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APPENDIX: ANALYTICITY AND UPPER BOUND OF $G(s,z)$

In this appendix we investigate the analytic properties and the upper bound on $G(s,z)$ defined by Eq. (2.10). For this purpose we consider

$$G^i(s,z)=\int_{-1}^{1} K^i(s,z)F(s,z')dz', \tag{A1}$$

where

$$K^0(z,z')=(1-z')^{\lambda-\mu}(1+z')^{\lambda+\mu}\sum_J z^{J-\lambda}P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(z')$$
$$=(-)^{\lambda+\mu}z^{-2\lambda}R^{-1}(R+z-1)^{\lambda-\mu}\times(R-z-1)^{\lambda+\mu} \tag{A2}$$

and

$$K^1(z,z')=(1-z')^{\lambda-\mu}(1+z')^{\lambda+\mu}\times\sum_J\left(\frac{J+a}{J+b}\right)z^{J-\lambda}P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(z') \tag{A3}$$

with $R=(1-2zz'+z^2)^{1/2}$. In deriving Eq. (A2) we have made use of the generating function of the Jacobi polynomials. Using the closed expression for $K^0(z,z')$, we see that $G^0(s,z)$ is analytic and bounded by some fixed power of s in a domain in which $|z|<1+\epsilon$, except for cuts from $-\infty$ to $-x-(x^2-1)^{1/2}$, and from $x+(x^2-1)^{1/2}$ to ∞, where ϵ is a finite positive constant determined by the analyticity domain of $F(s,z)$. The relation between $G^0(s,z)$ and $G^1(s,z)$ is

$$G^1(s,z)=\left(z\frac{d}{dz}+a-\lambda\right)z^{-\lambda-b}\int_0^z dz'z'^{\lambda+b-1}G^0(s,z'). \tag{A4}$$

Therefore, $G^1(s,z)$ has the same analytic properties and bounds as $G^0(s,z)$, provided that

$$\lambda+b-1\geqslant 0 \tag{A5}$$

The condition (A5) always holds because of (2.3). Repeating the operation in Eq. (A4), we arrive at the conclusion that $G(s,z)$ has the same analytic properties and bounds as $G^0(s,z)$.

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Angular Correlations in the Three-Pion System and the Diffractive Dissociation of a Pion

P. DENNERY
Laboratoire de Physique Théorique et Hautes Energies, Orsay, France

AND

A. KRZYWICKI
Laboratoire de Physique, Ecole Polytechnique, Paris, France
and
Laboratoire de Physique Théorique et Hautes Energies, Orsay, France

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We study the process of diffractive dissociation of a pion, $\pi+N\rightarrow 3\pi+N'$, without assuming that the target interacts coherently. We have found a general form of some angular correlations of the three pions when they are in a state of total angular momentum less than 3. It is shown that these angular correlations either depend solely on the dynamics of the crossed channel or, if they depend also on the dynamics of the 3π system, they do so in a straightforward way.

1. INTRODUCTION

EXPERIMENTS are now being carried out by the heavy-liquid bubble chamber group of the Ecole Polytechnique as well as by other groups on the so-called diffractive dissociation of pions interacting with nuclei:

$$\pi+N\rightarrow X+N' \qquad \searrow 3\pi, \tag{1}$$

where the momentum transferred to the nucleus N is

$$\Delta q_{11}\lesssim 1/R, \tag{2}$$

R standing for the nuclear radius, so that the pion interacts with the nucleus as a whole. In this paper we would like to discuss reaction (1), when X is an unstable state of spin less than 3 which disintegrates rapidly into three pions. Our discussion will be kept quite general insofar as we shall not make any specific dynamical assumption about the target.

We shall not enter into a detailed discussion of the properties of the process (1). Let us merely comment that although it appears that we understand the very general characteristics of diffractive dissociation, a more

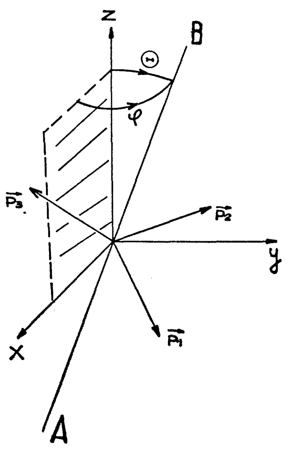

FIG. 1. The angles are defined in the 3π rest frame. Z is along the incident direction; the XZ plane is the production plane and the vectors p_i determine the decay plane. AB is perpendicular to the decay plane.

