Equivalence Between Four-Fermion and Yukawa Coupling, and the $Z_3=0$ Condition for Composite Bosons*

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The principal objective of the present work is the derivation of conditions for equivalence (A) between a four-Fermi theory with pseudoscalar coupling and a Yukawa theory with pseudoscalar bosons, and (B) in the corresponding vector case. In general, two conditions are found to be both necessary and sufficient. The first is a condition for the existence of an appropriate boson bound state in the four-Fermi theory. The second condition is that the boson wave function renormalization constant Z_3 in the Yukawa theory be equal to zero. We first derive our results by consideration of fermion-fermion scattering in the chain approximation, and proceed afterwards to prove them valid to all orders in perturbation theory. We also discuss the degenerate vacuum theories of Nambu and Jona-Lasinio and of Bjorken in the chain approximation. In each of these four-Fermi theories, the existence of boson bound states (massless pseudoscalar and massive scalar bosons in the former case, massless vector bosons in the latter) follows automatically from self-consistency conditions. Hence to have equivalence to Yukawa theories in which the bosons are described by elementary fields, we need only impose the $Z_3=0$ conditions on the Yukawa theories. Finally, we comment on Birula's theory of quantum electrodynamics without electromagnetic field. It is to be stressed that we deal throughout with full-scale relativistic field theories of physical consequence.

INTRODUCTION

In recent years, considerable attention has been given to the problem of describing a composite particle in quantum field theory.[1] One particularly promising idea which has been exploited by Salam[2] centers on the use of the criterion that the wave-function renormalization constant Z_3 be equal to zero for a composite particle. The criterion is suggested by inspection of various model situations.[3] In each of these models, one starts with a field theory whose Lagrangian involves an elementary boson field ϕ. In the limit of the vanishing of the wave-function renormalization constant associated with ϕ, predictions of the theory coincide with those of a theory in whose Lagrangian ϕ does not appear explicitly and in which the particles associated with ϕ emerge as bound states. Further support for use of the $Z_3=0$ condition for compositeness has been provided by recent work[4] on the bound-state problem within the framework of the N/D method[5] and the method of vertex equations.[6]

The principal aim of the present paper is to investigate conditions under which there can be equivalence between a Lagrangian field theory of a single fermion field with self-coupling of the four-Fermi type to a Lagrangian field theory of the same fermion field with coupling of a Yukawa type to a neutral-boson field.

We have considered two distinct cases, both of physical importance. The first case involves pseudoscalar four-Fermi coupling and a Yukawa theory with pseudoscalar bosons, while the second involves vector four-Fermi coupling and a Yukawa theory with vector bosons. Treatment of the vector case is very similar in spirit to that of the pseudoscalar case, except for complications associated with tensor indices. In each case, two conditions are found to be necessary and sufficient for equivalence. The first is a condition for the existence in the four-Fermi theory of a bound state with the same quantum numbers as the bosons of the Yukawa theory. The second condition is that the wave-function renormalization constant Z_3 associated with the boson of the Yukawa theory be equal to zero. The condition for the existence of the boson bound state in the four-Fermi theory can, in each of the cases under consideration, be converted into a relation between the unrenormalized four-Fermi and Yukawa coupling constants, respectively, g_0 and G_0, and the boson self-mass $\delta\mu^2$. In the pseudoscalar theory this relation reads as $\delta\mu^2=-G_0^2/(2g_0)$, while in the vector case[7] it is

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[1] R. Haag, Phys. Rev. **112**, 669 (1958); K. Nishijima, Phys. Rev. **111**, 995 (1958); **133**, B204 (1964); W. Zimmermann, Nuovo Cimento **10**, 597 (1958).

[2] A. Salam, Nuovo Cimento **25**, 224 (1962); Phys. Rev. **130**, 1287 (1963); A. Salam and R. Delbourgo, Phys. Rev. **135**, B1398 (1964). See also the review article by M. Cini, Proceedings of the International Conference on Elementary Particles at Sienna, 1963 (to be published).

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[4] R. M. Rockmore, Phys. Rev. **132**, 878 (1963); J. S. Dowker and J. E. Paton, Nuovo Cimento **30**, 450 (1963); B. W. Lee, K. T. Mahanthappa, I. S. Gerstein, and M. L. Whippman, Ann. Phys. (N. Y.) **28**, 466 (1964); E. G. P. Rowe, Nuovo Cimento **32**, 1422 (1964).

[5] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960); F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[6] L. S. Liu, Phys. Rev. **125**, 761 (1962).

[7] By omitting the factor $\frac{1}{2}$ from the Lagrangian density of Eq. (2.27), we could make the two conditions identical. However, we prefer to retain the factor $\frac{1}{2}$, as this seems to be standard practice in the current literature.

$\delta\mu^2=-G_0^2/g_0$. We derive the conditions for equivalence first by consideration of the chain approximation to fermion-fermion scattering. We then prove that the same conditions for equivalence, namely, $Z_3=0$ and the appropriate $\delta\mu^2$ formulas, obtain to all orders of perturbation theory by comparing the sets of integral equations for the basic Green's functions of the two theories.

Before specifying the content of the remainder of this paper, we comment briefly on the status of the results just described and their relationship to previous work. We are quite clearly placing ourselves in the perspective of the papers of Ref. 3. We are, however, dealing with full-fledged relativistic field theories, and the attempt to manufacture bosons as bound states of a fermion system is of immediate physical interest. Previous work on the subject dates back to the pioneering paper of Fermi and Yang.[8] Also related is the work of Heisenberg and collaborators on four-Fermi theories.[9] More recently, Jouvet[10] has examined the case of pseudoscalar four-Fermi coupling and its possible equivalence to pseudoscalar meson theory. In the third of his cited papers he has, on the basis of an elementary argument, foreseen the need for the relation $Z_3=0$ in the Yukawa theory as a condition for equivalence.[11] Further, we note that Birula[12] has given an incomplete discussion of the case of vector coupling in which the neutral vector boson has zero physical mass and the Yukawa theory is a form of quantum electrodynamics.

All of the related work so far cited involves the ordinary perturbative solutions of the Lagrangian field theories dealt with. We wish also to consider the theories of Nambu and Jona-Lasinio[13] and of Bjorken.[14] In each of these four-Fermi theories one studies, a nonperturbative solution involving (a) a vacuum degenerate with respect to a continuous group of transformation,[15] and (b) zero mass boson bound states, in accordance with Goldstone's theorem.[16] The former theory also contains massive scalar boson bound states. We discuss equivalence in the chain approximation between these theories and Yukawa theories in which the bosons are described by elementary fields. In each case, our discussion follows the pattern employed in our previous discussion of theories of conventional type. Since, in each case, no conditions for the existence of boson bound states in the four-Fermi theory have to be imposed, the $Z_3=0$ conditions on the corresponding Yukawa theories are sufficient for equivalence. Our main contribution to the work on these theories, in addition to their presentation within a unified framework, is the observation that the Yukawa theories to which they are equivalent are themselves based on nonperturbative solutions. We also discuss briefly the possibility of emergence of massive vector-boson bound states and the role of the Fierz identities[17] in the theory of Nambu and Jona-Lasinio. Our formulation of Bjorken's theory differs considerably from the original and is in close analogy with the discussion of the theory of Nambu and Jona-Lasinio.

As a third application of our general methods, we consider Birula's work on quantum electrodynamics without electromagnetic field.[12] We believe Birula's work to be incomplete, however, in that the need for the relation $Z_3=0$ as a condition for equivalence is not observed.

The material of the paper is originized as follows. In Sec. 2A, we derive and discuss the conditions which are necessary and sufficient for the equivalence in the chain approximation of a four-Fermi theory with pseudoscalar coupling to a Yukawa theory with pseudoscalar bosons. After observing that the case of scalar coupling can be given identical treatment, we attend in Sec. 2B to such modifications as are necessary for the development of a parallel discussion of the vector case. In Sec. 3, we prove with reference to the pseudoscalar case that the conditions for equivalence, $Z_3=0$ and $\delta\mu^2=-G_0^2/(2g_0)$, found in the chain approximation are indeed the conditions for equivalence to all orders in perturbation theory. Our work on the theories of Nambu and Jona-Lasinio, of Bjorken, and of Birula, respectively, constitute Secs. 4A, 4B, and 4C.

2. EQUIVALENCE CONDITIONS IN THE CHAIN APPROXIMATION

A. Pseudoscalar Case

The theories under consideration here are a Yukawa theory with interaction Lagrangian density

$$\mathcal{L}_y=iG_0\bar{\psi}\gamma_5\psi\phi, \tag{2.1}$$

[8] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949).

[9] H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, and K. Yamazaki, Z. Naturforsch. **14a**, 441 (1959). This paper contains references to previous work.

[10] B. Jouvet, J. Math. **33**, 201 (1954); Nuovo Cimento Suppl. 2, 941 (1955); Nuovo Cimento **3**, 1133 (1956); **5**, 1 (1957).

[11] To the best of our knowledge, this is the first mention in the literature of the $Z_3=0$ condition for compositeness. It has, however, seldom been referred to by subsequent workers.

[12] I. Bialynicki-Birula, Phys. Rev. **130**, 465 (1963). See also Proceedings Summer Seminar on Unified Theories of Elementary Particles, edited by D. Lurié and N. Mukunda, University of Rochester Report URPA-1, 1963 (unpublished).

[13] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1960); **124**, 246 (1961). See also, V. G. Vaks and A. I. Larkin, Zh. Eksperim. i Teor. Fiz. **40**, 282 and 1392 (1962) [English transls.: Soviet Phys.—JETP **13**, 192 and 979 (1962)].

[14] J. D. Bjorken, Ann. Phys. (N. Y.) **24**, 174 (1963). See also G. S. Guralnik, Ph.D. thesis, Harvard University, 1964 (unpublished).

[15] For a general discussion, see the article by B. Zumino, in *Werner Heisenberg und die Physik in Unserer Zeit* (Frederick Vieweg und Sohn Braunschweig, Germany, 1961).

[16] J. S. Goldstone, Nuovo Cimento **19**, 154 (1961); J. S. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962); J. C. Taylor, *Proceedings of the 1962 International Conference of High Energy Physics at CERN* (CERN, Geneva, 1962), p. 670; S. A. Bludman and A. Klein, Phys. Rev. **131**, 2364 (1963). See also, M. Baker, K. Johnson and B. W. Lee, Phys. Rev. **133**, B209 (1964); Y. Nambu, Phys. Letters **9**, 210 (1964).

[17] M. Fierz, Z. Physik **102**, 572 (1936). See also H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956), p. 118.

