section is smaller than predicted by the OPEM by a factor of the order of 2. The same result has also been obtained by the CERN group[8,10] at 3.0 and 4.0 GeV/c and by Baltay *et al.*[9] at 3.25 GeV/c.

(5) Double pion production in the reaction $\bar{p}p\rightarrow\bar{p}p\pi^+\pi^-$ agrees perfectly with the OPEM for all values of the four-momentum transfer, if all possible one-pion-exchange graphs are taken into account. In particular, this holds for $d\sigma/d\Delta^2$, for the distribution of the (anti) nucleon-pion effective mass, for the Treiman-Yang angular distribution and for the decay angular distribution of the pion-nucleon isobar. The main contribution is given by the "double isobar diagram" [Fig. 7(c)]. Similar conclusions were reached by Baltay *et al.*[9] at 3.25 GeV/c. The total cross section found is 3.80±0.22 mb, the OPEM predicts 3.55 mb.

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Asymptotic Self-Consistency of Propagators and Form Factors

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From a consideration of the elastic unitarity equations for the propagators Δ and vertex function Γ, it is argued that the following high-energy behaviors are self-consistent:

$$\Gamma^{-1}(s)\approx s^{\frac{1}{2}(a+b+c-1)}\ln s,\ \Delta_A(s)\approx s^{a-1}\ln s,\ \cdots$$

for the interactions among three particles A, B, and C (spins a, b, and c, respectively).

1. INTRODUCTION

A GREAT deal of attention during recent years has been devoted to the asymptotic region of the four-point function, but by comparison the two- and three-point functions have suffered neglect in this respect. Physically, the explanation is obvious: The scattering amplitude is directly accessible to experiment, whereas, the two- and three-point functions are not. Nevertheless, even in practical S-matrix theory calculations, the propagators Δ and vertex function Γ often make an appearance when the "pole approximation" is invoked, and since a knowledge of the asymptotic characteristics of Δ and Γ is more basic in many respects than that of the higher n-point functions because Δ and Γ are the "building blocks" of field theory, we wish in this paper, by adopting the simplest possible set of assumptions, to make some more definite predictions regarding their high-energy behavior using as little information as possible about many-particle S-matrix elements. Essentially all that has previously been stated on this subject is that the complete propagator is more singular than the Feynman propagator[1] and that Γ must fall off to zero at high-momentum transfer.[2]

Our method for deriving such asymptotic properties hinges on the unitarity equations for Δ and Γ where we restrict our attention to two-particle intermediate states (elastic unitarity). By approximating to the elastic scattering at large energy and momentum transfer by the one-particle exchange $\Gamma\Delta\Gamma$, this leads to a set of relations among Δ, Γ, and their imaginary parts, themselves connected to Δ and Γ by dispersion relations. Our "simplest set of assumptions" consists in assuming power-law behaviors of the type s^n for $\mathrm{Im}\Delta(s)$ and $\mathrm{Im}\Gamma^{-1}(s)$ *at large s*, and demanding that in this region the propagator depend only on the spin of the particle. Then by requiring that these asymptotic forms reproduce one another when inserted into the unitarity equations we deduce the following self-consistent behaviors in the $(a,0)$ Weinberg field representations.[3] $\Delta_A(p)\approx\pi^a(p)s^{-1}\ \ln s,\quad \tilde{\Gamma}_j^{-1}(s)\approx s^{\frac{1}{2}(a-1)}\ \ln s;\quad s=p^2.$ $\pi^a(p)$ is a monomial of degree $2a$ in p and $\tilde{\Gamma}_j(s)$ is the jth multipole form factor with B and C placed on the mass shell. Passing to the more familiar tensor representation the above implies for the proper vertex function,

$$\Gamma(\tfrac{1}{2}p+q,\tfrac{1}{2}p-q)\approx p^{-(a+b+c-1)}(\ln p^2)^{-1}.$$

On the above basis, the vertex and wave function renormalization constants must vanish (for renormaliza-

* Supported in part by the National Science Foundation.

[1] H. Lehmann, Nuovo Cimento **11**, 342 (1954); O. Steinman, J. Math. Phys. **4**, 583 (1963).

[2] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **2**, 425 (1955).

[3] S. Weinberg, Phys. Rev. **133**, B1318 (1964); **134**, B882 (1964). Our metric is different from his and our states are normalized differently.

ble theories where such constants are definable), a conjecture which has recently gained in favor since it is probably the field-theoretic statement that every particle in nature is equally elementary.[4] Neglecting logarithmic factors the above characteristics can be generalized to yield

$$\Delta_A{}^{1/2}\Delta_B{}^{1/2}\Delta_C{}^{1/2}\Gamma\approx p^{-2},$$

the stability criterion which guarantees the finiteness of a field theory based on the Dyson equations.[5] As another by-product, Ward's identity,

$$\partial\Delta_B{}^{-1}/\partial p=\Gamma(p)\,,$$

is automatically satisfied for the case of electrodynamics ($B=C$, $A=$photon), and as another theoretical consequence, we deduce (with reservations as explained in Sec. 4) that our results are in accord with Regge behavior $s^{\alpha(t)}$ of the scattering amplitude providing $\alpha(0)=1$ and $\alpha'(0)=O(g^2)$, g being the magnitude of the ABC coupling. Also by examination of the Bethe-Salpeter equation we find that the Fredholm kernel cannot lead to highly singular potentials.

