deduced from the Kemmer equation and the spin operators considered here.

The classical equations of motion of the spin-one operator, in the four-vector and antisymmetric forms, are considered recently by Young and Bludman[8] and Ford and Hirt.[9] In this paper we have made no attempt to generalize the operators for external fields; nor are the operators used in any specific application. However, this problem will be considered separately in another publication.

SPIN OPERATORS

The Kemmer equation is given by[10,11]

$$[\beta_\mu \partial_\mu + m]\psi = 0, \tag{1}$$

where the β_μ obeys the Duffin-Kemmer commutation rules

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\rho} + \beta_\rho\delta_{\nu\mu}, \tag{2}$$

which characterize the equations as belonging to spin-1 and spin-0 particles. The Hamiltonian form of Eq. (1) is given by

$$[\boldsymbol{\alpha}\cdot\mathbf{p} + \beta_4 m]\psi = i(\partial\psi/\partial t), \tag{3}$$

where

$$\alpha_i = i(\beta_4\beta_i - \beta_i\beta_4) \tag{4}$$

and Eq. (3) is governed by the time-independent initial condition

$$[H\beta_4 - m]\psi = 0. \tag{5}$$

The Klein-Gordon equation can be derived from Eq. (1) and hence Eq. (1) has plane-wave solutions of the form

$$\psi = u\exp(ip_\mu x_\mu), \tag{6}$$

where u is a 10-component column matrix and $p_4 = iE$ and $x_4 = it$. The β_μ's are singular matrices with the eigenvalues $+1$ (threefold), -1 (threefold), and 0 (fourfold). Hence the number of linearly independent solutions are only six for given momentum $\mathbf{p}$. Now we can write down the symmetrized energy-momentum tensor as

$$T_{\mu\nu} = -im[\bar{\psi}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\psi - \delta_{\mu\nu}\bar{\psi}\psi]. \tag{7}$$

Hence the energy density is given by

$$T_{44} = m\psi^+\psi \tag{8}$$

since $\bar{\psi} = i\psi^+(2\beta_4^2 - 1)$ and $(2\beta_4^2 - 1)^2 = 1$. Then it follows that the energy density is positive definite and so the energy is the same for all six states. However it changes the sign with respect to the charge. It is well known that the current-density four-vector is given by

$$S_\mu = e\bar{\psi}\beta_\mu\psi \tag{9}$$

and charge density is given by

$$S_4 = ie\psi^+\beta_4\psi, \tag{10}$$

with the eigenvalues $\pm e$. Thus the particle and antiparticle correspond to the energy eigenvalues $+E$ and $-E$. Then in the momentum representation we can write Eq. (1) as

$$[\pm i\beta_\mu p_\mu + m]U_\pm(p) = 0, \tag{11}$$

corresponding to the particle and antiparticle with the column vector $U_+(p)$ and $U_-(p)$, respectively. This equation is given here as it will be used later in the discussion of energy projection operators.

A three-vector spin operator is required to[12,13]:

(i) commute with the Hamiltonian and hence is defined separately on the energy manifolds,

(ii) have the reflection properties of an angular-momentum operator,

(iii) be well defined in the rest system, and

(iv) obey the usual angular-momentum commutation relations.

For a free particle such an operator is found to be[14,15]

$$\boldsymbol{\Sigma} = \frac{m}{E}\mathbf{s} + \frac{(\boldsymbol{\beta}\times\mathbf{p})}{E} + \frac{(\mathbf{s}\cdot\mathbf{p})\mathbf{p}}{E(E+m)}. \tag{12}$$

This operator obeys the usual angular-momentum commutation relations for spin 1:

$$[\Sigma_i, \Sigma_j] = i\epsilon_{ijk}\Sigma_k \tag{13}$$

and

$$\Sigma_i\Sigma_j\Sigma_k + \Sigma_k\Sigma_j\Sigma_i = \Sigma_i\delta_{jk} + \Sigma_k\delta_{ji}. \tag{14}$$