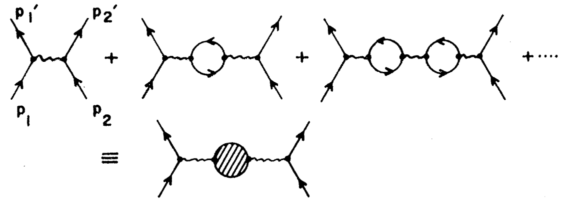

FIG. 1. Feynman diagrams that contribute to the chain approximation to the fermion-fermion scattering amplitude of the Yukawa theory.

and a four-Fermi theory with interaction Lagrangian density

$$\mathfrak{L}_f = -g_0 \bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi. \tag{2.2}$$

We assume that the physical fermion mass m is the same in each theory. However, we see after the discussion of the general equivalence given in the next section that when the equivalence conditions are satisfied the fermion self-masses also are equal.

We turn first to the Yukawa theory to evaluate the fermion-fermion scattering amplitude M_y generated by the infinite set of diagrams displayed in Fig. 1 (the chain approximation). If we omit from it factors like $\bar{u}(p')\gamma_5 u(p)$ which refer to the external fermion lines, M_y is related by

$$M_y(q^2) = G_0^2 \Delta'_F(q^2), \tag{2.3}$$

where $q = p_1 - p_1'$ is the momentum transfer, to the boson propagator Δ'_F, dressed as in Fig. 1. The dressed propagator Δ'_F satisfies the Dyson equation

$$\Delta'_F(q^2) = \Delta_F(q^2) + G_0^2 \Delta_F(q^2)\Pi(q^2)\Delta'_F(q^2), \tag{2.4}$$

where $\Delta_F(q^2) = (q^2+\mu_0^2)^{-1}$ and μ_0^2 is the bare boson mass, or, equivalently

$$\Delta'_F(q^2) = [q^2+\mu^2 - G_0^2\Pi(q^2) + G_0^2\Pi(-\mu^2)]^{-1}, \tag{2.5}$$

where μ^2 is the physical boson mass,[18] boson mass renormalization having been performed. The quantity $\Pi(q^2)$ refers to the basic fermion loop in Fig. 1 and is given by[19]

$$\Pi(q^2) = \frac{i}{(2\pi)^4}\int d^4p \,\mathrm{Tr}\left(\gamma_5 \frac{1}{\mathbf{p}-\mathbf{q}-im}\gamma_5\frac{1}{\mathbf{p}-im}\right)$$
$$= \frac{-4i}{(2\pi)^4}\int d^4p \frac{m^2+p^2-p\cdot q}{[(p-q)^2+m^2][p^2+m^2]}. \tag{2.6}$$

It follows from relativistic invariance that $\Pi(q^2)$ is a function of q^2 only. From Eqs. (2.3) and (2.5), we get the desired expression for M_y, namely

$$M_y(q^2) = G_0^2[q^2+\mu^2 - G_0^2\Pi(q^2) + G_0^2\Pi(-\mu^2)]^{-1}. \tag{2.7}$$

The quadratic divergence of the quantity $\Pi(q^2)$ has already been removed by boson mass renormalization. The remaining logarithmically divergent part can be isolated by means of the expansion[20]

$$\Pi(q^2) - \Pi(-\mu^2) = (q^2+\mu^2)\Pi'(-\mu^2) + \Pi_c(q^2), \tag{2.8}$$

wherein $\Pi_c(q^2)$ is convergent, and removed by renormalization of the coupling constant. For this purpose, we note that in the approximation in which we are working Z_1 and Z_2 are both equal to unity, so that the renormalized coupling constant G_R is given by[21]

$$G_R^2 = Z_3 G_0^2, \tag{2.9}$$

where $Z_3(G_R)$ is given by

$$Z_3(G_R) = 1 + G_R^2\Pi'(-\mu^2). \tag{2.10}$$

We may now use Eqs. (2.8)–(2.10) to express M_y in terms of G_R and the renormalized propagator Δ'_{F1}. A simple manipulation yields

$$M_y = Z_3 G_0^2 Z_3^{-1}\Delta'_F(q^2) = G_R^2\Delta'_{F1}(q^2)$$
$$= G_R^2[q^2+\mu^2 - G_R^2\Pi_c(q^2)]^{-1}. \tag{2.11}$$

We turn next to the four-Fermi theory and evaluate the fermion-fermion scattering amplitude M_f generated by the infinite set of diagrams shown in Fig. 2. Some comment on the "double-dot" notation is necessary. We represent a four-Fermi vertex by an adjacent pair of dots at each of which the matrix γ_5 acts.[22] This enables us to keep track of spinor indices simply by following along fermion lines. As can easily be seen from Wick's theorem, the diagrams shown in Fig. 2 are a very special subset of the diagrams that can occur in the four-Fermi theory. Various other possible diagrams are shown in Fig. 3. These diagrams, and their Yukawa counterparts, occur in higher orders in the iterative solution of the basic integral equations of the theory based on the chain approximation as lowest order solution. This will be clarified by the work of Sec. 3.

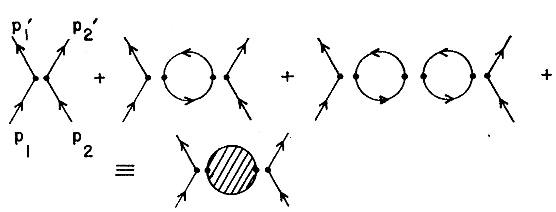

FIG. 2. Feynman diagrams that contribute to the chain approximation to the fermion-fermion scattering amplitude of the four-Fermi theory.

[18] We assume that the boson is stable, i.e., $\mu^2 < 4m^2$.

[19] For convenience in what follows, we have separated from $\Pi(q^2)$ the factor G_0^2 which is usually included in the expression for the fermion loop.

[20] Here $\Pi'(-\mu^2)$ means $\partial\Pi(q^2)/\partial q^2$ evaluated at $q^2 = -\mu^2$.

[21] F. J. Dyson, Phys. Rev. **75**, 1736 (1949); A. Salam, *ibid.* **82**, 217 (1951); **84**, 426 (1951); J. C. Ward, *ibid.* **84**, 897 (1951). See also, S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961), Sec. 16f.

[22] For different four-Fermi couplings, of course, correspondingly different matrices will act at each dot.

It can easily be seen[23] that M_f can be written as

$$M_f = 2g_0[1-2g_0\Pi(q^2)]^{-1}. \tag{2.12}$$

Since the fermion loop that occurs in Fig. 2 is the same as that which occurs in Fig. 1, $\Pi(q^2)$ in Eq. (2.12) is exactly as given by Eq. (2.6). We now seek the conditions that must be satisfied for M_y and M_f to be equal. First, M_f must have a pole at $q^2=-\mu^2$, μ^2 being the physical boson mass in the Yukawa theory, implying the existence of a bound state in the four-Fermi theory, with the same quantum numbers as the boson of the Yukawa theory. The condition for this is

$$\Pi(-\mu^2) = 1/(2g_0). \tag{2.13}$$

We may use Eqs. (2.8) and (2.13) to write Eq. (2.12) in the form

$$M_f = -[(q^2+\mu^2)\Pi'(-\mu^2)+\Pi_c(q^2)]^{-1}, \tag{2.14}$$

and it is to be noted that the (bare) four-Fermi coupling constant g_0 has disappeared completely from M_f. In the next section, we see that such a situation obtains in any order of perturbation theory: Results in any order are independent of the value of the (bare) four-Fermi coupling constant that satisfies the bound-state condition in that order.

While Eq. (2.13) ensures that M_y and M_f each have a pole at $q^2=-\mu^2$, it follows from Eqs. (2.11) and (2.14) that we must also impose

$$G_R^2 = -[\Pi'(-\mu^2)]^{-1}, \tag{2.15}$$

if the two amplitudes are able to be equal for all q^2. We thus obtain from Eqs. (2.15) and (2.10) the condition

$$Z_3(G_R) = 0. \tag{2.16}$$

The condition (2.13) can be rewritten in terms of the bare boson mass μ_0 and bare coupling constant G_0 in the Yukawa theory. We use the standard boson mass renormalization procedure to derive

$$\mu^2-\mu_0^2 = \delta\mu^2 = -G_0^2\Pi(-\mu^2) = -G_0^2/(2g_0). \tag{2.17}$$

We have been led to two conditions, Eqs. (2.16) and (2.17), which must be satisfied if M_y and M_f are to be equal. From the condition $Z_3=0$, we deduce from Eq. (2.9) that G_0^2 must be infinite if G_R^2 is finite, and then, from Eq. (2.17), that μ_0^2 must be infinite, if μ^2 and g_0 are finite.

So far we have glossed over the vital role of the cutoff in the above analysis. Clearly, use of a cutoff has tacitly been assumed, if Eq. (2.15) is to lead to finite G_R^2, and we turn now to the task of exhibiting the role of the cutoff in a precise fashion. For this purpose it is convenient to introduce the spectral representation of $\Pi(q^2)$[23]:

$$\Pi(q^2) = \int_{4m^2}^{\Lambda^2} \frac{\rho(\kappa^2)d\kappa^2}{q^2+\kappa^2}, \tag{2.18}$$

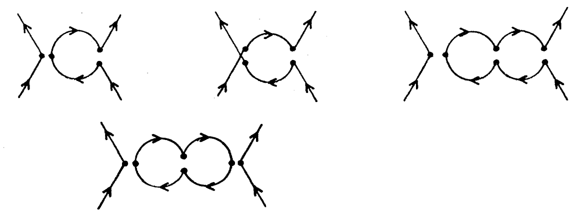

FIG. 3. Various other fermion-fermion scattering diagrams in the four-Fermi theory.

where

$$\rho(\kappa^2) = (1-4m^2/\kappa^2)^{1/2}\kappa^2/(8\pi^2), \tag{2.19}$$

thereby exhibiting the dependence on q^2. Since the integral is quadratically divergent, introduction of the cutoff Λ^2 into Eq. (2.18) is essential for it to be meaningful. We may now write Eq. (2.10) in the form

$$Z_3(G_R) = 1-G_R^2\int_{4m^2}^{\Lambda^2} \frac{\rho(\kappa^2)d\kappa^2}{(\kappa^2-\mu^2)^2}, \tag{2.20}$$

and then condition (2.16) becomes

$$G_R^2 = \left[\int_{4m^2}^{\Lambda^2} \frac{\rho(\kappa^2)d\kappa^2}{(\kappa^2-\mu^2)^2}\right]^{-1}. \tag{2.21}$$

The $Z_3=0$ condition is thus seen to result from a definite relationship, expressed by Eq. (2.21), between the renormalized Yukawa coupling constant, the renormalized fermion and boson masses, and the cutoff Λ.