Finally, we make a comparison of the predicted asymptotic behavior for the Sachs form factors[6] of the nucleon, $G_{E,M}\approx\alpha_{E,M}s^{-1}+\beta_{E,M}(\ln s)^{-1}$; α, β constant, with the recent experiments of Chen *et al.*[6] The term αs^{-1} represents the possible effect of resonance poles while the term $\beta(\ln s)^{-1}$ is the behavior predicted on the basis of our simple theory, and is expected to dominate at sufficiently large s. This is not in conflict with the experiments (though small β/α is suggested), nor is the further prediction that

$$\lim_{s\to\infty}\frac{G_E(s)}{G_M(s)}=\text{constant}\,,$$

although the individual form factors vanish in this limit.

2. THE $(j,0)$ REPRESENTATION OF THE FIELDS

We are visualizing a theory for which there are only three particles A, B and C of spins a, b, c that interact via some Lagrangian $L=gABC$. Vertices such as BAB are excluded *ab initio*, and for extra simplicity we assume the fields real since this makes no essential difference in what follows. The results to be given later depend crucially on the spins and in order to give a precise formulation to this aspect of the problem we specialize to the simplest possible representation for our fields, the $(j,0)$ representation of the Lorentz group. This has been extensively studied by Weinberg[3] and, being irreducible, it does away with unwanted degrees of freedom which would have necessitated appending various subsidiary conditions. Only after solving the problem of asymptotic self-consistency do we translate our results into the more familiar (tensor, γ matrix,...) representation of the Lorentz group. A resumé of the essential properties of the $(j,0)$ fields is therefore in order.

The basic fields $\phi_\sigma(x)$ satisfy local commutativity $[\phi_\rho(x),\phi_\sigma(y)]_\pm=0$ for $(x-y)^2<0$ and under a homogeneous Lorentz transformation Λ,

$$U[\Lambda]\phi_\sigma(x)U^{-1}[\Lambda]=\sum_\rho D_{\sigma\rho}[\Lambda^{-1}]\phi_\rho(\Lambda x)\,. \tag{1}$$

For a particle of spin j, the usual vector generators $\mathbf{J}$ and $\mathbf{K}$ of the Lorentz group are given in the $(j,0)$ representation by $\mathbf{J}\to\mathbf{J}^{(j)}$ and $\mathbf{K}\to-i\mathbf{J}^{(j)}$. We generate a general helicity state $|\mathbf{p}j\lambda\rangle$ from a standard state $|j\lambda\rangle$ (the rest state for massive particles or a state with specified momentum along the z axis for massless particles) by application of a pure Lorentz transformation along the z axis, described by the hyperbolic angle χ, followed by a pure rotation R:

$$|\mathbf{p}j\lambda\rangle=L(\mathbf{p})|j\lambda\rangle=R(\varphi,\theta,0)e^{-i\chi K_3}|j\lambda\rangle\,, \tag{2}$$

$$D_{\rho\sigma}{}^j[L(\mathbf{p})]=D_{\rho\sigma}{}^j[R]e^{-\sigma\chi}. \tag{3}$$

It is easily shown that for arbitrary Λ,

$$U[\Lambda]|\mathbf{p}j\lambda\rangle=\sum_\mu|\Lambda\mathbf{p}j\mu\rangle D_{\mu\lambda}{}^j[r]\,, \tag{4}$$

where $r=L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})$ is the "little group rotation." Hereafter we write our formulas for massive particles when $D[r]$ is a simple rotation matrix. The massless particle case can be obtained on multiplication of the fields by the factor m^j and taking the limit $m\to 0$.

(i) To the general state $|\mathbf{p}j\lambda\rangle$ is associated the wave function $u_\sigma(\mathbf{p}j\lambda)$ which transforms according to

$$\sum_\rho D_{\sigma\rho}[\Lambda]u_\rho(\mathbf{p}j\lambda)=\sum_\mu u_\sigma(\Lambda\mathbf{p}j\mu)D_{\mu\lambda}[r]\,. \tag{5}$$

In the $(j,0)$ representation the wave function is given by

$$u_\sigma(\mathbf{p}j\lambda)=D_{\sigma\lambda}{}^j[L(\mathbf{p})]=D_{\sigma\lambda}{}^j(\varphi,\theta,0)e^{-\lambda\chi} \tag{6}$$

with $\tanh\chi=|\mathbf{p}|/p_0$ and $p^2=m^2$, since the particle lies on the mass shell. Suppressing the spin label j, we will write $|p\lambda\rangle$, $|-p\lambda\rangle$ when p is directed, respectively, parallel and antiparallel to the z axis. The $(j,0)$ representation provides the explicit forms

$$u_\sigma(p\lambda)=\delta_{\sigma\lambda}e^{-\lambda\chi},\quad u_\sigma(-p,\lambda)=\delta_{\sigma-\lambda}(-1)^{j-\lambda}e^{-\lambda\chi}. \tag{7}$$

