Shrinkage of the Effective Core in the Scattering of Particles of Arbitrary Spin*

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A previous theorem, proving the shrinkage of the effective core, is extended to the scattering of particles of arbitrary spin with the result that any matrix element of the scattering amplitude has at high energy no significant contribution from the "partial waves" whose total angular momenta are larger than const $\ln^2 s/\sin\theta$. It is also shown that any matrix element of the scattering amplitude is bounded by min (const $\ln^{3/2} s/\sin^2\theta$, const $s \ln^2 s$) at high energy.

1. INTRODUCTION AND RESULTS

In a previous paper,[1] one of us (K.Y.) proved that when the scattering amplitude $T(s,\cos\theta)$ is divided into an upper and a lower sum for the partial waves by definition

$$T_U(s,\cos\theta)=\sum_{l=0}^{\infty}(2l+1)a_l(s)[1-f_l(s)]P_l(\cos\theta) \quad (1.1)$$

and

$$T_L(s,\cos\theta)=\sum_{l=0}^{\infty}(2l+1)a_l(s)f_l(s)P_l(\cos\theta), \quad (1.2)$$

using the "step function"

$$f_l(s)=\{1-\exp[-\alpha\ln^2 s/(l+1)\sin\theta_0]\}^{\beta\ln s}, \quad (1.3)$$

and $T(s,\cos\theta)$ is analytic in a particular region as a function of $\cos\theta$ and is bounded by a power of s within this region, then for α sufficiently large,

$$|T_U(s,\cos\theta)|<s^{-N} \quad \text{for} \quad |\sin\theta|>|\sin\theta_0|, \quad (1.4)$$

where N is an arbitrarily large positive number. The dependence on $\sin\theta_0$ in the definition (1.3) of $f_l(s)$ is important only when the angle θ_0 has an s dependence and converges to 0 or π at $s\to\infty$. The inequality (1.4) is also valid for such an s dependent angle.

By Eq. (1.3)

$$|1-f_l(s)|<s^{-N} \quad \text{for} \quad l<(N/\alpha)\ln s/\sin\theta_0 \quad (1.5)$$

and

$$|f_l(s)|<s^{-N} \quad \text{for} \quad l>\alpha_0\ln^2 s/\sin\theta_0, \quad (1.6)$$

where $\alpha_0=-\alpha/\ln[1-\exp(-N/\beta)]$. Therefore, from the inequality (1.4) and (1.6) one can see that, in the high-energy limit, the partial-wave amplitude $a_l(s)$ for $l>$const $\ln^2 s/\sin\theta$ does not contribute substantially to the amplitude $T(s,\cos\theta)$.

The purpose of this note is to show that the result of I can be obtained for the scattering amplitudes of particles with spin. In this case, instead of the partial-wave expansion of the amplitude $T(s,\cos\theta)$, we have the following expansion for the helicity amplitude[2]:

$$\langle\theta,\phi,\lambda_3,\lambda_4|T(s)|0,0,\lambda_1,\lambda_2\rangle=\sum_J(2J+1)\times\langle\lambda_3,\lambda_4|T^J(s)|\lambda_1\lambda_2\rangle e^{i(\lambda-\mu)\phi}d^J{}_{\lambda\mu}(\theta), \quad (1.7)$$

where $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$, λ_i is the quantum number for the helicity, J is the total angular momentum, and $d_{\lambda\mu}{}^J(\theta)$ is

$$d_{\lambda\mu}(\theta)=\langle J\lambda|\exp(-i\theta J_y)|J\lambda\rangle=(-)^{\mu-\lambda}2^{-\lambda}\left[\frac{(J+\lambda)!(J-\lambda)!}{(J+\mu)!(J-\mu)!}\right]^{1/2}\times(1+\cos\theta)^{(\lambda+\mu)/2}(1-\cos\theta)^{(\lambda-\mu)/2}\times P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(\cos\theta). \quad (1.8)$$

Here, $P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}$ is a Jacobi polynomial. Using the expansion (1.7), we can define an upper sum by[3]

$$\langle\theta,\phi,\lambda_3,\lambda_4|T_U(s)|0,0,\lambda_1,\lambda_2\rangle=\sum_J(2J+1)[1-f_J(s)]\times\langle\lambda_3,\lambda_4|T^J(s)|\lambda_1,\lambda_2\rangle e^{-i(\lambda-\mu)\phi}d_{\lambda\mu}{}^J(\theta). \quad (1.9)$$