We wish to make a clear distinction between the present result and the well-known argument[24] that $Z_3=0$ when computed to lowest order in G_R^2 using the Lehmann formula[25]

$$Z_3^{-1} = 1+\int_{3\mu^2}^{\infty} \sigma(\kappa^2)d\kappa^2, \tag{2.22}$$

where $\sigma(\kappa^2)$ is the spectral function of the renormalized boson propagator. In this approach one evaluates $\sigma(\kappa^2)$ to lowest order in G_R^2 and thereby obtains from Eq. (2.22) the result

$$Z_3^{-1} = 1+G_R^2\int_{4m^2}^{\infty} \frac{\rho(\kappa^2)d\kappa^2}{(\kappa^2-\mu^2)^2}, \tag{2.23}$$

where $\rho(\kappa^2)$ is given by Eq. (2.19).[26] One then argues that Z_3^{-1} is infinite since the integral in Eq. (2.23) diverges, and hence that $Z_3=0$. In the present approach, however, the limit $\Lambda^2\to\infty$ cannot be taken,[27] as this would give $G_R^2=0$ by Eq. (2.21).

[23] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1960).

[24] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954). See also, P. Olesen, Phys. Letters **9**, 277 (1964); and C. R. Hagen (to be published).

[25] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[26] The relation $\sigma(\kappa^2)=G_R^2\rho(\kappa^2)(\kappa^2-\mu^2)^{-2}$ is true only to lowest order in the renormalized coupling constant.

[27] Compare with the work of Sec. 4A, where the cutoff also plays a crucial role.

The difference between the two points of view can be exhibited even more sharply by expanding Z_3, as given by Eq. (2.23) in powers of $G_R{}^2$, and keeping only the two lowest terms. The resulting expression for Z_3

$$Z_3=1-G_R{}^2\int_{4m^2}^{\infty}\frac{\rho(\kappa^2)d\kappa^2}{(\kappa^2-\mu^2)^2} \tag{2.24}$$

is correct to order $G_R{}^2$, as inclusion of higher order corrections to the Lehmann weight $\rho(\kappa^2)$ gives rise to terms of higher order. Equation (2.24) becomes identical with Eq. (2.20) when a cutoff Λ^2 is introduced. In summary, the point of view stressed here is that the $Z_3=0$ condition for composite "elementary" particles must be regarded as arising from the cancellation of the two (finite) terms of Eq. (2.24) in the presence of a cutoff, rather than from the divergence of the integral in Eq. (2.22).

We conclude Sec. 2A by making the almost self-evident remark that the case of scalar coupling can be given identical treatment.

B. Vector Case

The Yukawa and four-Fermi interaction Lagrangian densities under consideration now are

$$\mathcal{L}_y=iG_0\bar{\psi}\gamma_\mu\psi A_\mu\,, \tag{2.25}$$

where A_μ describes a neutral vector boson,[28] and

$$\mathcal{L}_f=-\tfrac{1}{2}g_0\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma_\mu\psi\,. \tag{2.26}$$

We turn first to the Yukawa theory to evaluate the fermion-fermion scattering amplitude $M_{y\mu\nu}$ generated by the infinite set of diagrams shown in Fig. 1. It is related in the manner of Eq. (2.3) to the vector boson propagator $D'_{F\mu\nu}$, dressed as in Fig. 1. This dressed propagator obeys a tensorial analog of the Dyson equation (2.4), namely,

$$D'_{F\mu\nu}(q)=D_{F\mu\nu}(q)+G_0{}^2D_{F\mu\kappa}(q)\bar{\Pi}_{\kappa\lambda}(q)D'_{F\lambda\nu}(q)\,. \tag{2.27}$$

Herein, $D_{F\mu\nu}$ is the bare propagator, which is given, after vector boson mass renormalization, in terms of the physical mass μ by[29]

$$D_{F\mu\nu}(q)=\delta_{\mu\nu}[q^2+\mu^2]^{-1}. \tag{2.28}$$

The quantity $\bar{\Pi}_{\mu\nu}(q)$ is obtained by vector boson mass renormalization from the well-known vacuum-polarization tensor $\Pi_{\mu\nu}(q)$, which describes the basic fermion loop of the present context and which is of the form[30]

$$\Pi_{\mu\nu}(q)=\Pi_{\mu\nu}(0)+\tau_{\mu\nu}(q)\Pi_V(q^2)\,, \tag{2.29}$$

where $\Pi_{\mu\nu}(0)=A\delta_{\mu\nu}$, and

$$\tau_{\mu\nu}(q)=[\delta_{\mu\nu}-q_\mu q_\nu/(q^2)]\,. \tag{2.30}$$

The function $\Pi_V(q^2)$ has the spectral representation

$$\Pi_V(q^2)=-q^2\int_{4m^2}^{\Lambda^2}\frac{\rho_V(\kappa^2)d\kappa^2}{(q^2+\kappa^2)\kappa^2}\,, \tag{2.31}$$

with weight function

$$\rho_V(\kappa^2)=(1/12\pi^2)(\kappa^2+2m^2)[1-(4m^2/\kappa^2)]^{1/2}. \tag{2.32}$$

The Dyson mass-renormalization procedure gives

$$\delta\mu^2=-G_0{}^2[A+\Pi_V(-\mu^2)]\,, \tag{2.33}$$

and hence we have

$$\bar{\Pi}_{\mu\nu}(q)=\tau_{\mu\nu}(q)\Pi_V(q^2)-\delta_{\mu\nu}\Pi_V(-\mu^2)\,. \tag{2.34}$$

We could now use Eq. (2.28) and Eq. (2.34) to solve Eq. (2.27) for $D'_{F\mu\nu}$. However, it is convenient, although not necessary, to introduce into Eq. (2.27) the term $-q_\mu q_\nu[q^2(q^2+\mu^2)]^{-1}$, which is effectively zero, and then solve for $D'_{F\mu\nu}$. This leads to the result

$$\begin{aligned}M_{y\mu\nu}(q)&=G_0{}^2D'_{F\mu\nu}(q)\\&=G_0{}^2\tau_{\mu\nu}(q)[q^2+\mu^2-G_0{}^2\Pi_V(q^2)+G_0{}^2\Pi_V(\mu^2)]^{-1}.\end{aligned} \tag{2.35}$$

Now, apart from the kinematic factor $\tau_{\mu\nu}(q)$, $M_{y\mu\nu}$ is identical in form to the corresponding pseudoscalar amplitude M_y, given by Eq. (2.7), so that we can proceed as for M_y to derive the result

$$M_{y\mu\nu}(q)=G_R{}^2\tau_{\mu\nu}(q)[q^2+\mu^2-G_R{}^2\Pi_{Vc}(q^2)]^{-1}\,, \tag{2.36}$$

where Π_{Vc} is defined by an expansion like Eq. (2.8), and

$$G_R{}^2=Z_3G_0{}^2\,, \tag{2.37}$$

$$Z_3(G_R)=1+G_R{}^2\Pi_V'(-\mu^2)\,. \tag{2.38}$$

We next consider the fermion-fermion scattering amplitude $M_{f\mu\nu}$ of the four-Fermi theory corresponding to Fig. 2. It satisfies the equation

$$M_{f\mu\nu}(q)=g_0\delta_{\mu\nu}+g_0\delta_{\mu\kappa}\Pi_{\kappa\lambda}(q)M_{f\lambda\nu}(q)\,. \tag{2.39}$$

Again it is convenient to introduce into Eq. (2.39) the term $-g_0q_\mu q_\nu/(q^2)$, which is effectively zero. Then,

[28] P. T. Matthews, Phys. Rev. **76**, 1657 (1949). We make use of a later formulation of the theory of vector bosons in interaction with fermions due to F. Coester, Phys. Rev. **83**, 798 (1951). See also, R. J. N. Phillips, Phys. Rev. **96**, 1678 (1954).

[29] As we are using the formulation (Ref. 28) of vector boson theory which goes over continuously into quantum electrodynamics as $\mu\to 0$, we omit the customary term $q_\mu q_\nu[\mu^2(q^2+\mu^2)]^{-1}$. In any case, inclusion of this term would simply modify our final result Eq. (2.36) by an additional term which is effectively zero.

[30] We retain the non-gauge-invariant term $\Pi_{\mu\nu}(0)$ solely for the sake of generality. Relativistic invariance requires that it be of the form $A\delta_{\mu\nu}$. J. Schwinger [Phys. Rev. **74**, 1439 (1948)] has justified the result $A=0$ on the basis of an appropriate method of manipulation of divergent integrals, but G. Wentzel [*ibid.* **74**, 1070 (1948)] has criticized this method. Also W. Pauli and F. Villars [Rev. Mod. Phys. **21**, 434 (1949)] and R. Jost and J. Rayski [Helv. Phys. Acta **22**, 457 (1949)] have obtained the result $A=0$ by the method of regularization. More recently, K. Johnson [Nucl. Phys. **25**, 431 (1961)] has shown that an automatic subtraction of the $\Pi_{\mu\nu}(0)$ term follows when one redefines the fermion current operator in such a way that it be invariant under local gauge transformations. See also, O. Hara [Progr. Theoret. Phys. (Kyoto) **30**, 370 (1963)] where the result $A\neq 0$ is assumed.

with the aid of Eq. (2.29), we obtain the solution

$$M_{f\mu\nu}(q)=g_0\tau_{\mu\nu}(q)[1-g_0(A+\Pi_V(q^2))]^{-1} \quad (2.40)$$

of Eq. (2.39). The discussion of equivalence is hereafter identical to that of the pseudoscalar case. The amplitude $M_{f\mu\nu}$ has a pole at $q^2=-\mu^2$, if

$$A+\Pi_V(-\mu^2)=1/g_0, \quad (2.41)$$

and can then be written in the form

$$M_{f\mu\nu}(q)=\tau_{\mu\nu}(q)[-(q^2+\mu^2)\Pi_V'(-\mu^2)-\Pi_{Vc}(q^2)]^{-1}. \quad (2.42)$$

The condition

$$G_R^2=-[\Pi_V'(-\mu^2)]^{-1} \quad (2.43)$$

for the equivalence of $M_{f\mu\nu}$ with the expression (2.36) for $M_{y\mu\nu}$ thus implies the condition

$$Z_3(G_R)=0 \quad (2.44)$$

on the Yukawa theory.