In the language of Feynman diagrams we associate with the line representing an incoming particle of momentum p and helicity λ the wave function $u_\sigma(\mathbf{p}j\lambda)$ while to a similar outgoing particle line is attached $u_\sigma{}^*(\mathbf{p}j\lambda)$. The σ indices are, of course, characteristic of the Lorentz group representation and are to be contracted out on

[4] A. Salam, Nuovo Cimento **25**, 224 (1962); S. Weinberg, Phys. Rev. **130**, 776 (1963). More recent references are to be found in the preprint of B. W. Lee, K. T. Mahanthappa, I. S. Gerstein, and M. L. Whippman.

[5] A full discussion of this question is to be found in A. Salam and R. Delbourgo, Phys. Rev., **135**, B1398 (1964).

[6] R. G. Sachs, Phys. Rev. **126**, 2256 (1962); K. W. Chen, A. A. Cone, J. R. Dunning, Jr., S. G. F. Frank, N. F. Ramsey, J. K. Walker, and Richard Wilson, Phys. Rev. Letters **11**, 561 (1963).

indices of the "blob" representing all the possible interaction diagrams.

(ii) According to the usual definition, the complete propagator is given by

$$i\Delta_{\sigma\rho}(p)=\int\langle 0|T[\phi_\sigma(\tfrac{1}{2}x),\phi_\rho^+(-\tfrac{1}{2}x)]|0\rangle e^{ip\cdot x}d^4x \quad (8)$$

and it is easily shown from Eq. (1) that it transforms as

$$\Delta_{\sigma\rho}(p)=\sum_{\mu\lambda}D_{\sigma\mu}[\Lambda]\Delta_{\mu\lambda}(p)D_{\lambda\rho}^+[\Lambda]. \quad (9)$$

A rotation about the z axis proves that in the frame where $p_1=p_2=0$, $\Delta_{\sigma\rho}(p_0,p_3)\propto\delta_{\sigma\rho}$. Indeed, if $p^2>0$, we may proceed to the rest frame where we define

$$\Delta_{\sigma\rho}(p_0,\mathbf{0})=\delta_{\sigma\rho}(m^{-2}p^2)^j\tilde{\Delta}(p^2). \quad (10)$$

$\tilde{\Delta}(p^2)$ is a scalar function and from Eq. (9) we deduce that in general

$$\Delta_{\sigma\rho}(p)=\pi_{\sigma\rho}(p)\tilde{\Delta}(p^2), \quad (11)$$

where

$$\pi_{\sigma\rho}(p)=\sum_\lambda D_{\sigma\lambda}[L(\mathbf{p})]D_{\lambda\rho}^+[L(\mathbf{p})](m^{-2}p^2)^j \quad (12)$$

is a well-defined function whose properties have been listed by Weinberg.[3] In particular, when $p_1=p_2=0$,

$$\pi_{\sigma\rho}(p_0,p_3)=(p_0+p_3)^{-2\sigma}p^{2\sigma}(m^{-2}p^2)^j\delta_{\sigma\rho}. \quad (13)$$

The positive definiteness of the spectral function for $\tilde{\Delta}$ is established in the frame $\mathbf{p}=0$. There,

$$-(m^{-2}p^2)^j\,\mathrm{Im}\tilde{\Delta}(p^2)=\sum_n\int d^4xe^{ip\cdot x}\langle 0|\phi_\sigma(\tfrac{1}{2}x)|n\rangle\times\langle n|\phi_\sigma^+(-\tfrac{1}{2}x)|0\rangle$$

or

$$-\mathrm{Im}\tilde{\Delta}(p^2)=\sum_{p=p_n}(2\pi)^4(m^{-2}p^2)^{-j}|\langle 0|\phi_\sigma(0)|n\rangle|^2.$$

Hence $\tilde{\Delta}(p^2)$ will possess the standard Lehmann-Källen[7] representation and

$$\Delta_{\sigma\rho}(p)=\pi_{\sigma\rho}(p)\left[\frac{1}{p^2-m^2}+\frac{1}{\pi}\int_{s_0}\frac{\mathrm{Im}\tilde{\Delta}(x)dx}{x-p^2-i\epsilon}\right]. \quad (14)$$

$\tilde{\Delta}(s)$ may develop a single real zero at s_1 in the range $m^2<s_1<s_0$. If this zero does exist, it must be explicitly inserted as a C.D.D. (Castillejo-Dalitz-Dyson) pole in the representation for $\Delta^{-1}(s)$, viz.:

$$\tilde{\Delta}^{-1}(s)=(s-m^2)\left[1+\frac{c(s-m^2)}{s-s_1}+\frac{(s-m^2)}{\pi}\times\int\frac{\mathrm{Im}\tilde{\Delta}^{-1}(x)dx}{(x-s)(x-m^2)^2}\right]. \quad (15)$$

[7] G. Källén, Helv. Phys. Acta **25**, 416 (1952); H. Lehmann, Ref. 2.