The main object of the next section is to prove, then, that if we choose α sufficiently large,

$$|\langle\theta,\phi,\lambda_3,\lambda_4|T_U(s)|0,0,\lambda_1,\lambda_2\rangle|<s^{-N}. \quad (1.10)$$

In Sec. 3 we use the unitarity condition to estimate the upper bound of the helicity amplitude. The results are

$$|\langle\theta,\phi,\lambda_3,\lambda_4|T(s)|0,0,\lambda_1,\lambda_2\rangle|<\text{const}\ln^{3/2}s/\sin^2\theta \quad (1.11)$$

or

$$|\langle\theta,\phi,\lambda_3,\lambda_4|T(s)|0,0,\lambda_1,\lambda_2\rangle|<\text{const}\, s\ln^2 s. \quad (1.12)$$

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[1] K. Yamamoto, Phys. Rev. **135**, B567 (1964), hereinafter referred to as I.

[2] M. Jacob and G. Wick, Ann. Phys. (N.Y.) **7**, 404 (1959).

[3] One might think that it is better to use the orbital angular momentum in dividing $T(s,\cos\theta)$, instead of the total angular momentum, since the orbital angular momentum has a direct connection with impact parameter. However, we think that the use of total angular momentum is better, since only the total angular momentum is conserved through a collision in our case. However, even if we use the orbital angular momentum, we can prove the inequality (1.10). At any rate, the difference between total and orbital angular momentum is finite for any case.

These are the most general extensions to particles with spin of the bound of Kinoshita, Loeffel, and Martin.[4] The inequalities (1.11) and (1.12) give us the result that any matrix element of the scattering amplitude of particles with arbitrary spin is bounded by $\min[\text{const}\ln^{3/2}s/\sin^2\theta, \text{const}s\ln^2 s]$.

The contents of this note are not the mere generalization to particles with spin, of I. We expect that, in a final stage, the results of I could be generalized to include the multiple production process. For this generalization the contents of this note will be useful.

2. PROOF

In this section we outline the proof of the inequality (1.10).

As is well known, helicity amplitudes may include singular kinematical factors. Therefore it is convenient to use the auxiliary function

$$F(s,\cos\theta)=\sum_J (2J+1)a^J(s)P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(\cos\theta), \quad (2.1)$$

which is free from these kinematical singularities, where

$$a^J(s)=\left[\frac{(J+\lambda)!(J-\lambda)!}{(J+\mu)!(J-\mu)!}\right]^{1/2}\langle\lambda_3,\lambda_4|T^J(s)|\lambda_1,\lambda_2\rangle. \quad (2.2)$$

In Eq. (2.1) we assume that

$$\lambda-|\mu|\geq 0 \quad (2.3)$$

without loss of generality, because we know that[5]

$$d_{\lambda\mu}{}^J(\cos\theta)=(-)^{\lambda-\mu}d_{\mu\lambda}{}^J(\cos\theta)=d_{-\mu,-\lambda}{}^J(\cos\theta). \quad (2.4)$$

Apart from a bounded s-independent factor, the only difference between the helicity amplitude (1.7) and $F(s,\cos\theta)$ is in the factor

$$(1+\cos\theta)^{(\lambda+\mu)/2}(1-\cos\theta)^{(\lambda-\mu)/2}. \quad (2.5)$$

In order to prove the inequality (1.10), it is sufficient to show that

$$|F_U(s,\cos\theta)|<s^{-N}, \quad (2.6)$$

where

$$F_U(s,\cos\theta)=\sum_J (2J+1)a^J(s)[1-f_J(s)] \times P_{J-\lambda}{}^{\lambda+\mu,\lambda-\mu}(\cos\theta). \quad (2.7)$$