Additional remarks like those made after Eq. (2.16) in the pseudoscalar theory can be made here also. From Eqs. (2.33) and (2.41) we have

$$\delta\mu^2=-G_0^2/g_0, \quad (2.45)$$

and, since Eq. (2.44) implies G_0^2 infinite, then the vector boson bare mass μ_0^2 must also be infinite. Finally, we mention that a calculation of Z_3^{-1}, to lowest order in G_R^2 using the Lehmann formula, gives the result

$$Z_3^{-1}=1+G_R^2\int\frac{\rho_V(\kappa^2)d\kappa^2}{(\kappa^2-\mu^2)^2}, \quad (2.46)$$

where $\rho_V(\kappa^2)$ is given by Eq. (2.32). This result is the exact analog of Eq. (2.23).

3. EQUIVALENCE CONDITIONS IN ALL ORDERS OF PERTURBATION THEORY

In the preceding section we obtained conditions for the equivalence between four-Fermi and Yukawa theories in the chain approximation. We now derive conditions valid to all orders in perturbation theory. We treat only the case of pseudoscalar coupling explicitly, but it will be apparent that an analogous treatment of the vector case can be carried out. Attention in the Yukawa theory is confined to graphs without external boson lines.[31] The method of derivation of equivalence conditions is the comparison of the integral equations for the basic Green's functions in the two theories. In the Yukawa theory, the Green's function in question are the fully dressed boson and fermion propagators, Δ'_F and S'_F, and the fully dressed vertex function Γ, and the equations are the Dyson equations.[21,32] In the four-Fermi theory, one exploits the one-to-one correspondence of its Feynman graphs to those of the Yukawa theory, and one is thereby led to a basic set of Green's functions whose relation to the Yukawa set is developed below. They obey equations similar in structure to the Dyson equations, but identical to them only when certain conditions on the two theories are satisfied. These are the conditions we seek.

We first consider the Yukawa theory. We recall that any Green's function of the theory can be expressed as a functional of the basic set of three Green's functions Δ'_F, S'_F, and Γ, which satisfy a system of coupled integral equations. The equation for Δ'_F generalizes Eq. (2.4) and can be written as

$$G_0^2\Delta'_F(q^2)=G_0^2\Delta'_F(q^2)+G_0^2\Delta_F(q^2)\times\Pi^*(q^2)G_0^2\Delta'_F(q^2). \quad (3.1)$$

Here $\Delta_F(q^2)=(q^2+\mu_0^2)^{-1}$ and $\Pi^*(q^2)$ is given in terms of S'_F and Γ, according to

$$\Pi^*(q^2)=i(2\pi)^{-4}\,\mathrm{Tr}\left[\gamma_5\int d^4kS'_F(k-q)\times\Gamma(k-q,k)S'_F(k)\right]. \quad (3.2)$$

Some comments concerning the distribution of G_0^2 factors must be made. Here, just as the preceding section, explicit association of a factor G_0^2 with Δ'_F will aid in the comparison of Yukawa with four-Fermi expressions. Also, the convention of introducing $\Pi(q^2)$ by Eq. (2.6) without the customary G_0^2 factor has been maintained in the definition (3.2). An equivalent form of Eq. (3.1) is

$$G_0^2\Delta'_F(q^2)=G_0^2[q^2+\mu^2-G_0^2\Pi^*(q^2)+G_0^2\Pi^*(-\mu^2)]^{-1}. \quad (3.3)$$

The equation for S'_F is

$$S'_F(p)=[\gamma\cdot p-im-\Sigma^*(p)+\Sigma^*(p)|_{\gamma\cdot p=im}]^{-1}, \quad (3.4)$$

where

$$\Sigma^*(p)=-i(2\pi)^{-4}\int d^4k\gamma_5S'_F(p-k)\times\Gamma(p-k,k)G_0^2\Delta'_F(k^2), \quad (3.5)$$

and the equation for Γ is

$$\Gamma(p_1,p_2)=\gamma_5+\textstyle\sum_i\Lambda^{(i)}(p_1,p_2). \quad (3.6)$$

The summation in Eq. (3.6) is over irreducible vertex parts, the substitutions $G_0^2\Delta'_F$, S'_F, and Γ for $G_0^2\times(q^2+\mu_0^2)^{-1}$, $(\gamma\cdot p-im_0)^{-1}$, and γ_5, respectively, having been made throughout the expression for each such

[31] It should be possible to remove this restriction. Although the four-Fermi theory does not allow direct description of processes like boson-fermion scattering in terms of graphs with external boson lines, such processes can, nevertheless, be described in both theories by regarding the real bosons as a limiting case of virtual bosons, and including the source and sink of the bosons in the analysis of the scattering process. See R. P. Feynman, Phys. Rev. **80**, 440 (1950), App. B.

[32] See also J. Schwinger, Proc. Natl. Acad. Sci. **37**, 452 and 455 (1951).

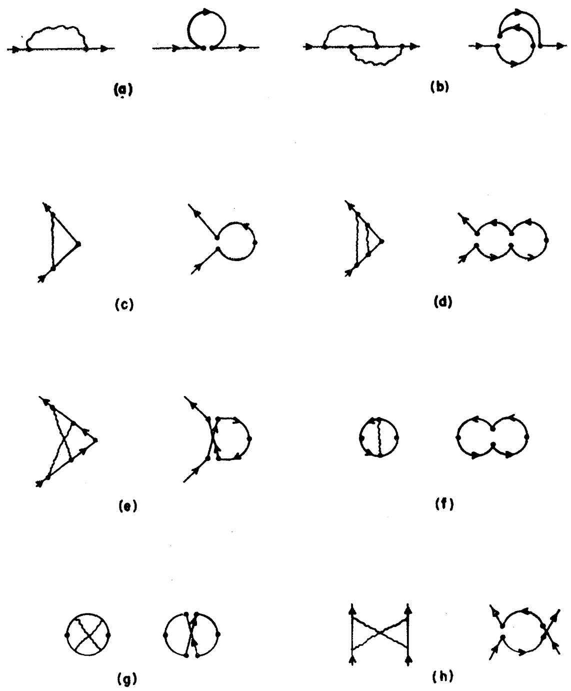

FIG. 4. Corresponding pairs of Yukawa and four-Fermi theory Feynman diagrams. The Yukawa diagrams (left) represent fermion self-energy parts in (a) and (b), vertex parts in (c), (d), and (e), boson self-energy parts in (f) and (g), and Møller scattering in (h).

part. For example, the lowest order vertex part $\Lambda^{(1)}$ has the form

$$\Lambda^{(1)}(p_1,p_2) = -i(2\pi)^{-4}\int d^4k G_0^2\Delta'_F(k^2)\Gamma(p_1,\, p_1-k)S'_F(p-k) \times\Gamma(p_1-k,\, p_2-k)S'_F(p_2-k)\Gamma(p_2-k,\, p_2)\,. \quad (3.7)$$

The above equations were originally written down by Dyson[21] on the basis of the perturbation expansion of the S matrix. This expansion can in turn be recovered iteratively from the Dyson equations by use of the zero-order approximation

$$\begin{aligned} \Delta'_F{}^{(0)} &= (q^2+\mu^2)^{-1}\,, \\ S'_F{}^{(0)} &= (\gamma\cdot p-im)^{-1}\,, \\ \Gamma^{(0)} &= \gamma_5\,. \end{aligned} \quad (3.8)$$

Contact with the work of Sec. 2A follows from noting that the insertion of $S'_F{}^{(0)}$ and $\Gamma^{(0)}$ into Eq. (3.2) reproduces the approximation of Eq. (2.6).

Before turning to the four-Fermi theory, we wish to rewrite the Dyson equation for the boson propagator in a form that may be regarded as a generalization of that given by Eq. (2.11). For this purpose, we must introduce the renormalized functions $\Delta'_{F1}(G_R)$, $S'_{F1}(G_R)$, and $\Gamma_1(G_R)$ which are related to Δ'_F, S'_F, and Γ by[21]

$$\begin{aligned} \Delta'_F &= Z_3\Delta'_{F1}(G_R)\,, \\ S'_F &= Z_2 S'_{F1}(G_R)\,, \\ \Gamma &= Z_1^{-1}\Gamma_1(G_R)\,, \end{aligned} \quad (3.9)$$

where G_R is given by

$$G_R = Z_1^{-1}Z_2Z_3^{1/2}G_0\,. \quad (3.10)$$

We also require the function $\Pi^*{}_1(q^2)$, which is obtained from $\Pi^*(q^2)$ by replacing S'_F and Γ in Eq. (3.2) by S'_{F1} and Γ_1, respectively, and removing the so-called "b divergences."[21] The relation between $\Pi^*{}_1$ and Π^* has been established by Dyson,[21] Salam,[21] and Ward.[21] We write their result in the form

$$G_0^2\Pi^*(q^2) = Z_3^{-1}G_R^2\Pi^*{}_1(q^2)\,. \quad (3.11)$$

If we make an expansion

$$\Pi^*{}_1(q^2) = \Pi^*{}_1(-\mu^2) + (q^2+\mu^2)\Pi^*{}_1{}'(-\mu^2) + \Pi^*{}_{1c}(q^2) \quad (3.12)$$

of $\Pi^*{}_1$, then $\Pi^*{}_{1c}$ is finite. We also have, from the work of Dyson,[21] Salam,[21] and Ward,[21] the following expression for Z_3:

$$Z_3(G_R) = 1 + G_R^2\Pi^*{}_1{}'(-\mu^2)\,. \quad (3.13)$$

We may now use Eqs. (3.11)–(3.13) to write Eq. (3.3) in the form

$$G_0^2\Delta'_F(q^2) = Z_3G_0^2[q^2+\mu^2-G_R^2\Pi^*{}_{1c}(q^2)]^{-1}\,, \quad (3.14)$$

which clearly generalizes Eq. (2.11). It is more convenient to work with this result in the form

$$G_0^2\Delta'_F(q^2) = Z_3G_0^2[q^2+\mu^2-Z_3G_0^2\Pi^*{}_c(q^2)]^{-1}\,, \quad (3.15)$$

where the quantity $\Pi^*{}_c$ is defined either by an expansion like (3.12) of Π^* or else given in terms of $\Pi^*{}_{1c}$ by an equation like (3.11).[33] We note that, in the approximation of Sec. 2A, $Z_1=Z_2$, $G_R^2=Z_3G_0^2$, and $\Pi^*{}_1=\Pi^*=\Pi$, with Π given by Eq. (2.6). Then Eqs. (3.13) and (3.14) reduce to Eqs. (2.10) and (2.11), respectively.