Analogously to the treatment for scalar particles, we may define a wave function renormalization constant by

$$Z^{-1}=\lim_{s\to\infty}s\tilde{\Delta}(s)=1-\frac{1}{\pi}\int\mathrm{Im}\Delta(x)dx$$
$$=\left[1+c-\frac{1}{\pi}\int\frac{\mathrm{Im}\tilde{\Delta}^{-1}(x)dx}{(x-m^2)^2}\right]^{-1}. \quad (16)$$

(iii) Now consider the three-point vertex ABC to which are associated the helicities λ, μ, ν and the Lorentz indices α, β, γ. Typically with B and C placed on the mass shell we are led to consider the quantity

$$\langle 0|\phi_\alpha(0)|\tfrac{1}{2}\mathbf{p}+\mathbf{q},\mu;\tfrac{1}{2}\mathbf{p}-\mathbf{q},\nu\rangle$$
$$\equiv\sum i\Delta_{\alpha\alpha'}(p)\Gamma_{\alpha'\beta\gamma}(p,q)u_\beta(\tfrac{1}{2}\mathbf{p}+\mathbf{q},\mu)u_\gamma(\tfrac{1}{2}\mathbf{p}-\mathbf{q},\nu).$$

$\Gamma_{\alpha\beta\gamma}$ is the proper vertex function and exhibits the transformation property

$$\Gamma_{\alpha\beta\gamma}(\Lambda p,\Lambda q)=\sum_{\alpha'\beta'\gamma'}D_{\alpha\alpha'}{}^{a+}[\Lambda^{-1}]\Gamma_{\alpha'\beta'\gamma'}(p,q)D_{\beta\beta'}{}^b[\Lambda^{-1}]\times D_{\gamma'\gamma}{}^c(\Lambda^{-1}].$$

With $p^2>0$ we can, just like the propagator, deduce the general properties of Γ by examining its features in the Breit or brickwall frame ($\mathbf{p}=0$) with B and C directed along the z axis.

$$\langle 0|\phi_\alpha(0)|q\mu,-q\nu\rangle$$
$$=i(m_A{}^{-2}s)^a\Delta_A(s)\Gamma_{\alpha\mu-\nu}(s)e^{-\mu\chi_B}(-1)^{c-\nu}e^{-\nu\chi_C}, \quad (17)$$

where

$$\sinh\chi_{B,C}=qm_{B,C}{}^{-1}$$

and

$$4q^2s=[s-(m_B+m_C)^2][s-(m_B-m_C)^2]. \quad (18)$$

Noticing from Eq. (1) that $\phi_\alpha{}^+(0)$ behaves as a spherical tensor operator $T_\alpha{}^a$, we may make a multipole decomposition of the form factor in the manner of DeCelles, Durand, and Marr[8]

$$\langle 0|\phi_\lambda(0)|q\mu,-q\nu\rangle=i(m_A{}^{-2}s)^a\tilde{\Delta}(s)\times\sum_j\begin{pmatrix}j & b & c\\ -\lambda & \mu & -\nu\end{pmatrix}\tilde{\Gamma}_j(s).$$

Performing a rotation we obtain the general form in the c. m. frame,

$$\langle 0|\phi_\alpha(0)|q\mu,-q\nu\rangle=i(m_A{}^{-2}s)^2\tilde{\Delta}(s)\times\sum_j D_{\alpha\lambda}{}^a(q)\begin{pmatrix}j & b & c\\ -\lambda & \mu & -\nu\end{pmatrix}\tilde{\Gamma}_j(s). \quad (19)$$

In the limit as $q\to 0$, $\Gamma_j(s)\approx q^{|j-a|}$ and in the static

[8] L. Durand, III, P. DeCelles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).

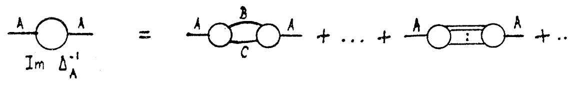

FIG. 1. Contributions to the absorptive part of the A propagator.

case the only nonvanishing coupling is the term

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}.$$

The perturbation diagrams suggest that $\tilde{\Gamma}_j$ satisfies the dispersion relation

$$\tilde{\Gamma}_j(s)=\tilde{\Gamma}_j(\infty)+\frac{1}{\pi}\int_s^\infty \frac{\mathrm{Im}\tilde{\Gamma}_j(x)dx}{x-s}. \tag{20}$$