detailed description of the mechanism of interaction is still lacking. It is desirable to examine the properties of the many-pion system produced. It should also be interesting to see what one can say experimentally about the families of states exchanged in (1). It seems indeed that a description of (1) by means of Regge-pole exchange is very plausible. In fact it turns out that the two questions raised above are closely connected in view of an interesting fact noted by Gottfried and Jackson.[1] They have shown on very general grounds that the helicity of X in the c.m. system of the crossed process $N+\bar{N}'\rightarrow\bar{\pi}+X$ is equal to the component of the spin of X along the direction of motion of the incident particle in the X rest frame, in the direct channel.

This makes it possible to express the density-matrix elements describing the polarization of X in terms of the analytically continued helicity amplitudes for the crossed process.

By studying the angular correlations between the pions in (1), one can get information not only about the spin of X, but also about the nature of the exchanged objects, and more generally about the interaction mechanism. These correlations were studied by Berman and Drell[2] and by Zemach[3] who, however, made use of more or less restrictive conditions. It was assumed by these authors, for instance, that the target acts only as a spectator, and thus possible effects of a nuclear spin or of nuclear excitations were ignored.

We may note, however, that the condition (2) used by experimentalists to select events guarantees that the nucleus acts as a whole, although not necessarily elastically. In fact, experimental results of Fretter and Huson[4] indicate there may be an appreciable amount of nuclear excitation. From an examination of the experimental correlations one cannot deduce whether deviations from theoretical predictions are due to the mixing of several spin states of X or to the influence of the target. It seems to us worthwhile to write down the most general form of certain angular correlations involving the elements of the density matrix describing the polarization of X, without making any *a priori* assumptions. The empirical determination of the structure of this density matrix will probably throw some new light on the details of the interaction. The comparison of the structure of the density matrix for different targets (including the free nucleon) and for different classes of events may be particularly instructive.

The angular distribution of the normal to the plane determined by the momenta of the three produced pions in the X rest frame is, for obvious reasons, the least dependent on the details of dynamics of the three pion system; for that reason we shall focus our attention on it. In Sec. 2 we give these angular distributions. Section 3 contains a general discussion of the results.

2. ANGULAR CORRELATIONS

The notation used is explained in Fig. 1. We work in the X rest frame. Axis z is chosen along the momentum of the incident pion. The momentum of the incident nucleon lies in the xz plane—we shall call this plane the production plane. $\mathbf{p}_i$ are the momenta of the produced pions and AB is a line perpendicular to the plane defined by the vectors $\mathbf{p}_i$—hereafter this plane will be called the decay plane. Θ and φ are the polar and the azimuthal angles defining the direction AB in the xyz frame; in order to avoid ambiguity we assume[5]

$$0\leqslant\Theta\leqslant\pi\,,\qquad 0\leqslant\varphi\leqslant\pi\,. \tag{3}$$

Let $\rho_{mm'}$ denote the element of the density matrix describing the polarization of X. Since parity conservation requires invariance for reflections with respect to the xz plane,

$$\rho_{mm'}=(-1)^{m-m'}\rho_{-m-m'}\,. \tag{4}$$

The differential probability for finding the produced pions in a particular energy-momentum state (in the X c.m. system) is

$$dP\sim\sum_{-J\leqslant m,'\leqslant J}{}_{mm'}M_m(p_i)M_{m'}{}^*(p_i)\delta_3(\sum_j\mathbf{p}_j)\times\delta(W-\sum_j\omega_j)\textstyle\prod_j d^3p_j/\omega_j\,, \tag{5}$$

where M_m is the amplitude for the $X\rightarrow 3\pi$ transition when the z component of the spin of X equals m, J and W are the spin and the mass of X, respectively, and ω_j are the pion energies.

The rules for writing the amplitudes M in tensor form have been given by Zemach.[3] The M_m's can be obtained from these M's by using the well-known linear relations between the components of a spherical tensor and the

[1] K. Gottfried and J. D. Jackson, CERN, 1964 (to be published).
[2] S. M. Berman and S. D. Drell, Phys. Rev. Letters **11**, 220 (1963).
[3] C. Zemach, Phys. Rev. **133**, B1200 (1964).
[4] W. Fretter and R. Huson, Berkeley, 1964 (to be published).