Our approach to the four-Fermi theory is based on the important fact that its Feynman diagrams can be placed in one-to-one correspondence with the Feynman diagrams of the Yukawa theory with no external boson lines, by means of the replacement of four-Fermi vertices by bare boson propagators. More precisely, we realize the correspondence by associating the two dots of the "double-dot symbol" for a four-Fermi vertex to a pair of Yukawa vertices with a bare virtual boson connecting them. Examples of the correspondence are displayed in Fig. 4. Expressions for corresponding four-Fermi and Yukawa diagrams can be obtained, one from the other, by means of the replacements

$$2g_0 \rightleftarrows G_0^2(q^2+\mu_0^2)^{-1} \quad (3.16)$$

[33] It should be noted that although $\Pi^*{}_{1c}$ is finite, $\Pi^*{}_c$ is not (Ref. 21).

for each four-Fermi vertex and corresponding pair of Yukawa vertices, q being the momentum transfer carried by the boson line connecting the two Yukawa vertices.

We exploit the correspondence of Feynman diagrams by introducing a set of three basic Green's functions $\mathbf{\Delta}'_F$, $\mathbf{S}'_F$, and $\mathbf{\Gamma}$ in the four-Fermi theory. Their significance follows, in the light of the correspondence, from that of Δ'_F, S'_F, and Γ in the Yukawa theory, and they satisfy a set of coupled integral equations whose structure parallels the equations written above for the Yukawa theory.[34] The difference between the two theories is expressed by (3.16) and manifests itself in the equation for $\mathbf{\Delta}'_F$. This equation can be written as

$$2g_0\mathbf{\Delta}'_F(q^2)=2g_0+2g_0\mathbf{\Pi}^*(q^2)2g_0\mathbf{\Delta}'_F(q^2)\,, \quad (3.17)$$

or else as

$$2g_0\mathbf{\Delta}'_F(q^2)=2g_0[1-2g_0\mathbf{\Pi}^*(q^2)]^{-1}\,, \quad (3.18)$$

where $\mathbf{\Pi}^*$ is given by an equation formally identical to Eq. (3.2)

$$\mathbf{\Pi}^*(q^2)=i(2\pi)^{-4}\,\mathrm{Tr}\left[\gamma_5\int d^4k\mathbf{S}'_F(k-q)\times\mathbf{\Gamma}(k-q,k)\mathbf{S}'_F(k)\right]. \quad (3.19)$$

The remaining equations are formally identical to their Yukawa counterparts with $G_0^2\Delta'_F$, wherever it occurs, replaced by $2g_0\mathbf{\Delta}'_F$. As in the Yukawa theory, the perturbation expansion of the four-Fermi theory can be obtained by iteration of the basic integral equations.

In order that the Yukawa and four-Fermi theories be equivalent, it is necessary that we be able to identify $G_0^2\Delta'_F$, S'_F, and Γ, respectively with $2g_0\mathbf{\Delta}'_F$, $\mathbf{S}'_F$, and $\mathbf{\Gamma}$. Such an identification is not, in general, possible, in view of the difference between Eqs. (3.15) and (3.18). Conversely when these two equations (and, consequently, the two sets of integral equations) become identical, the above identification can be made. By expanding $\mathbf{\Pi}^*(q^2)$ about $q^2=-\mu^2$ as in Eqs. (2.8) and (3.12), we find that, if the conditions

$$2g_0\mathbf{\Pi}^*(-\mu^2)=1\,, \quad (3.20)$$

$$Z_3G_0^2=-[\mathbf{\Pi}^{*\prime}(-\mu^2)]^{-1} \quad (3.21)$$

are satisfied, Eq. (3.18) takes the form

$$2g_0\mathbf{\Delta}'_F(q^2)=Z_3G_0^2[q^2+\mu^2-Z_3G_0^2\mathbf{\Pi}^*_c(q^2)]^{-1}\,, \quad (3.22)$$

which is formally identical to Eq. (3.15) and, in fact, independent of g_0. The identifications[35]

$$2g_0\mathbf{\Delta}'_F=G_0^2\Delta'_F\,,\quad \mathbf{S}'_F=S'_F\,,\quad \mathbf{\Gamma}=\Gamma\,,$$
$$\mathbf{\Pi}^*=\Pi^*\,,\quad \mathbf{\Sigma}^*=\Sigma^*\,,\quad \mathbf{\Lambda}^{(i)}=\Lambda^{(i)}\,, \quad (3.23)$$

can thus be made and inserted into Eqs. (3.20) and (3.21), giving

$$2g_0\Pi^*(-\mu^2)=1\,, \quad (3.24)$$

$$Z_3G_0^2=-[\Pi^{*\prime}(-\mu^2)]^{-1}. \quad (3.25)$$

Using the relation

$$\delta\mu^2=-G_0^2\Pi^*(-\mu^2)\,, \quad (3.26)$$

and Eqs. (3.11) and (3.13), we can rewrite the equivalence conditions (3.24) and (3.25) in the form

$$\delta\mu^2=-G_0^2/(2g_0)\,, \quad (3.27)$$

and

$$Z_3(G_R)=0\,. \quad (3.28)$$

These results are true to all orders of perturbation theory. This completes our extension of the range of validity of the conditions (2.17) and (2.16) for equivalence. The $Z_3=0$ condition has arisen from the cancelation of the two terms in Eq. (3.13), and, since $\Pi^*_1{}'(-\mu^2)$ is logarithmically divergent, this implies the necessity of a cutoff. This is, of course, exactly parallel to the situation discussed in Sec. 2A.

We note that Eqs. (3.10) and (3.25) do not necessarily imply that an infinite value of G_0^2 is required to insure that G_R^2 be finite.[36] However, we can still deduce that μ_0^2 must be infinite. From Eqs. (3.26) and (3.11), we have

$$\delta\mu^2=-Z_3^{-1}G_R^2\Pi^*_1(-\mu^2)\,, \quad (3.29)$$

which imples that μ_0^2 is infinite when $Z_3=0$ and G_R^2 and μ^2 are finite.[37] It then follows from Eq. (3.27) that either G_0^2 is infinite or g_0 is equal to zero.

It is of interest to note that the equivalence conditions (3.27) and (3.28) have previously been obtained by Jouvet[38] on the basis of a somewhat incomplete argument. Jouvet observes the one-to-one correspondence of Feynman graphs in the two theories and seeks conditions under which the correspondence relation (3.16) may be regarded formally as an equality

$$2g_0=G_0^2(q^2+\mu_0^2)^{-1}. \quad (3.30)$$

He states that the above equality holds, provided that μ_0^2 and G_0^2 are infinite and $G_0^2/\mu_0^2=2g_0$, and deduces $Z_3=0$ from μ_0^2 infinite. Aside from its incompleteness, Jouvet's procedure suffers from the disadvantage that it does not clarify the nature of the equivalence conditions. For example, the relation of Eq. (3.27) to the condition in the four-Fermi theory for a pseudoscalar boson bound state is not exhibited. Moreover, the essential role of the cutoff is not made clear. We could have argued in close analogy to the method of Jouvet,

[34] I. B. Birula (Ref. 12) has written down the integral equations of a four-Fermi theory with vector interaction by means of the functional method of Schwinger (Ref. 32).

[35] The identification $\mathbf{\Sigma}^*=\Sigma^*$ gives the equality of the fermion self-masses of the two theories when the equivalence conditions are satisfied.

[36] A finite value of G_R^2 can be achieved when $Z_3=0$ without having G_0^2 infinite in the special case when Z_1 but not Z_2 is vanishing.

[37] We draw attention to the fact that, in our approach, $Z_3=0$ does not imply $\delta\mu^2=0$ as suggested by Salam (Ref. 2).

[38] B. Jouvet (Ref. 12), third paper cited. We have corrected Jouvet's results by including in Eq. (3.16) the factor 2 on the left-hand side which he omitted.

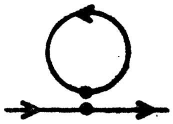

FIG. 5. Four-Fermi theory tadpole diagram.

by considering the condition under which Eqs. (3.1) and (3.17) are identical, which is that the formal equality (3.30) hold. Such a procedure, however, suffers from the same drawbacks as that of Jouvet.

Another argument,[39] which is closely related to and has the same disadvantages as Jouvet's argument, is based on a direct comparison of the Lagrangian equations of motion of the two theories. The Lagrangian densities of the two theories are

$$\mathcal{L}_y = -\bar{\psi}(\gamma_\mu\partial_\mu + m_0)\psi - \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi - \tfrac{1}{2}\mu_0{}^2\phi^2 + iG_0\bar{\psi}\gamma_5\psi\phi \quad (3.31)$$

and

$$\mathcal{L}_f = -\bar{\psi}(\gamma_\mu\partial_\mu + m_0)\psi - g_0\bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi. \quad (3.32)$$

The equation of motion for the boson field ϕ is

$$(\Box - \mu_0{}^2)\phi = -iG_0\bar{\psi}\gamma_5\psi. \quad (3.33)$$

When $\mu_0{}^2$ and G_0 are infinite, one may write

$$\phi = i(G_0/\mu_0{}^2)\bar{\psi}\gamma_5\psi, \quad (3.34)$$

and eliminate ϕ from $\mathcal{L}_y$, obtaining

$$\mathcal{L}_y = -\bar{\psi}(\gamma_\mu\partial_\mu + m_0)\psi - (G_0{}^2/2\mu_0{}^2)\bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi. \quad (3.35)$$

This coincides with $\mathcal{L}_f$, given by (3.32), when $g_0 = G_0{}^2/(2\mu_0{}^2)$.