When $j\neq a$, the threshold condition must be separately imposed to restrict the acceptable form of $\mathrm{Im}\tilde{\Gamma}_j$. Inverting the representation (20), with C.D.D. poles and their appropriate residues having to be explicitly added if real zeros of $\tilde{\Gamma}_j$ are present, it follows that

$$\tilde{\Gamma}_j^{-1}(s)=\tilde{\Gamma}_j^{-1}(m_A^2)\left[1+\frac{\Gamma_j(m_A^2)(s-m_A^2)}{\pi}\times\int\frac{\mathrm{Im}\tilde{\Gamma}_j^{-1}(x)dx}{(x-m_A^2)(x-s)}\right]. \tag{21}$$

The dispersion relation can alternatively be stated in terms of the multipole moment functions $Q_j(s)=q^{-|j-a|}\tilde{\Gamma}_j(s)$ with no particular boundary constraint at $s=(m_B+m_C)^2$. These Q_j are related to the various static moments and correspond to the presence of fundamental derivative couplings in the interaction Lagrangian for $j\neq a$. Of particular interest is the "charge moment" $j=a$ which plays the role of the basic (nonderivative) coupling. Since all the fields have been accorded the dimensions of a mass, $\Gamma_a(m_A^2)=gM$ where g is dimensionless and M is a typical mass, say $\frac{1}{3}(m_A+m_B+m_C)$. For conventionally renormalizable theories the vertex renormalization constant is associated with just this form factor and the bare $Q_j(m_A^2)$ are supposed to vanish for $j\neq a$ in a "minimal interaction" theory. Since field theory gives $\Gamma_a(\infty)=gMZ_1$, Eq. (20) may be recast as

$$\tilde{\Gamma}_a(s)=gM+\frac{(s-m_A^2)}{\pi}\int\frac{\mathrm{Im}\tilde{\Gamma}_a(x)dx}{(x-s)(x-m_A^2)}, \tag{22}$$

with

$$\begin{aligned} Z_1&=1-\frac{1}{\pi gM}\int\frac{\mathrm{Im}\Gamma_a(x)dx}{x-m_A^2}=\lim_{s\to\infty}(gM)^{-1}\Gamma_a(s) \\ &=\left[1-\frac{gM}{\pi}\int\frac{\mathrm{Im}\Gamma_a^{-1}(x)dx}{x-m_A^2}\right]^{-1} \end{aligned} \tag{23}$$

from Eq. (21).

(iv) Finally, we compute the BC scattering diagram, in which particle A is exchanged, for later reference. Working in the over-all c.m. frame with the final particles directed along the z axis, we must discuss

$$M_{\mu\nu\mu'\nu'}(q)=\langle q\mu|\phi_\alpha(0)|-\mathbf{q}\nu'\rangle\Delta_{\alpha\alpha'}{}^{-1}\langle -q\nu|\phi_{\alpha'}{}^{+}(0)|\mathbf{q}\mu'\rangle.$$

q is related to the total energy $s^{1/2}$ as before in Eq. (18). Vertex functions with t (the squared momentum carried by line A) spacelike made their appearance and these can again be related to the brickwall frame functions by suitable Lorentz transformation—a simple rotation will not suffice. Letting Λ_+ be that transformation which carries q into q' and $-\mathbf{q}$ into $-q'$, Λ_- that transformation which $\mathbf{q}$ into q' and $-q$ into $-q'$, we obtain, after some straightforward manipulations from basic Eq. (5), that

$$\begin{aligned} M_{\mu\nu\mu'\nu'}(q)=\sum_{j\alpha\beta\gamma j'\alpha'\beta'\gamma'} & D_{\beta\mu}{}^*(r_{B+})D_{\beta'\mu'}(r_{B-})D_{\gamma\nu}{}^*(r_{C-}) \\ & \times D_{\gamma'\nu'}(r_{C+})\tilde{\Gamma}_j(t)\tilde{\Delta}(t)\tilde{\Gamma}_{j'}(t) \\ & \times\begin{pmatrix} j & b & c \\ -\alpha & \beta & -\gamma' \end{pmatrix}\pi_{\alpha\alpha}D_{\alpha\alpha'}{}^{a+} \\ & \times[\Lambda_-{}^{-1}\Lambda_+]\begin{pmatrix} j' & b & c \\ -\alpha' & \beta' & -\gamma \end{pmatrix}. \end{aligned} \tag{24}$$

$r_{B,C\pm}$ denote the appropriate little group rotations of B and C states corresponding to $\Lambda_\pm$ and we have used the crossing property of the Weinberg fields to write

$$\langle q'\mu|\phi_\alpha(0)|-q'\nu\rangle=i\sum_j\Delta_{\alpha\lambda}(t)\begin{pmatrix} j & b & c \\ -\lambda & \mu & -\nu \end{pmatrix}\tilde{\Gamma}_j(t)$$

and furthermore that $\Delta_{\alpha\alpha'}=\pi_{\alpha\alpha}\delta_{\alpha\alpha'}$ in the Breit frame.