[5] We shall calculate the probability that either $\mathbf{p}_1\times\mathbf{p}_2$ or $\mathbf{p}_2\times\mathbf{p}_1$ has the coordinate $\varphi\leqslant\pi$.

components of a symmetric, traceless, Cartesian tensor of rank J.[6] After performing the necessary integrations one obtains from Eq. (5) a probability distribution $P(\Theta,\varphi)d\cos\Theta d\varphi$. Let us consider separately the different spin-parity assignments for X.

0^- case

This is indeed a trivial case:

$$P(\Theta,\varphi)=\text{const.} \tag{6}$$

1^- case

M has the form

$$M\sim\mathbf{q}\cdot F,\quad \mathbf{q}=\mathbf{p}_1\times\mathbf{p}_2. \tag{7}$$

One gets

$$P(\Theta,\varphi)\sim\rho_{00}\cos^2\Theta+\rho_{11}\sin^2\Theta-\rho_{1-1}\sin^2\Theta \times\cos2\varphi-\sqrt{2}\,\mathrm{Re}\rho_{10}\sin2\Theta\cos\varphi, \tag{8}$$

where

$$\rho_{00}=1-2\rho_{11}.$$

All the parameters are real, owing to the Hermiticity of the density matrix.

1^+ case

M has the form

$$M\sim F_1\mathbf{p}_1+F_2\mathbf{p}_2+F_3\mathbf{p}_3. \tag{9}$$

It turns out that F_i enters only as a multiplicative factor.

$$P(\Theta,\varphi)\sim\rho_{00}\sin^2\Theta+\rho_{11}(1+\cos^2\Theta)+\rho_{1-1}\sin^2\Theta \times\cos2\varphi+\sqrt{2}\,\mathrm{Re}\rho_{10}\sin2\Theta\cos\varphi, \tag{10}$$

$$\rho_{00}=1-2\rho_{11}.$$

2^+ case

M has the form (we use Zemach's notation)

$$M\sim F_1T(1q)+F_2T(2q)+F_3T(3q), \tag{11}$$

with

$$T_{ij}(nq)=\tfrac{1}{2}(p_{ni}q_j+p_{nj}q_i)$$

and

$$\sum_n T(nq)=0.$$

Here again the dependence on F_i factorizes:

$$\begin{aligned}P(\Theta,\varphi)\sim{}&\tfrac{1}{4}\rho_{00}\sin^22\Theta+(\sqrt{\tfrac{1}{6}})\,\mathrm{Re}\rho_{10}\sin4\Theta\cos\varphi-(\sqrt{\tfrac{1}{6}})\,\mathrm{Re}\rho_{20}\sin^22\Theta\cos2\varphi\\&+\tfrac{1}{3}\rho_{11}(\cos^2\Theta+\cos^22\Theta)+\tfrac{1}{3}\rho_{1-1}(\cos^2\Theta-\cos^22\Theta)\cos2\varphi-\tfrac{2}{3}\,\mathrm{Re}\rho_{12}\cos^2\Theta\sin2\Theta\cos\varphi\\&-\tfrac{2}{3}\,\mathrm{Re}\rho_{1-2}\sin^2\Theta\sin2\Theta\cos\varphi(4\sin^2\varphi-1)+\tfrac{1}{3}\rho_{22}\sin^2\Theta(1+\cos^2\Theta)-\tfrac{1}{3}\rho_{2-2}\sin^4\Theta\cos4\varphi,\end{aligned} \tag{12}$$

$$\rho_{00}=1-2\rho_{11}-2\rho_{22}.$$

2^- case

In this case the dependence of $P(\Theta,\varphi)$ on form factors is no longer trivial. M can be written as