4. SPECIFIC EXAMPLES OF EQUIVALENCE IN THE CHAIN APPROXIMATION

A. Theory of Nambu and Jona-Lasinio

The theory of Nambu and Jona-Lasinio[13] is based cn the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \\ \mathcal{L}_0 &= -\bar{\psi}\gamma_\mu\partial_\mu\psi, \\ \mathcal{L}_{\text{int}} &= g_0[\bar{\psi}\psi\bar{\psi}\psi - \bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi], \end{aligned} \quad (4.1)$$

which is invariant under the γ_5 transformation $\psi \to e^{i\alpha\gamma_5}\psi$. Ordinary perturbation theory with $\mathcal{L}_{\text{int}}$ as perturbation yields a theory with a nondegenerate vacuum and a propagator corresponding to zero physical fermion mass. We are interested here in the theory with a degenerate vacuum, and this is obtained by writing the Lagrangian density $\mathcal{L}$ of Eq. (4.1) in the form

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \mathcal{L}_0' + \mathcal{L}_{\text{int}}', \\ \mathcal{L}_0' &= \mathcal{L}_0 - m\bar{\psi}\psi, \\ \mathcal{L}_{\text{int}}' &= \mathcal{L}_{\text{int}} + m\bar{\psi}\psi. \end{aligned} \quad (4.2)$$

In the perturbation theory based on Eq. (4.2), the unperturbed fermion has a mass m which is determined self-consistently by the requirement that the self-energy due to $\mathcal{L}_{\text{int}}'$ vanish. To lowest order, the relevant self-energy graph is the one displayed in Fig. 5, and the self-consistency requirement gives rise to the equation[23]

$$m = \frac{-ig_0}{2\pi^4}\int d^4p \frac{m}{p^2+m^2}, \quad (4.3)$$

the quadratically divergent integral being made finite by use of a relativistic cutoff. The $m=0$ solution of Eq. (4.3) corresponds to the ordinary perturbation theory based on Eq. (4.1), while the $m\neq 0$ solution of

$$1 = \frac{-ig_0}{2\pi^4}\int d^4p \frac{1}{p^2+m^2} \quad (4.4)$$

corresponds to the degenerate vacuum theory. One notices that Eq. (4.4) can be written in terms of the function $\Pi(q^2)$ of Eq. (2.6) according to[23]

$$1 = 2g_0\Pi(0), \quad (4.5)$$

provided that the same cutoff is used now as in Eq. (2.6).

One now uses perturbation theory based on Eq. (4.2) to compute the pseudoscalar contribution M_f to the fermion-fermion scattering amplitude in the chain approximation. This leads to the result Eq. (2.12). The most important feature of the present subsection now emerges. We do not need to impose a condition for the existence of a pole; a pole at $q^2=0$, corresponding to a zero-mass pseudoscalar boson,[23,16] is automatically present by virtue of the self-consistency condition written in the form of Eq. (4.5). Comparison of M_f with the corresponding amplitude M_y in a Yukawa theory in which the massless boson is described by an elementary field now proceeds exactly as in Sec. 2A. Setting $\mu^2=0$ in Eq. (2.7), we find that the condition

$$G_R{}^2 = -[\Pi'(0)]^{-1}, \quad (4.6)$$

for equality of M_f to M_y, again implies that, in the Yukawa theory, we must have

$$Z_3(G_R) = 0. \quad (4.7)$$

We next turn to the scalar boson bound state that occurs in the theory of Nambu and Jona-Lasinio as a result of the self-consistency condition. The scalar contribution M_{Sf} to the fermion-fermion scattering amplitude generated by the infinite set of diagrams shown in Fig. 2 can be written as

$$M_{Sf}(q^2) = 2g_0[1 - 2g_0\Pi_S(q^2)]^{-1}, \quad (4.8)$$

where $\Pi_S(q^2)$ has the spectral form

$$\Pi_S(q^2) = \int_{4m^2}^{\Lambda^2} \frac{\rho_S(\kappa^2)d\kappa^2}{q^2+\kappa^2}, \quad (4.9)$$

$$\rho_S(\kappa^2) = (1 - 4m^2/\kappa^2)^{1/2}(\kappa^2 - 4m^2)/8\pi^2. \quad (4.10)$$

[39] The authors thank Drs. C. R. Hagen and S. Okubo for informing them of this argument in private discussion.

Use of Eqs. (2.18), (2.19), and (4.5) now leads to the result[23]

$$\Pi_S(-4m^2)=\Pi(0)=1/(2g_0)\,, \qquad (4.11)$$

and hence there is a pole corresponding to a scalar boson of mass $2m$ in M_{Sf}. Equivalence of M_{Sf} to the corresponding amplitude in a Yukawa theory in which the scalar boson is described by an elementary field and in which $Z_{S3}=0$ follows exactly as in the pseudoscalar case.

As in Sec. 2A, we find, for the pseudoscalar boson, that[40]

$$\delta\mu^2=-G_0{}^2\Pi(0)=-G_0{}^2/(2g_0)\,, \qquad (4.12)$$

and for the scalar boson, that

$$\delta\mu_S{}^2=-G_{S0}{}^2\Pi_S(-4m^2)=-G_{S0}{}^2/(2g_0)\,, \qquad (4.13)$$

and that both bosons have infinite bare mass and infinite bare coupling to the fermion field.

It must be stressed that we have assumed the physical fermion mass m to be the same in both theories. It might be argued, therefore, that the two theories are essentially different, inasmuch as in the theory of Nambu and Jona-Lasinio one starts from zero bare fermion mass and generates the physical mass self-consistently. We now show, therefore, that one can obtain the fermion mass in the equivalent Yukawa theory in the same way.

To see this, we consider the self-energy diagram of Fig. 6 involving the bare (scalar) boson propagator $(q^2+\mu_{S0}{}^2)^{-1}$ with $q^2=0$. This clearly corresponds, in the Yukawa theory, to the fermion self-energy diagram of Fig. 5.[41] Computing the self mass $\delta m(m)$ arising from this diagram, we get

$$\delta m=\frac{-i}{2\pi^4}\frac{G_{S0}{}^2}{2\mu_{S0}{}^2}\int d^4p\frac{m}{p^2+m^2}\,. \qquad (4.14)$$

Using now Eq. (4.13) with μ_{S0} infinite, we find

$$\delta m=\frac{-i}{2\pi^4}g_0\int d^4p\frac{m}{p^2+m^2}\,.$$

It follows now from Eq. (4.3) that $\delta m=m$, so that $m_0=0$. Thus the physical fermion mass in the Yukawa theory which is equivalent to the Nambu-Jona-Lasinio theory is also generated self-consistently.[42]

In addition to scalar and pseudoscalar bound states, Nambu and Jona-Lasinio also find vector boson bound states to be present in their theory. To exhibit these states, they use the Fierz identities[17] to rewrite the four-Fermi coupling term in (4.1) and (4.2) as

$$\mathcal{L}_{\text{int}}=-\tfrac{1}{2}g_0[\bar\psi\gamma_\mu\psi\bar\psi\gamma_\mu\psi-\bar\psi\gamma_\mu\gamma_5\psi\bar\psi\gamma_\mu\gamma_5\psi]\,. \qquad (4.15)$$

We follow a different procedure, however, and work directly from (4.2). This enables us to exhibit the manner in which the Fierz identities manifest themselves in the perturbation expansion. Let us compute, for example, the matrix elements corresponding to the first two graphs in Fig. 7. They are, respectively,

$$\sum_{i=1}^{2}\bar u(p_2{}')M_iu(p_1)\bar u(p_1{}')M_iu(p_2) \qquad (4.16)$$

and

$$\sum_{i,j=1}^{2}\int d^4k\bar u(p_1{}')M_i\frac{1}{\mathbf{k}-\mathbf{q}-im}\times M_ju(p_2)\bar u(p_2{}')M_j\frac{1}{\mathbf{k}-im}M_iu(p_1)\,, \qquad (4.17)$$

where we have introduced the abbreviated notation $M_i=(1,i\gamma_5)$ to take account of the two terms in the interaction Lagrangian density of Eq. (4.1). Introducing explicit spinor indices and using the relation

$$\sum_{i=1}^{2}M_i{}^{\alpha\beta}M_i{}^{\gamma\delta}=\tfrac{1}{2}\sum_{i=1}^{2}N_i{}^{\gamma\beta}N_i{}^{\alpha\delta}\,, \qquad (4.18)$$

with $N_i\equiv(\gamma_\mu,i\gamma_\mu\gamma_5)$, we can rewrite (4.16) and (4.17) as

$$\tfrac{1}{2}\sum_{i=1}^{2}\bar u(p_1{}')N_iu(p_1)\bar u(p_2{}')N_iu(p_2) \qquad (4.19)$$

and

$$\tfrac{1}{4}\sum_{i,j=1}^{2}\int d^4k\bar u(p_1{}')N_iu(p_1)\times\mathrm{Tr}\left[N_i\frac{1}{\mathbf{k}-\mathbf{q}-im}N_j\frac{1}{\mathbf{k}-im}\right]\bar u(p_2{}')N_ju(p_2)\,, \qquad (4.20)$$

respectively. In terms of graphs, what we have done is to convert the first two diagrams of Fig. 7, where factors of 1 and $i\gamma_5$ operate at each dot, into the first two diagrams of Fig. 2, where factors of γ_μ and $i\gamma_\mu\gamma_5$ operate at each dot. A similar transformation may be

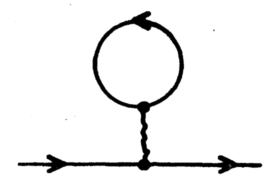

FIG. 6. Yukawa theory tadpole diagram.

[40] We continue to omit a label P when dealing with quantities that refer to pseudoscalar bosons.

[41] The correspondence of Figs. 5 and 6 can be regarded as a special case of that employed by Jouvet (Ref. 10) to obtain the equivalence conditions (3.27) and (3.28). It should be noted, however, that, because the boson line carries no momentum transfer, we are dealing with the one case where one need not employ the doubtful procedure of neglecting q^2 with respect to $\mu_0{}^2$ in the inverse propagator.

[42] We note that the physical fermion mass in the Yukawa theory can be thought of as arising from a nonvanishing vacuum expectation value $\langle\phi_S\rangle_0$ of the scalar field operator (Ref. 16).

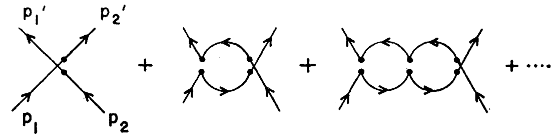

FIG. 7. Fermion-fermion scattering graphs which correspond to exchange of vector bound states.

applied to each diagram in Fig. 7, and the resulting series can then be identified as the chain approximation for fermion-fermion scattering based on the interaction (4.15). This example effectively illustrates the manner in which the Fierz identities manifest themselves in the perturbation expansion, the main point being that any diagram can be considered either as part of the original four-Fermi theory or, alternatively, as part of the Fierz transformed theory.

Nambu and Jona-Lasinio now evaluate the matrix element $M_{f\mu\nu}$ corresponding to the sum of diagrams in Fig. 2 by the procedure described in Sec. 2B, and adopt the choice $A=0$. They then show that, in the vector case, the resulting condition

$$\Pi_V(-\mu^2)=1/g_0 \qquad (4.21)$$

for the existence of a vector boson pole in $M_{f\mu\nu}$ is satisfied by virtue of the self-consistency condition (4.3) for $8m^2/3<\mu^2<4m^2$ and for sufficiently small cutoff. On the basis of the work of Sec. 2B, we can conclude the equality of $M_{f\mu\nu}$ with the corresponding amplitude in a Yukawa theory in which the vector boson is described by an elementary field with $Z_3=0$. For the pseudovector case, on the other hand, Nambu and Jona-Lasinio find that the corresponding condition for the existence of a pseudovector bound state is not satisfied for μ^2 in the range 0 to $4m^2$.