For this we define a new function $G(s,z)$ by

$$G(s,z)=\sum_J (2J+1)a^J(s)z^{J-\lambda}, \quad (2.8)$$

with $J-\lambda$ a non-negative integer. The connection between $F(s,z)$ and $G(s,z)$ is obtainable from a procedure of Cornille.[6] F may be obtained from G, using the integral representation, derivable from Rodrigues' formula of the Jacobi polynomials, as

$$F(s,\cos\theta)=\frac{1}{2\pi}\int_0^{2\pi}d\phi G(s,\cos\theta+i\sin\theta\cos\phi) \times\left(1-\frac{i\sin\theta e^{i\phi}}{1-\cos\theta}\right)^{\lambda-\mu}\left(1+\frac{i\sin\theta e^{i\phi}}{1+\cos\theta}\right)^{\lambda+\mu}, \quad (2.9)$$

and G may be obtained from F as

$$G(s,z)=\int_{-1}^{1}K(z,z')F(s,z')dz', \quad (2.10)$$

where

$$K(z,z')=2^{-2\lambda-1}(1+z')^{\lambda+\mu}(1-z')^{\lambda-\mu}\sum_J (2J+1) \times\frac{(J+\lambda)!(J-\lambda)!}{(J+\mu)!(J-\mu)!}z^{J-\lambda}P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(z'). \quad (2.11)$$

The expression (2.11) is rather complicated but still simple enough to allow an investigation into the analyticity and upper bound of $G(s,z)$. We assume that $F(s,z)$ is analytic and bounded by some fixed power of s in an s-independent complex neighborhood of the real segment $(-1, 1)$ except for its intersection with cuts from $-\infty$ to $-x(s)$ and from $x(s)$ to ∞, $x(s)$ being an arbitrary function of s with $x(s)>1$. From this assumption, using Eq. (2.10), it may be shown (see Appendix) that $G(s,z)$ is also analytic and bounded by some fixed power of s in a domain where $|z|<1+\epsilon$ for some finite $\epsilon>0$, except for cuts from $-\infty$ to $-x-(x^2-1)^{1/2}$ and from $+x+(x^2-1)^{1/2}$ to ∞.

We divide $G(s,z)$ in two parts, $G_1(s,z)$ and $G_2(s,z)$,[7] that, using this analyticity and the Cauchy integral, are

$$G_i(s,z)=\frac{1}{2\pi i}\int_{C_i}\frac{G(s,z')}{z'-z}dz', \quad (2.12)$$

where C_1 plus C_2 forms the boundary of the analyticity domain, and C_2 is the part of the boundary along the cuts. From $G_i(s,z)$, using Eq. (2.9), one defines $F_i(s,z)$, and with these, using Eq. (2.1), or from $G_i(s,z)$ using Eq. (2.8), one defines the "partial-wave amplitude" $a_i{}^J(s)$. Finally, in a similar way to that used for Eq. (2.4), we define $F_{iU}(s,z)$ and $G_{iU}(s,z)$. Among these new quantities there are many relations, listed in part below:

$$F_U(s,t)=F_{1U}(s,t)+F_{2U}(s,t), \quad (2.13)$$

$$a^J(s)=a_1{}^J(s)+a_2{}^J(s), \quad (2.14)$$

$$G_i(s,z)=\sum_J (2J+1)a_i{}^J(s)z^{J-\lambda}, \quad (2.15)$$

[4] T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1963).

[5] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), p. 54.

[6] H. Cornille (to be published).

[7] This division of $G(s,z)$ corresponds to that of $T(s,z)$ in Eq. (30) and (31) of I, but this division is better than that of I.

$$a_i^J(s)=\frac{1}{2\pi i}\frac{1}{2J+1}\int_{C_i}\frac{G(s,z')}{(z')^{J-\lambda}}dz', \tag{2.16}$$

and

$$G_{iU}(s,z)=\sum_J (2J+1)[1-f_J(s)]a_i^J(s)z^{J-\lambda}. \tag{2.17}$$

By virtue of Eq. (2.13), we can investigate $F_{1U}(s,z)$ and $F_{2U}(s,z)$ separately in the proof of the inequality (2.6). First, for $F_{1U}(s,z)$, from Eq. (2.9), because on the boundary $C_1|G(s,z)|<s^{N'}$, one has

$$|a_1^J(s)|<[1/(2J+1)]s^{N'}(1+\epsilon)^{-J+\lambda}, \tag{2.18}$$

an inequality corresponding to (34) of I. For $P_{J-\lambda}^{\lambda-\mu,\lambda+\mu}\times(\cos\theta)$ we have[8]

$$|P_{J-\lambda}^{\lambda-\mu,\lambda+\mu}(\cos\theta)|\leq\frac{(J-\lambda+q)!}{(J-\lambda)!q!}\sim J^q, \tag{2.19}$$

where $q=\max(\lambda-\mu,\lambda+\mu)$. Therefore, we can prove

$$|F_{1U}(s,\cos\theta)|<s^{-N} \tag{2.20}$$

by the same technique used in Sec. III of I. For an upper bound to $F_{2U}(s,\cos\theta)$, it is sufficient to show