3. THE SELF-CONSISTENCY RELATIONS

Having set up the basic spin formalism in Sec. 2, we now attack the problem of obtaining the asymptotic characteristics of Δ and Γ in the optimistic hope that it is indeed possible to extract such information purely by examination (in the asymptotic limit) of the elastic unitarity equations connecting Δ and Γ with their imaginary parts. (That the ensuing results receive qualitative support at the experimental level and have reasonable theoretical repercussions does perhaps justify this hope.) In the same spirit we propose the

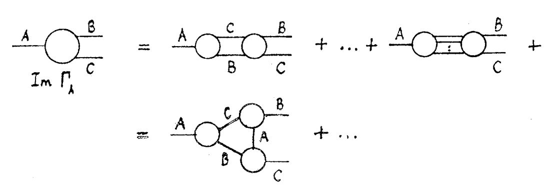

FIG. 2. Contributions to the absorptive part of the ABC form factor.

simplest possible set of assumptions to establish this asymptotic self-consistency since nothing definite is known in this respect at present. Our cause becomes hopeless unless we do so.

Referring to Figs. 1 and 2, and to Eqs. (10) and (19) the absorptive parts of Δ and Γ are given in the usual way by the relations:

$$\mathrm{Im}\Delta_A(s) = -\frac{q m_A{}^{2a}}{16\pi^2 s^{\frac{1}{2}+a}} \int d\varphi d(\cos\theta) \times \sum_{\mu,\nu} |\langle 0|\phi_\lambda(0)|\mathbf{q}\mu, -\mathbf{q}\nu\rangle|^2 + \cdots$$

$$\mathrm{Im}\langle 0|\phi_\lambda(0)|q\mu, -q\nu\rangle = \frac{q\,\mathrm{Re}}{16\pi^2 s^{1/2}} \int d\varphi d(\cos\theta) \sum_{\mu'\nu'} \langle 0|\phi_\lambda(0)|\mathbf{q}\mu', -\mathbf{q}\nu'\rangle^* \times M_{\mu\nu\mu'\nu'}(q) + \cdots$$

or

$$\mathrm{Im}\Delta_A{}^{-1}(s) = \frac{q s^{a-\frac{1}{2}}}{4\pi m_A{}^2} \sum_j \frac{|\tilde{\Gamma}_j(s)|^2}{2j+1} + \cdots, \tag{25}$$

$$\mathrm{Im}\tilde{\Gamma}_j(s) = \frac{q\,\mathrm{Re}}{16\pi^2 s^{1/2}} \int d\varphi d(\cos\theta) \times \sum_{\substack{j'\mu'\nu' \\ \mu\nu}} \frac{D_{\lambda\lambda'}{}^a(q)}{2j+1} \begin{pmatrix} j & b & c \\ -\lambda & \mu & -\nu \end{pmatrix} M_{\mu\nu\mu'\nu'} \times \begin{pmatrix} j' & b & c \\ -\lambda' & \mu' & -\nu' \end{pmatrix} \tilde{\Gamma}_{j'}{}^*(s) + \cdots, \tag{26}$$

and M is to be approximated by the Born amplitude for BC scattering, viz., the exchange of A with complete propagator and vertices.

Further progress is impossible unless we make the above-mentioned simple (though drastic) assumptions.

(i) We suppose the $\Delta_A(s)$ depends, *for large s*, only on the spin a, and that the character of this dependence is universal. Since the spirit of this work implicitly presumes that the higher (than two) particle intermediate states give the same (or more convergent) asympotics than the elastic terms and that their inclusion results in a matching of coefficients as well as of high-energy characteristics, it immediately follows from Eq. (25) that $\tilde{\Gamma}_j$ is then independent of b and c. As to the nature of the spin dependence, we place our faith in the conventionally renormalizable theories where $(s-m_A{}^2)\tilde{\Delta}_A(s)$ diverges logarithmically. This suggests that we take

$$\mathrm{Im}\tilde{\Delta}(s) = O(s^{-1}),$$

$$\mathrm{Im}\tilde{\Delta}^{-1}(s) = O(s(\ln s)^{-2})$$

and

$$\Delta_A(p) \approx \pi^a(p) s^{-1} \ln s, \tag{27}$$

i.e., all the spin dependence is carried in the monomial π and $Z=0$ is an automatic consequence of Eq. (16). (See the Appendix for the details concerning the asymptotic behaviors of dispersion integrals.)

(ii) Likewise assume that $\mathrm{Im}\tilde{\Gamma}_j{}^{-1}(s) = 0(s^\gamma)$ with γ integral $\geqslant 0$ as suggested by lowest order perturbation theory. Barring pathological oscillations (which we hope to be physically nonexistent),

$$\tilde{\Gamma}_j{}^{-1}(s) \approx s^\gamma \ln s \quad \text{and} \quad \mathrm{Im}\tilde{\Gamma}_j(s) \approx s^{-\gamma}(\ln s)^{-2} \tag{28}$$

as shown in the Appendix. From Eq. (23), $Z_1=0$. Notice that power-law behavior has been assumed for $\mathrm{Im}\tilde{\Gamma}_j{}^{-1}$, not for $\mathrm{Im}\tilde{\Gamma}_j$, and the necessity for this will be immediately apparent.