We conclude this section with a few general remarks regarding the significance to be attached to the emergence of vector boson bound states in the theory defined by (4.2). Let us assume that one could somehow extend the program of Nambu and Jona-Lasinio to all orders of perturbation theory and establish the existence in the theory of pseudoscalar and scalar boson bound states. By the work of Sec. 3, such a result would imply the complete equivalence of the four-Fermi theory (4.2) with the corresponding Yukawa theory with $Z_3=Z_{S3}=0$. The question then arises as to whether such a result can be reconciled with the existence of vector boson bound states in the theory. One possibility, perhaps the most likely, is that the bound-state conditions cannot be satisfied simultaneously for the original and Fierz transformed couplings; this would mean that the emergence of a vector boson bound state is simply a lowest order phenomenon and has no real significance. If, on the other hand, the bound-state conditions could be satisfied for both the original and the Fierz-transformed couplings, one would have to consider the possibility that the corresponding Yukawa theories might be directly equivalent to each other for the special values of the masses and coupling constants satisfying the equivalence conditions. We are, of course, unable to answer, at present, the question as to which, if either, of these possibilities is actually realized.

Finally, we wish to draw attention to a model for which the above problem does not arise. Consider the following Fierz-invariant and γ_5-invariant interaction:

$$\mathcal{L}_{\text{int}}=g_0[\bar\psi\psi\bar\psi\psi-\bar\psi\gamma_5\psi\bar\psi\gamma_5\psi-\bar\psi\gamma_\mu\psi\bar\psi\gamma_\mu\psi]. \qquad (4.22)$$

The theory based on (4.22) is easily seen to give rise to the same self-consistent fermion mass and to the same spectrum of pseudoscalar, scalar, and vector bound states as the theory of Nambu and Jona-Lasinio.[43] There is, of course, no pseudovector coupling in the theory due to its Fierz-invariant nature, and in particular, no admixture of pseudovector coupling of the massless boson to the fermion. This admixture ordinarily arises from mixed graphs of the type displayed in Fig. 8 and is present in the theory of Nambu and Jona-Lasinio.[13]

B. Theory of Bjorken

Our discussion of the theory of Bjorken[14] employs the Lagrangian density[44]

$$\begin{aligned}\mathcal{L}&=\mathcal{L}_0+\mathcal{L}_{\text{int}},\\ \mathcal{L}_0&=-\bar\psi(\gamma_\mu\partial_\mu+m)\psi,\\ \mathcal{L}_{\text{int}}&=-\tfrac{1}{2}g_0\bar\psi\gamma_\mu\psi\bar\psi\gamma_\mu\psi.\end{aligned} \qquad (4.23)$$

Ordinary perturbation theory with $\mathcal{L}_{\text{int}}$ as perturbation leads to a theory with a Lorentz-invariant vacuum. To obtain the degenerate vacuum theory, we write $\mathcal{L}$ in the form

$$\begin{aligned}\mathcal{L}&=\mathcal{L}_0'+\mathcal{L}_{\text{int}}',\\ \mathcal{L}_0'&=\mathcal{L}_0-\bar\psi i\gamma_\mu Q_\mu\psi,\\ \mathcal{L}_{\text{int}}'&=\mathcal{L}_{\text{int}}+\bar\psi i\gamma_\mu Q_\mu\psi,\end{aligned} \qquad (4.24)$$

where Q_μ is a constant four-vector to be determined self-consistently. In the perturbation theory based on (4.24), an unperturbed fermion line carries momentum

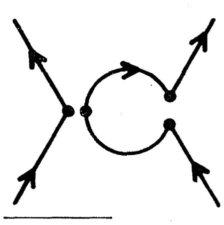

FIG. 8. "Mixed" graph representing PS–PV or S–V interference. The latter vanishes by Furry's theorem.

[43] There is a trivial difference in the vector case owing to the absence of the factor $\frac{1}{2}$ in (4.22).

[44] The Lagrangian density (4.23) differs from Bjorken's choice in that we have replaced $\frac{1}{2}[\bar\psi,\gamma_\mu\psi]=N(\bar\psi\gamma_\mu\psi)$ by $\bar\psi\gamma_\mu\psi$. This change allows a different method of setting up the degenerate vacuum theory. Instead of using the method of Bogoliubov [Physica **26**, S1 (1960)] as did Bjorken, we follow a method analogous to that employed in the theory of Nambu and Jona-Lasinio.

$(k+Q)_\mu$ rather than k_μ simply, and we determine Q_μ by demanding that the effect of the first-order tadpole diagram of Fig. 6 be canceled by the Q_μ term of $\mathcal{L}_{\text{int}}'$. This leads to the self-consistency condition

$$Q_\mu=\frac{-ig_0}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\left[\gamma_\mu\frac{1}{\mathbf{k}+\mathbf{Q}-im}\right], \tag{4.25}$$

which agrees with Bjorken's condition to the order in which we are working. According to Bjorken, a suitably cutoff version of Eq. (4.25) can be written as

$$Q_\mu=g_0Q_\mu\phi(Q^2)\,, \tag{4.26}$$

and the $Q_\mu\neq 0$ solution of

$$1=g_0\phi(Q^2) \tag{4.27}$$

corresponds to the theory of present interest.

We now use perturbation theory based on Eq. (4.24) to compute the fermion-fermion scattering amplitude $M_{f\mu\nu}$ generated by the infinite set of diagrams shown in Fig. 2. It satisfies an equation like Eq. (2.40), namely,

$$M_{f\mu\nu}(q)=g_0\delta_{\mu\nu}+g_0\delta_{\mu\kappa}\Pi_{B\kappa\lambda}(q)M_{f\lambda\nu}(q)\,. \tag{4.28}$$

The expression $\Pi_{B\mu\nu}(q)$ for the basic fermion loop now is

$$\Pi_{B\mu\nu}(q)=\frac{i}{(2\pi)^4}\int d^4k \times\mathrm{Tr}\left[\gamma_\mu\frac{1}{\mathbf{k}-\mathbf{q}+\mathbf{Q}-im}\gamma_\nu\frac{1}{\mathbf{k}+\mathbf{Q}-im}\right]. \tag{4.29}$$

We use Bjorken's result[14] that $\Pi_{B\mu\nu}$ is of the form

$$\Pi_{B\mu\nu}(q)=\Pi_{B\mu\nu}(0)+\tau_{\mu\nu}(q)\Pi_V(q^2)\,, \tag{4.30}$$

with $\Pi_V(q^2)$ given by Eq. (2.31). Although one cannot here argue that $\Pi_{B\mu\nu}(0)$ is of the form $A\delta_{\mu\nu}$ by relativistic invariance, since the vector Q_μ is now at one's disposal, one can use the self-consistency condition (4.25) to evaluate $\Pi_{B\mu\nu}(0)$. This gives[14]

$$\Pi_{B\mu\nu}(0)=\frac{\partial}{\partial Q_\nu}\left[\frac{-i}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\left(\gamma_\mu\frac{1}{\mathbf{k}+\mathbf{Q}-im}\right)\right] =[\delta_{\mu\nu}+2g_0Q_\mu Q_\nu\phi'(Q^2)]/g_0\,, \tag{4.31}$$

where the argument of ϕ' is set, after differentiation, equal to the value of Q^2 which satisfies Eq. (4.27). Insertion of Eqs. (4.30) and (4.31) into (4.28) leads directly to the solution

$$M_{f\mu\nu}(q)=\frac{-1}{\Pi_V(q^2)}\left[\delta_{\mu\nu}-\frac{q_\mu Q_\nu+Q_\mu q_\nu}{(q\cdot Q)}+\frac{Q^2q_\mu q_\nu}{(q\cdot Q)^2}\right] -\frac{q_\mu q_\nu}{2(q\cdot Q)^2\phi'(Q^2)}, \tag{4.32}$$

where all terms but the $\delta_{\mu\nu}$ term are effectively zero.[45] Since $\Pi_V(q^2)$ vanishes for $q^2=0$, one need not impose any condition for the emergence of a pole in $M_{f\mu\nu}$. This is as in Sec. 4A. A pole at $q^2=0$ corresponding to a vector boson with zero mass[16] has been obtained automatically after using the self-consistency condition on the theory.

Comparison with a Yukawa theory in which the massless vector boson is described by an elementary field must now be undertaken. We must, however, devote careful attention to the specification of the Yukawa theory. Since the physical mass of the vector boson is zero, one is tempted to regard the Yukawa theory as quantum electrodynamics. However, for reasons that become clear below, we must deal with a Yukawa theory of fermions in interaction with neutral vector bosons of physical mass zero and bare mass nonzero and, in fact, infinite.[28] Further, we do not deal with the ordinary perturbative solution of such a theory, but rather with a degenerate vacuum solution in which a Q_μ is determined self-consistently in the same way as in the Bjorken four-Fermi theory itself. This allows us to use the same description of unperturbed fermions in the Yukawa theory as in the four-Fermi theory, and, in particular, to have the same external line factors for the fermion-fermion scattering amplitudes in the two theories.

We obtain the degenerate vacuum solution to the Yukawa theory, by applying the same procedure as we have used above in the four-Fermi theory, and determine Q_μ self-consistently by the analogous demand that the Q_μ-counter term cancel the effect of the first-order tadpole diagram of Fig. 6. This gives the self-consistency condition

$$Q_\mu=\frac{-i}{(2\pi)^4}\frac{G_0^2}{\mu_0^2}\int d^4k\,\mathrm{Tr}\left(\gamma_\mu\frac{1}{\mathbf{k}+\mathbf{Q}-im}\right) =(G_0^2/\mu_0^2)Q_\mu\phi(Q^2). \tag{4.33}$$

This coincides with Eq. (4.26), and hence has the same solution, if

$$g_0=G_0^2/\mu_0^2\,. \tag{4.34}$$

We therefore impose this as a condition on the Yukawa theory. Having thereby insured the same description of unperturbed fermions in the Yukawa theory as in the four-Fermi theory, we can turn now to the amplitude $M_{y\mu\nu}$, which is to be compared with $M_{f\mu\nu}$ computed above. It should be stressed that no condition on the finiteness of the G_0 and μ_0 which obey Eq. (4.34) has yet been imposed.