$$|G_{2U}[s,\exp(\pm i\theta)]|<s^{-N}, \tag{2.21}$$

since there is a relationship identical to (2.9) between $G_{2U}(s,\cos\theta)$ $F_{2U}(s,\cos\theta)$; and to obtain the inequality (2.21) one needs only the analytic properties of $a_2^J(s)$ in the left-hand half-plane. From Eq. (2.16), we can easily see that if one writes

$$a_2^J(s)=a_2'^J(s)+(-)^{J-\lambda}a_2''^J(s), \tag{2.22}$$

where $a_2'^J(s)$ and $(-)^{J-\lambda}a_2''^J(s)$ are the contribution from left- and right-hand cuts, respectively, then $a_2'^J(s)$ and $a_2''^J(s)$ are analytic in the left-hand half-plane of J plane and Eq. (2.22) exactly corresponds to Eq. (50) in I. Although we do not repeat the discussion of I, the above discussion is sufficient to prove the inequality (1.10).

3. UPPER BOUND ON THE SCATTERING AMPLITUDE

Here we estimate the upper bound of the scattering amplitude. For this we already have a general theory.[9] However, in that theory only a weak bound for small angles was obtained, and also it was not possible to estimate the bound of the following scattering amplitude:

$$N+\bar{N}=\pi+\pi, \tag{3.1}$$

for example. Therefore, it is worthwhile to estimate here the upper bound of the general two-body scattering amplitude.

In Sec. II we have proved that only $J<\text{const}\ln^2 s/\sin\theta$ contribute to $F(s,\cos\theta)$. Using this result, the unitarity condition

$$|a^J(s)|<\left[\frac{(2\lambda)!}{(\lambda-\mu)!(\lambda+\mu)!}\right]^{1/2}\times|\langle\lambda_3,\lambda_4|T^J(s)|\lambda_1\lambda_2\rangle|<\text{const}, \tag{3.2}$$

and the bounds of the Jacobi polynomial at $n\to\infty$,[10]

$$|P_n^{\alpha,\beta}(\cos\theta)|\leqslant\begin{cases}\text{const}n^{-\frac{1}{2}}\sin^{-\alpha-\frac{1}{2}}\theta, & 0\leq\theta\leqslant\pi/2\\ \text{const}n^{-\frac{1}{2}}\sin^{-\beta-\frac{1}{2}}\theta, & \pi/2\leqslant\theta\leqslant\pi,\end{cases} \tag{3.3}$$

we obtain

$$|\langle 0,\phi,\lambda_3,\lambda_4|T(s)|0,0,\lambda_1,\lambda_2\rangle|<\text{const}\ln^3 s/\sin^2\theta. \tag{3.4}$$

These results are not satisfactory, however, since we have obtained a stronger bound for the case of scattering of scalar particles.

To obtain a stronger bound we must work along similar lines to those taken by Kinoshita *et al.*[4] $G(s,z)$ has exactly the same analyticity properties and bounding conditions as in the scalar case. The only differences due to the spin are the factor

$$\left(1-\frac{i\sin\theta e^{i\phi}}{1-\cos\theta}\right)^{\lambda-\mu}\left(1+\frac{i\sin\theta e^{i\phi}}{1+\cos\theta}\right)^{\lambda+\mu} \tag{3.5}$$

in Eq. (2.9) and the factor (2.5). The product of the factors (2.5) and (3.5) is

$$[(1-\cos\theta)^{1/2}-i(1+\cos\theta)^{1/2}e^{i\phi}]^{\lambda-\mu}\times[(1+\cos\theta)^{1/2}+i(1-\cos\theta)^{1/2}e^{i\phi}]^{\lambda+\mu}, \tag{3.6}$$

and we can easily see that this is finite over the integration region of ϕ in Eq. (2.9), however. Thus all effects due to the spin disappear, and we obtain the bound (2.11).

Finally, we discuss the scattering amplitude in the forward or backward directions. In this case, the high-energy behavior of the branch point $x(s)$ in the z plane is necessary. Although it has not been explicitly remarked in the paper of Kinoshita *et al.*, if we assume

$$x(s)\sim 1+\text{const}/s, \tag{3.7}$$

it is possible to prove that[11]

$$|G(s,z)|<s\ln^2 s \tag{3.8}$$

in $|z|\leq 1$. Then we get (1.12).