(iii) We now ask that substitution of Eqs. (27) and (28) into the elastic terms of Eqs. (25) and (26) be self-reproductive. A detailed evaluation of the integral over the Born term is not in fact necessary; we only need to recognize that $M_{\mu\nu\mu'\nu'}$ may be written as some product of d^J matrices times $\tilde{\Gamma}_j(t)\tilde{\Delta}(t)\tilde{\Gamma}_{j'}(t)$ and a polynomial $P_a(s,t)$ of degree a in s and t, arising from the complete A propagator. Due to the asymptotic nature of $\tilde{\Gamma}\tilde{\Delta}\tilde{\Gamma}P_a \approx t^{-1-2\gamma}(\ln t)^{-1}P_a(s,t)$, the leading behavior[9] of the integral in (26) gives

$$s^{-\gamma}(\ln s)^{-2} \approx s^{-\gamma}(\ln s)^{-1} \times s^{a-2\gamma-1}(\ln s)^{-1} + \cdots$$

and

$$s(\ln s)^{-2} \approx s^{a-2\gamma}(\ln s)^{-2} + \cdots$$

from Eq. (25). Thus, rather surprisingly, both propagator and vertex equations are satisfied if $\gamma = \frac{1}{2}(a-1)$, so that we finally have as our self-generating functions,

$$\Delta_A(p) = \pi^a(p) s^{-1} \ln s, \quad \tilde{\Gamma}_j(s) \approx s^{\frac{1}{2}(1-a)}(\ln s)^{-1}. \tag{29}$$

Recall that $\tilde{\Gamma}_j$ is the jth multipole form factor with B and C placed on the mass shell.[10]

To translate these results into the more familiar tensor representation of the Lorentz group (times a γ matrix in case the spin is half odd integral), we observe that the Feynman propagator invariably takes the form $\Delta(p) = \pi(p)(p^2-m^2)^{-1}$, where $\pi(p)$ is a monomial of degree $2j$ in the momentum p, and as in Eq. (12), $\pi(p) = \sum_\lambda u_\lambda(p) u_\lambda{}^+(p)$. We may therefore imagine a monomial in p of degree j associated with each free-particle solution $u(p)$ in so far as computations of high-energy behavior are concerned. Since

$$\tilde{\Gamma}(p^2) \approx u_B{}^+(\tfrac{1}{2}p+q)\Gamma(p,q)u_C(\tfrac{1}{2}p-q),$$

[9] The asymptotic behavior of the logarithmic integral

$$\int_{-s}^0 dt\, t^{-\epsilon}(\ln t)^{-1} \approx s^{1-\epsilon}(\ln s)^{-1}; \quad \epsilon < 1$$

may be established by partial integration. Making use of the differential properties of the d^j matrices, a parallel procedure leads to the desired result.

[10] When $a-1$ is odd the final result contradicts the initial assumption that γ is integral. However, to the extent that logarithms can be neglected, (29) remains valid. Verification of the Regge behavior in the next section then only entails that $\mathrm{Im}\Gamma/|\Gamma| \approx (\ln s)^{-1}$, as for integer γ. We shall discuss such questions fully in future publications by examining some special theories where the spins a, b, c are fixed at some low values.

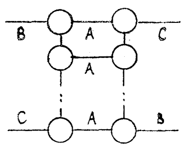

FIG. 3. Typical ladder diagram for BC scattering.

we deduce from Eq. (29) that for large values of p the proper vertex function behaves as

$$\Gamma(p,q)\approx p^{-(a+b+c-1)}(\ln p^2)^{-1},$$

and as before,

$$\Delta_A(p)\approx p^{2a-2}(\ln p^2)^{-1}. \tag{30}$$

4. THEORETICAL CONSEQUENCES

While the simple-minded assumptions which have led to Eq. (30) are not free of defects and are certainly wide open to criticism, the final formulas possess at least the following redeeming virtues:

(i) They are in agreement with the well-known Regge behavior of the scattering amplitude $s^{\alpha(t)}$ at large s and small t. To effect this demonstration, we note from Eq. (28) the important information that $\mathrm{Im}\Gamma(s)/|\Gamma(s)| \approx (\ln s)^{-1}$, (for all partial waves in fact) so that if we insert this into the elastic unitarity equation for Γ we deduce that

$$\frac{s}{\ln s}\approx\int_{-s}^{0}P_l\left(\frac{t}{s}\right)|M(s,t)|\,dt;\quad \mathbf{l}+\mathbf{b}+\mathbf{c}=\mathbf{a}.$$