$$M\sim F_1T(11)+F_2T(22)+F_3T(33), \tag{13}$$

where

$$T_{ij}(nn)=p_{ni}p_{nj}-\tfrac{1}{3}\delta_{ij}p_n^2.$$

The angular distribution is

$$\begin{aligned}P(\Theta,\varphi)\sim{}&\rho_{00}[(\tfrac{3}{2}\sin^4\Theta-2\sin^2\Theta+\tfrac{2}{3})+\tfrac{2}{3}\alpha(3\sin^2\Theta-1)]+(\sqrt{\tfrac{2}{3}})\,\mathrm{Re}\rho_{10}\sin2\Theta\cos\varphi\\&\times[1-3\cos^2\Theta+2\alpha]+(\sqrt{\tfrac{2}{3}})\,\mathrm{Re}\rho_{20}\sin^2\Theta\cos2\varphi[3\cos^2\Theta-1+4\alpha]+2\rho_{11}\sin^2\Theta[\cos^2\Theta+\alpha]\\&-2\rho_{1-1}\sin^2\Theta\cos2\varphi[\cos^2\Theta-\alpha]-\mathrm{Re}\rho_{12}\sin2\Theta\cos\varphi[\sin^2\Theta-4\alpha]-\mathrm{Re}\rho_{1-2}\sin^2\Theta\sin2\Theta\cos\varphi\\&\times[2\cos2\varphi-1]+\tfrac{1}{2}\rho_{22}[\sin^4\Theta+8\alpha\cos^2\Theta]+\tfrac{1}{2}\rho_{2-2}\sin^4\Theta\cos4\varphi,\end{aligned} \tag{14}$$

$$\rho_{00}=1-2\rho_{11}-2\rho_{22},$$

$$\alpha=(\beta+\gamma)/(3\beta+\gamma), \tag{15}$$

where

$$\beta=\int d\omega_1d\omega_2|\sum_n F_np_n^2|^2,$$

$$\gamma=4\int d\omega_1d\omega_2\,\mathrm{Re}\{\sum_{m<n}F_mF_n^*(p_mp_n\sin\chi_{mn})^2\}, \tag{16}$$

where $\chi_{mn}<\pi$ and

$$\chi_{mn}=|\cos^{-1}[\mathbf{p}_m\cdot\mathbf{p}_n/p_mp_n]|. \tag{17}$$

By straightforward algebra one can show that

$$0\leqslant\alpha\leqslant1. \tag{18}$$

In order to get a more precise prediction for the value of α, one has to make rather specific assumptions about the dynamics of the three-pion system.

When X has a spin of 3 or more, the angular distributions become vastly more complicated, owing to the appearance of a much large number of elements of the density matrix and of form factors. We feel that the improbability of the occurrence of such large-spin states of X, coupled with the fact that the statistics of reaction (1) are still too poor to allow an unambiguous de-

[6] L. D. Landau and E. M. Lifschitz, *Quantum Mechanics* (Pergamon Press, Inc., New York, 1962), p. 202.

termination of all the parameters involved, justify our limiting ourselves to spins of X less than 3.

3. DISCUSSION OF RESULTS

We may note that all of the above angular distributions have one characteristic in common: they contain terms that are antisymmetric with respect to the transformations $\Theta\to\pi-\Theta$ and $\varphi\to\pi-\varphi$. This means that if, for a given Θ (or for a given φ), one sums over events at angles φ and $\pi-\varphi$ (or Θ and $\pi-\Theta$), one can eliminate from the angular distributions a certain number of parameters of the density matrix. Thus for the spin-1 case there will remain 2 parameters after symmetrization, instead of 3, and for the spin-2 case, there will remain 5 parameters instead of 8. In the latter case this means a considerable simplification.

Let us now note some immediate implications of the Gottfried-Jackson result. Assume that a state with spin s is exchanged in (1). If the parity of this state is $(-1)^s$ we have, as has been observed by these authors

$$\begin{aligned}&\rho_{0i}=0(i=\pm1,0)\text{ in the case }1^-,\\&\rho_{0i}=0(i=\pm2,\pm1,0)\text{ in the case }2^+,\end{aligned}\tag{19}$$

because if states $|\bar{\pi}X\rangle$ have total angular momentum s and if the helicity of X, $\lambda_X=0$, then their parity is $(-1)^{s+1}$. In the cases 1^+ and 2^- one gets the (experimentally) uninteresting result:

$$\begin{aligned}&\rho_{0i}=0(i=\pm1,0)\text{ in the case }1^+,\\&\rho_{0i}=0(i=\pm2,\pm1,0)\text{ in the case }2^-,\end{aligned}\tag{20}$$

when only the exchange of states with parity $(-1)^{s+1}$ is allowed.

Consider now the relation between our results and those of Zemach, and of Berman and Drell. When the target acts merely as a spectator, the exchanged states have parity $(-1)^s$, and hence, taking into account (19), one gets for the cases 1^- and 2^+ (after integrating with respect to φ) the distributions given in Ref. 3. One also finds that the Θ distribution for the case 1^+ given in that reference is in fact quite general.