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[8] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939), p. 163.

[9] K. Yamamoto, Nuovo Cimento **27**, 1277 (1963).

[10] Reference 8, p. 164.

[11] The inequality (3.8) may be obtained by the three-circles theorem applied to G bounded by s^N in its analyticity domain, and by $C/(1-|z|)^2$ for $|z|<1$. The theorem is applied by putting the center of the circles at $|z|=(1-r_2)$, $r_2=\text{const}s^{-1/2}>r_1$, $r_3=\text{const}s^{-1/2}>r_2$, $r_2/r_1=1+\text{const}/\ln s$, in order to remain in the analyticity domain of G which has cuts starting from $1+\text{const}s^{-1/2}$.

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APPENDIX: ANALYTICITY AND UPPER BOUND OF $G(s,z)$

In this appendix we investigate the analytic properties and the upper bound on $G(s,z)$ defined by Eq. (2.10). For this purpose we consider

$$G^i(s,z)=\int_{-1}^{1} K^i(s,z)F(s,z')dz'\,, \tag{A1}$$

where

$$\begin{aligned}K^0(z,z')&=(1-z')^{\lambda-\mu}(1+z')^{\lambda+\mu}\sum_J z^{J-\lambda}P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(z')\\&=(-)^{\lambda+\mu}z^{-2\lambda}R^{-1}(R+z-1)^{\lambda-\mu}\\&\qquad\times(R-z-1)^{\lambda+\mu}\end{aligned} \tag{A2}$$

and

$$K^1(z,z')=(1-z')^{\lambda-\mu}(1+z')^{\lambda+\mu}\times\sum_J\left(\frac{J+a}{J+b}\right)z^{J-\lambda}P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(z') \tag{A3}$$

with $R=(1-2zz'+z^2)^{1/2}$. In deriving Eq. (A2) we have made use of the generating function of the Jacobi polynomials. Using the closed expression for $K^0(z,z')$, we see that $G^0(s,z)$ is analytic and bounded by some fixed power of s in a domain in which $|z|<1+\epsilon$, except for cuts from $-\infty$ to $-x-(x^2-1)^{1/2}$, and from $x+(x^2-1)^{1/2}$ to ∞, where ϵ is a finite positive constant determined by the analyticity domain of $F(s,z)$. The relation between $G^0(s,z)$ and $G^1(s,z)$ is

$$G^1(s,z)=\left(z\frac{d}{dz}+a-\lambda\right)z^{-\lambda-b}\int_0^z dz'z'^{\lambda+b-1}G^0(s,z')\,. \tag{A4}$$

Therefore, $G^1(s,z)$ has the same analytic properties and bounds as $G^0(s,z)$, provided that

$$\lambda+b-1\geqslant 0 \tag{A5}$$

The condition (A5) always holds because of (2.3). Repeating the operation in Eq. (A4), we arrive at the conclusion that $G(s,z)$ has the same analytic properties and bounds as $G^0(s,z)$.

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Angular Correlations in the Three-Pion System and the Diffractive Dissociation of a Pion

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We study the process of diffractive dissociation of a pion, $\pi+N\rightarrow 3\pi+N'$, without assuming that the target interacts coherently. We have found a general form of some angular correlations of the three pions when they are in a state of total angular momentum less than 3. It is shown that these angular correlations either depend solely on the dynamics of the crossed channel or, if they depend also on the dynamics of the 3π system, they do so in a straightforward way.

1. INTRODUCTION

EXPERIMENTS are now being carried out by the heavy-liquid bubble chamber group of the Ecole Polytechnique as well as by other groups on the so-called diffractive dissociation of pions interacting with nuclei:

$$\pi+N\rightarrow X+N' \quad \searrow 3\pi\,, \tag{1}$$

where the momentum transferred to the nucleus N is

$$\Delta q_{11}\lesssim 1/R\,, \tag{2}$$

R standing for the nuclear radius, so that the pion interacts with the nucleus as a whole. In this paper we would like to discuss reaction (1), when X is an unstable state of spin less than 3 which disintegrates rapidly into three pions. Our discussion will be kept quite general insofar as we shall not make any specific dynamical assumption about the target.

We shall not enter into a detailed discussion of the properties of the process (1). Let us merely comment that although it appears that we understand the very general characteristics of diffractive dissociation, a more