We consider then the amplitude $M_{y\mu\nu}$ corresponding to Fig. 2. It is given, as in Sec. 2B, by the expression

$$M_{y\mu\nu}(q)=G_0^2D'_{F\mu\nu}(q)\,, \tag{4.35}$$

[45] The spinor $u(p+Q)$, which describes an external fermion line in the perturbation theory based on Eq. (4.24), satisfies $(\mathbf{p}+\mathbf{Q}-im)u(p+Q)=0$ so that if $q=p-p'$, we still have $\bar{u}(p'+Q)\times\mathbf{q}u(p+Q)=0$.

where

$$D'_{F\mu\nu}(q)=D_{F\mu\nu}(q)+G_0^2D_{F\mu\kappa}(q)\times\bar{\Pi}_{B\kappa\lambda}(g)D'_{F\lambda\nu}(q)\,. \quad (4.36)$$

The quantity $\bar{\Pi}_{B\mu\nu}$ is obtained by mass renormalization from the quantity $\Pi_{B\mu\nu}$, given by Eqs. (4.29) and (4.30). As in the four-Fermi theory, we obtain

$$\Pi_{B\mu\nu}(0)=(\mu_0^2/G_0^2)\delta_{\mu\nu}+2Q_\mu Q_\nu\phi'(Q^2)\,, \quad (4.37)$$

which reduces to Eq. (4.31) when Eq. (4.34) is used. The boson self-mass is then given by

$$\delta\mu^2=-G_0^2[(\mu_0^2/G_0^2)+\Pi_V(-\mu^2)]\,. \quad (4.38)$$

Since $\Pi_V(0)=0$, this equation has the solution $\mu^2=0$ for the physical boson mass, in accordance with Goldstone's theorem.[46] Hence we have

$$\bar{\Pi}_{B\mu\nu}(q)=2Q_\mu Q_\nu\phi'(Q^2)+\tau_{\mu\nu}(q)\Pi_V(q^2)\,. \quad (4.39)$$

Also $D_{F\mu\nu}$ in Eq. (4.36) is the bare propagator, which, after mass renormalization, is given, in the Feynman gauge, by

$$D_{F\mu\nu}(q)=\delta_{\mu\nu}/(q^2)\,. \quad (4.40)$$

Insertion of Eqs. (4.39) and (4.40) into Eq. (4.36) now allows its solution for $D'_{F\mu\nu}$. We find a result of the form

$$M_{y\mu\nu}(q)=G_0^2[q^2-G_0^2\Pi_V(q^2)]^{-1}(\delta_{\mu\nu}+\cdots)\,, \quad (4.41)$$

where the coefficients of the $q_\mu q_\nu$, $(q_\mu Q_\nu+Q_\mu q_\nu)$, and $Q_\mu Q_\nu$ terms are complicated expressions which we do not write down. We simply state the result that, when G_0^2 is infinite or equivalently $Z_3=0$, the above expression for $M_{y\mu\nu}$ becomes identical to Eq. (4.32) for $M_{f\mu\nu}$. It also follows that μ_0^2 must be infinite if Eq. (4.34) is to be maintained with infinite G_0^2 and finite g_0.

C. Theory of Birula

Birula[12] considers the question of the equivalence of a four-Fermi theory with vector coupling to a Yukawa theory with neutral vector bosons. In either case, ordinary perturbative solutions and not degenerate vacuum solutions are dealt with, and the method of discussion is completely different from the present methods. We believe Birula's conclusion to be incorrect in that he has not observed the need for the $Z_3=0$ condition on the Yukawa theory. In order to justify this claim, we may restrict attention to his treatment, in the chain approximation, of the case where the vector boson has zero physical mass.

A consistent treatment of equivalence in the chain approximation in the case of zero physical vector boson mass can be obtained simply by putting $\mu^2=0$ in the discussion of Sec. 2B. However, for the purpose of comparison with Birula's work, we make explicit reference to certain points. First, we notice that Eq. (2.33) becomes

$$\delta\mu^2=-G_0^2A=-\mu_0^2\,, \quad (4.42)$$

in the case of physical mass zero, so that mass renormalization eliminates $\Pi_{\mu\nu}(0)$ and leaves $\bar{\Pi}_{\mu\nu}$ in the usual gauge-invariant form. We stress again that we can have a consistent theory with either $A\neq0$ or $A=0$. The remainder of the discussion of the Yukawa theory requires no comment. We next examine the condition for the existence of a $q^2=0$ pole in the amplitude $M_{f\mu\nu}$ of Eq. (2.41). When $A\neq0$, this condition is

$$Ag_0=1\,. \quad (4.43)$$

When $A=0$, g_0 infinite, without Eq. (4.43), is sufficient to ensure the existence of the pole. In either case, one is led to expression (2.43) for $M_{f\mu\nu}$, and the remainder of the discussion follows easily, as in Sec. 2B, to the conclusion that $Z_3=0$ or G_0^2 infinite must hold in the Yukawa theory as a condition for equivalence. In the $A\neq0$ formulation, we observe that G_0^2 infinite implies μ_0^2 infinite, and that Eqs. (4.42) and (4.43) imply the result $\delta\mu^2=-G_0^2/g_0$. In the $A=0$ formulation, however, we can make no statement regarding μ_0; and, moreover, since Eq. (4.43) does not hold, we do not have the relation $\delta\mu^2=-G_0^2/g_0$.

Birula follows, in principle, the second alternative of setting $A=0$ and g_0 infinite without Eq. (4.43), but fails to observe that $Z_3=0$ is required for equivalence. At the start of his discussion of the four-Fermi theory, Birula adopts the formal measure of introducing two parameters μ_0' and G_0' by means of the definition

$$g_0=G_0'^2/\mu_0'^2\,, \quad (4.44)$$

implying that an appropriate interpretation of the parameters can be made later. Also, he works in terms of a quantity $\mathcal{G}_{\mu\nu}$ related to our $M_{f\mu\nu}$ by

$$M_{f\mu\nu}(q)=G_0'^2\mathcal{G}_{\mu\nu}(q)\,. \quad (4.45)$$

From Eqs. (4.45), (2.40), and (4.44), we get

$$\mathcal{G}_{\mu\nu}(q)=\tau_{\mu\nu}(q)[\mu_0'^2-G_0'^2A-G_0'^2\Pi_V(q^2)]^{-1}. \quad (4.46)$$

As stated earlier, Birula now sets $A=0$ and $\mu_0=0$, thereby obtaining

$$M_{f\mu\nu}(q)=G_0'^2\mathcal{G}_{\mu\nu}(q)\,, \quad (4.45)$$

$$\mathcal{G}_{\mu\nu}(q)=\tau_{\mu\nu}(q)[-G_0'^2\Pi_V(q^2)]^{-1}\,, \quad (4.47)$$

which should be compared with the corresponding equations in the Yukawa theory obtained by setting $\mu=0$ in Eq. (2.35):

$$M_{y\mu\nu}(q)=G_0^2D'_{F\mu\nu}(q)\,, \quad (4.48)$$

$$D'_{F\mu\nu}(q)=\tau_{\mu\nu}(q)[q^2-G_0^2\Pi_V(q^2)]^{-1}. \quad (4.49)$$

Birula now identifies G_0 with G_0' and claims that the expressions (4.47) and (4.49) will be identical, without any further condition, when expressed in terms of observable quantities. Such a conclusion is incorrect.

[46] We also note that Eq. (4.38) is consistent with Eq. (4.34). Introducing the latter equation into the former, we get $\delta\mu^2=-G_0^2/g_0$, which agrees with Eq. (4.34) since $\delta\mu^2=-\mu_0^2$.

As was shown in Sec. 2B, the equality of $M_{f\mu\nu}$ and $M_{y\mu\nu}$ requires that the condition $Z_3=0$ on the Yukawa theory be satisfied.[47]

[47] We note that the identification of G_0' and G_0 is only justified when G_0 is infinite and $Z_3=0$. Moreover, identification of μ_0' with μ_0 is not justified at all since Birula has used $\mu_0'=0$ whereas μ_0 is strictly undetermined, as mentioned above after Eq. (4.43).

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Higher Symmetries for the Vector Mesons*

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To explain the existence of nine vector mesons and the degeneracy of the unmixed masses of the ϕ and ω, we postulate that the vector mesons possess a higher symmetry than SU_3. We discuss various properties of the adjoint representation of SU_n and propose three simple models for the symmetry-breaking interaction. We specialize the results to SU_4, and find good agreement between mass formulas and experiment. We introduce a new quantum number, "hyperstrangeness," and six hyperstrange vector mesons with fractional hypercharge but integral electric charge. These particles, which form an isotopic doublet and an isotopic singlet (together with their antiparticles) are approximately degenerate with the ρ and K^*, respectively, and are stable with respect to the strong and electromagnetic interactions. We also discuss the production and decay of these particles.

I. INTRODUCTION

ONE of the interesting problems in the assignment of strongly interacting particles to representations of SU_3 is the existence of at least nine vector mesons ρ, K^*, $\bar{K}^*$, ω, and ϕ, which seem to belong to an octet and a singlet in the limit of exact unitary symmetry. The symmetry-breaking interaction complicates the situation by allowing mixing between the singlet (which we shall call ϕ_0) and the neutral[1] member of the octet (ω_0). The mass operator (or more generally the inverse propagator) is then no longer diagonal in the (ω_0,ϕ_0) representation and the physically observed particles (ω,ϕ) are those linear combinations of ϕ_0 and ω_0 which diagonalize the mass operator. In this scheme there is no *a priori* relationship between the masses of the singlet and the octet. These masses can be calculated from the experimental data in a given model, however, and the remarkable result is obtained that they are approximately equal.[2,3]

It is of course possible that this equality is merely coincidental, or that it has some deep significance that can only be understood by a detailed dynamical calculation (for example one of the many triplet models[4]). The purpose of this paper is to explore the possibility that this may be explained on a purely group theoretical level by postulating some higher symmetry for the vector mesons.

As a first attempt we might look for a group possessing a nine-dimensional self-conjugate representation with the correct isotopic spin and hypercharge content. There is no simple Lie group with these properties, and it appears that it is not even possible to use a nonsimple group without some specific dynamical hypotheses. We are therefore led to consider the possibility that there are in fact more than nine vector mesons and that the observed nine form part of a representation of some larger group.

It has been shown that the usual interpretation of vector mesons as gauge particles requires us to assign them to the adjoint representation of the underlying symmetry group.[5] A dynamical bootstrap model leads to the same result.[6] We will restrict ourselves to simple groups and it follows that the rank (r) of the group must be equal to the number of neutral vector mesons,[7]

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[1] We use "neutral" in the extended sense of having all additive quantum numbers zero.

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