In the case of forward production, the collision has complete cylindrical symmetry with respect to rotations about the z axis. Consequently the ρ matrix is diagonal. As is well known, the scattering amplitude for the process $\pi+N\to X+N'$ can be written in terms of helicity amplitudes as[7]

$$T_{fi}\sim\textstyle\sum_J(2J+1)\langle\lambda_{N'},\lambda_X|T^J|\lambda_N\rangle D_{\lambda_N,\lambda_{N'}-\lambda_X}{}^J(\vartheta,\phi).\tag{21}$$

Let $\Delta\lambda$ be the helicity change of the target.

For $\vartheta\to0$,

$$D_{\lambda_N,\lambda_{N'}-\lambda_X}{}^J(\vartheta,\phi)\sim\vartheta^{|\lambda_X-\Delta\lambda|}.\tag{22}$$

Thus in the case of forward production in which the target acts only as a spectator ($\Delta\lambda=0$), the allowed transition with the smallest $|\lambda_X|$ will dominate. Unless $\langle\lambda_{N'},\lambda_X|T^J|\lambda_N\rangle$ vanishes, one can set equal to zero in $P(\Theta,\varphi)$ all elements of ρ with the exception of

$$\begin{aligned}&\rho_{11}=(-1)^{S_X+1}\cdot\rho_{1-1}\text{ in the cases }1^-,\,2^+,^8\\&\rho_{00}\text{ in the cases }1^+,\,2^-.\end{aligned}\tag{23}$$

The situation may be different, however, either if there is a nuclear excitation, or if the target undergoes a spin flip. As an illustration, consider the case of C^{12}. The ground state of C^{12} is 0^+. The low-lying excited states of C^{12} have spin-parity assignments 2^+, 0^+, 1^-. All states $|\bar{C}'C\rangle$ of spin s, where C' is one of the above excited states of carbon, such that $\Delta\lambda=0$, have parity $(-1)^s$. According to (22), if the target undergoes a $\Delta\lambda=0$ transition, then $\lambda_X=0$ is the most probable helicity of X in near-forward production. But by (19), $\lambda_X=0$ cannot be excited if X is in either of the states 1^- or 2^+. Therefore, for these spin-parity values of X, only ρ_{11} is nonzero when C' is 1^- or 0^+, while ρ_{11} and ρ_{22} are nonzero when C' is 2^+. The appearance of ρ_{22} when C' is 2^+ is a feature characteristic of incoherent production. Analogously, when the spin of X is 1^+ or 2^-, we have the following predominant elements of ρ:

$$\rho_{00},\,\rho_{11},\,(\rho_{22})\text{ when }C'\text{ is }2^+,\tag{24a}$$

$$\rho_{00}\quad\text{when }C'\text{ is }0^+,\tag{24b}$$

$$\rho_{00},\,\rho_{11}\quad\text{when }C'\text{ is }1^-.\tag{24c}$$

Of course, no distinction can be made at forward angles between coherent production and incoherent production in which the nucleus is left in the state 0^+. A difference does occur, however, if the nucleus is left in the excited states 2^+ or 1^-, that difference being manifested by the presence of terms proportional to ρ_{11} and ρ_{22}.

Consider now fluorine as a target. The ground state is $\frac{1}{2}^+$. In the event that F undergoes a spin flip, we can have $\Delta\lambda=0,\pm1$. But because F is a fermion, no state $|F\bar{F}'\rangle$ in which F and $\bar{F}'$ have given helicities, is an eigenstate of parity.[7] Therefore there is no selection rule as in the C^{12} case; all that can be said here is that for near-forward production the excitation of $|\lambda_X|=2$ is improbable.[9] The three-pion state with $|\lambda_X|=2$ can be produced, however, if one has an excitation of fluorine. In fact F^{19} has many low-lying excited states whose excitation may lead to $|\lambda_X|=2$, even for near-forward production.

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[7] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[8] S_X is the spin of X. When the nucleus is not excited or does not spin-flip, the production in the exactly forward direction is forbidden for these spin-parity assignments of X. Thus the symmetry argument fails and one gets the indicated nondiagonal terms in ρ, which, for $\vartheta\to0$, may easily be calculated from (2I).

[9] The same is true for a free nucleon.