The square of the operator $\boldsymbol{\Sigma}$ is two times the unit matrix

$$\boldsymbol{\Sigma}^2 = 2I. \tag{15}$$

Again it has the following property

$$(\boldsymbol{\Sigma}\cdot\hat{n})^3 = (\boldsymbol{\Sigma}\cdot\hat{n}), \tag{16}$$

where $\hat{n}$ is any arbitrary unit vector. Hence $\boldsymbol{\Sigma}\cdot\hat{n}$ has the eigenvalues ± 1 and 0. From the above discussion it is obvious that one can obtain a complete set of eigenfunctions of H and $\boldsymbol{\Sigma}\cdot\hat{n}$ for a given momentum. A system in an eigenstate of $\boldsymbol{\Sigma}\cdot\hat{n}$ is said to be polarized in the $\hat{n}$ direction. In fact one can consider $\boldsymbol{\Sigma}\cdot\hat{n}$ as a generalized helicity operator. As we will show later, another interesting property of this operator for a

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particular energy state is that the polarization thus defined is the same irrespective of the Lorentz frame from which the particle is looked at.

It is well known that[4–6,16] for a particle with momentum $\mathbf{p}$ and energy E one can define a four-vector r_μ in the laboratory system as

$$\mathbf{r}=\hat{n}+\frac{(\hat{n}\cdot\mathbf{p})\mathbf{p}}{m(E+m)}; \tag{17}$$

$$r_4=(i/m)(\hat{n}\cdot\mathbf{p}), \tag{18}$$

which is the Lorentz transform of $(r_\mu)_{\text{rest}}=(\hat{n},0)$ from the rest to the laboratory system. Similarly by Lorentz-transforming an antisymmetric tensor

$$(m_{\mu\nu})_{\text{rest}}=(\epsilon_{ijk}\hat{n}_k,0) \tag{19}$$

from the rest to the laboratory system one gets

$$m_{ij}=\epsilon_{ijk}\left[\frac{E}{m}\hat{n}-\frac{(\hat{n}\cdot\mathbf{p})\mathbf{p}}{m(E+m)}\right]_k. \tag{20}$$

$$m_{i4}=(i/m)(\hat{n}\times\mathbf{p})_i. \tag{21}$$

Now we shall consider $\mathbf{\Sigma}$ as an instantaneous rest-system operator for the Kemmer particle with the understanding that the rest frame of the particle means a coordinate system in which the particle is instantaneously at rest, which is obtained by a pure Lorentz transformation from a fixed laboratory system. Then, by making use of the Eqs. (17) to (21) we can easily write down the axial four-vector and antisymmetric tensor operators. They are given by

$$\mathbf{T}=\frac{m}{E}\mathbf{s}+\frac{(\boldsymbol{\beta}\times\mathbf{p})}{E}+\frac{(\mathbf{s}\cdot\mathbf{p})\mathbf{p}}{mE}, \tag{22}$$

$$T_4=(i/m)(\mathbf{s}\cdot\mathbf{p}), \tag{23}$$

and

$$R_{ij}=\epsilon_{ijk}[\mathbf{s}+(\boldsymbol{\beta}\times\mathbf{p})/m]_k, \tag{24}$$

$$R_{i4}=-i/E[(\mathbf{s}\times\mathbf{p})+(1/m)(\boldsymbol{\beta}\times\mathbf{p})\times\mathbf{p}]_i. \tag{25}$$

These operators T_μ and $R_{\mu\nu}$ have the following interesting properties. The scalar product of T_μ with the momentum four-vector turns out to be

$$T_\mu p_\mu=0, \tag{26}$$

guaranteeing that during the motion T_μ continues to have the same form in the rest system of the particle. Again the scalar product of T_μ with itself is a multiple of a unit matrix

$$T_\mu T_\mu=2, \tag{27}$$

guaranteeing that the magnitude of T_μ is constant during the motion. As required earlier, the components of T_μ commute with the Hamiltonian

$$[T_\mu,H]=0. \tag{28}$$

The commutator among the components of T_μ is given by

$$[T_\mu,T_\nu]=iR_{\mu\nu}. \tag{29}$$

The last equation shows that the four-vector and antisymmetric tensor operators are closely connected in a covariant way as we will again see later. The antisymmetric tensor operator $R_{\mu\nu}$ also satisfies the following relations:

$$R_{\mu\nu}p_\nu=0, \tag{30}$$

$$R_{\mu\nu}R_{\mu\nu}=2, \tag{31}$$

and

$$[R_{\mu\nu},H]=0. \tag{32}$$

Again if we denote the space-space part of $R_{\mu\nu}$ by $\mathbf{O}$ and the space-time part by $\mathbf{Q}$, then they obey the following commutation relations:

$$[O_i,O_j]=(E/m)i\epsilon_{ijk}T_k \tag{33}$$

and

$$[Q_i,Q_j]=-(i/m^2)\epsilon_{ijk}(\mathbf{s}\cdot\mathbf{p})p_k. \tag{34}$$

In between $\mathbf{O}$ and $\mathbf{Q}$ the following relations are obtained:

$$[O_i,Q_i]=1/m[\mathbf{T}\cdot\mathbf{p}-T_ip_i] \quad (i=1,2,3 \text{ no summation over } i), \tag{35}$$

$$[\mathbf{O}\cdot\mathbf{Q}-\mathbf{Q}\cdot\mathbf{O}]=(2/m)\mathbf{T}\cdot\mathbf{p}, \tag{36}$$

$$[O_i,Q_j]=-p_iT_j/m \quad (i\neq j). \tag{37}$$

From the relations (29) to (37), it is clear that T_μ and $R_{\mu\nu}$ are intimately connected and the connection is given by

$$T_\lambda=(i/2m)\epsilon_{\mu\nu\rho\lambda}p_\rho R_{\mu\nu}. \tag{38}$$

The conserved covariant tensor has been derived by Mathews and the author[17] by an entirely different procedure. It is also interesting to note the connection between the operators $\mathbf{\Sigma}$, $\mathbf{T}$, and $\mathbf{O}$ in the directions parallel and perpendicular to the momentum. Along the direction of momentum

$$\mathbf{\Sigma}_l=\mathbf{O}_l=(m/E)\mathbf{T}_l=(\mathbf{s}\cdot\hat{p})\hat{p}. \tag{39}$$

Perpendicular to the direction of momentum

$$\mathbf{\Sigma}_t=\mathbf{T}_t=(m/E)\mathbf{O}_t. \tag{40}$$

Then, using the relations (39) and (40), we can write

$$\mathbf{O}=\left(\mathbf{\Sigma}_l+\frac{E}{m}\mathbf{\Sigma}_t\right)=\frac{E}{m}\mathbf{T}_t+\frac{m}{E}\mathbf{T}_l \tag{41}$$

and

$$\mathbf{T}=\left(\frac{E}{m}\mathbf{\Sigma}_l+\mathbf{\Sigma}_t\right)=\frac{m}{E}\mathbf{O}_t+\frac{E}{m}\mathbf{O}_l. \tag{42}$$

Now using the relations $\hat{n}$ and r_μ, and $\hat{n}$ and $m_{\mu\nu}$ we get

$$\mathbf{\Sigma}\cdot\hat{n}=T_\mu r_\mu=R_{\mu\nu}m_{\mu\nu}. \tag{43}$$

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As we noted earlier these operators can be called generalized helicity operators. The covariant nature of this operator suggests that the polarization thus defined for a plane-wave state is the same irrespective of the Lorentz frame from which the particle is viewed. Equation (43) can be interpreted as follows: For a plane-wave state $\mathbf{\Sigma}$ can be considered as the laboratory-system spin operator corresponding to the direction of polarization $\hat{n}$ in the rest system of the particle and T_μ and $R_{\mu\nu}$ are the laboratory-system operators corresponding to the four-vector r_μ and tensor $m_{\mu\nu}$ which are the Lorentz transform of $(\hat{n},0)$ and $(\epsilon_{ijk}n_k,0)$ from the rest system, respectively.

PROJECTION OPERATORS

For plane-wave solutions the state of a single Kemmer particle with given momentum is fully specified by the eigenvalues of two commuting operators, the Hamiltonian and the generalized helicity operator. By using Eqs. (1), (16), and (43) and the fact $p_\mu=-i\partial_\mu$, we shall write the following equations.

$$[-(i/m)\beta_\mu p_\mu]\psi=\psi \tag{44}$$

and

$$T_\mu r_\mu\psi=\psi. \tag{45}$$

These equations are covariantly written and hence the covariantly defined energy and spin-projection operators are easily written down. The energy projection operators are given by

$$\Lambda_+=(-1/2m^2)[(\beta_\mu p_\mu)^2+im\beta_\mu p_\mu], \tag{46a}$$

$$\Lambda_-=(-1/2m^2)[(\beta_\mu p_\mu)^2-im\beta_\mu p_\mu], \tag{46b}$$

and

$$\Lambda_0=(1/m^2)[(\beta_\mu p_\mu)^2+m^2]. \tag{46c}$$

Now using the property

$$\left(-\frac{i}{m}\beta_\mu p_\mu\right)^3=\left(-\frac{i}{m}\beta_\mu p_\mu\right), \tag{47}$$

it is easily seen that

$$\Lambda_++\Lambda_-+\Lambda_0=1 \tag{48}$$

and

$$\Lambda_i\Lambda_j=\delta_{ij}\Lambda_j, \tag{49}$$

where i and j stand for $+$, $-$, and 0. Again for any solution ψ they have the following properties:

$$(-(i/m)\beta_\mu p_\mu)\Lambda_+\psi=\Lambda_+\psi, \tag{50a}$$

$$(-(i/m)\beta_\mu p_\mu)\Lambda_-\psi=-\Lambda_-\psi, \tag{50b}$$

and

$$(-(i/m)\beta_\mu p_\mu)\Lambda_0\psi=0. \tag{50c}$$

Now in terms of the projection operators, solutions $U_\pm(p)$ satisfying Eq. (11) can be written as

$$U_\pm(p)=im\Lambda_\pm\chi(\pm p), \tag{51}$$

where χ's are arbitrary 10-component column vectors in the momentum space. From the Eq. (50), it is obvious that the space defined by the column matrix ψ splits into three subspaces and the particle and antiparticle solutions of the Kemmer equation correspond to Λ_+ and Λ_-, as is obvious from the Eq. (51).

Now we turn to the consideration of spin-projection operators. As we noted earlier, $T_\mu r_\mu$ gives the covariantly defined helicity operator. Then from the Eq. (45), one can write down the following operators:

$$P_+=\tfrac{1}{2}T_\mu r_\mu[T_\mu r_\mu+1], \tag{52a}$$

$$P_-=\tfrac{1}{2}T_\mu r_\mu[T_\mu r_\mu-1], \tag{52b}$$

and

$$P_0=[1-(T_\mu r_\mu)^2]. \tag{52c}$$

That they represent the spin projection operators can easily be seen because of the following properties:

$$P_++P_-+P_0=1, \tag{53}$$

$$P_iP_j=\delta_{ij}P_j \quad (i,j=+,-,0), \tag{54}$$

by using the property

$$(T_\mu r_\mu)^3=(T_\mu r_\mu). \tag{55}$$

Again

$$T_\mu r_\mu P_+=P_+, \tag{56a}$$

$$T_\mu r_\mu P_-=-P_-, \tag{56b}$$

and

$$T_\mu r_\mu P_0=0. \tag{56c}$$

These equations again justify calling P_+, P_-, and P_0 as spin projection operators in the sense that they select the spin eigenstates. P_0 corresponds to the longitudinal polarization with the eigenvalue 0 and $P_\pm$ correspond to the transverse polarizations, perpendicular to each other with the eigenvalues ± 1. We can also define the spin projection operators as above just by replacing $T_\mu r_\mu$ either by $R_{\mu\nu}m_{\mu\nu}$ or by $\mathbf{\Sigma}\cdot\hat{n}$.

ACKNOWLEDGMENT

The author wishes to thank Professor R. H. Good, Jr., for his interest in the work.