For simplicity, consider the s-wave projection (what we have to say applies equally well to all l waves near $t=0$) when

$$\int_{-s}^{0}dt|M(s,t)|\,dt\approx\frac{s}{\ln s}. \tag{31}$$

Now for large s and large t (i.e., near the limit of integration $-s$) we have shown in Sec. 3 that if M is represented by the complete Born term, relation (31) is satisfied; however, at the upper limit ($t\approx 0$), where one should not really place any faith on the Born term, we must make sure that the asymptotic result is not violated if we replace M(Born) by M(Regge). In fact, it is not, for if we suppose that in this region of large and small t, M is well represented by

$$|M(s,t)|\approx O(g^2)s^{\alpha(0)+t\alpha'(0)},$$

we derive from the upper limit in Eq. (31) the totally unexpected result that

$$\alpha(0)=1 \quad\text{and}\quad \alpha'(0)=0(g^2) \tag{32}$$

if we are to have asymptotic self-consistency.

Some serious consequences arise if we follow through the above calculation more closely and examine the approximation of elastic unitarity.[1] In actual fact, it is known that M is almost absorptive at high energy, so if we employ elastic unitarity alone, we deduce that Γ is purely absorptive by Watson's theorem; but this is hardly likely for a function having a *single cut* as shown in the Appendix; therefore we are forced to step outside the framework of elastic unitarity and include many-particle intermediate states into our discussion in attempting to use the asymptotic form of M. Now result (32) follows from a Regge amplitude $\gamma(t)s^{\alpha(t)}$ which is *almost real* for $t<0$. Such an M could arise from the ladder diagrams of Fig. 3, whereas the complete M (which is almost imaginary) comprises the crossed ladder diagrams of Fig. 4 as well. Consequently, if we presume that (32) is more than a lucky coincidence, we deduce that the Fig. 4 contributions to $\mathrm{Im}\Gamma$ must be canceled in some way by the many-particle intermediate state contributions. The situation has a parallel in the Amati-Fubini-Stanghellini[11] theory of Regge cuts, where it has been shown by Polkinghorne and Mandelstam[12] that pure elastic unitarity used in conjunction with complete scattering amplitudes can be a very misleading procedure, and moreover, that these particular cuts do not in reality exist when inelastic intermediate states are added.

(ii) Γ can be a highly convergent quantity with convergence improving as the spins of the participating particles are increased. Indeed, if we attempt to extend formula (30) by anticipating that with all particles off the mass shell,

$$\Gamma(p,q)\approx k^{-(a+b+c-1)}, \tag{33}$$

for large k (in any direction in the hyperplane of p and q, and aside from logarithmic factors), then we obtain

$$\Delta_A{}^{1/2}\Delta_B{}^{1/2}\Delta_C{}^{1/2}\Gamma\approx k^{-2}.$$

This last condition is sufficient to ensure the success of any approximation scheme based on the Dyson equations for computation of S-matrix elements.[5] In particular, for the electrodynamics of scalar and vector mesons, a detailed and independent verification of relations (30) was directly obtained[5] (as a consequence of gauge identities) and an actual scheme for finite computation was proposed to replace the Feynman rules.

We may examine (33) further with a view to renormalizability. In conventional theory, renormalization of Δ_0 and Γ_0 is possible providing that

$$\Delta_0{}^{1/2}\Delta_0{}^{1/2}\Delta_0{}^{1/2}\Gamma_0\approx k^{-1-\epsilon};\quad \epsilon\gtrless 0$$

and if this condition is met (e.g., scalar and spinor

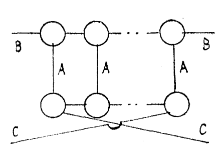

FIG. 4. Typical crossed ladder diagram for BC scattering.

[11] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters 1, 29 (1962); Nuovo Cimento 26, 896 (1962).

[12] J. C. Polkinghorne (to be published); S. Mandelstam, Nuovo Cimento 30, 1127 (1963).

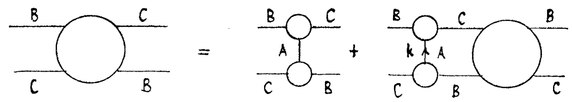

FIG. 5. Bethe-Salpeter equation for BC scattering.

electrodynamics, meson-nucleon interactions, etc.) Γ behaves at infinity like the bare vertex Γ_0, apart from logarithmic factors. However, for theories that are conventionally unrenormalizable we have an entirely different situation in that Γ is more convergent than Γ_0. Therefore, in setting up an iteration scheme for calculation of Feynman diagrams based on Dyson's equations, it is imperative to provide an initial Γ that does behave like (30), as otherwise the k dependence is fundamentally unstable. That is, the "lowest order" Γ must be chosen to exhibit the behaviors indicated in the following typical cases:

(a) Vector electrodynamics: A = Photon, $B=C$ = spin-1 meson. $\Gamma\approx k^{-1}$ although $\Gamma_0\approx k$.

(b) Scalar meson-lepton interactions: A = meson, $B=C$ = lepton. $\Gamma\approx 1$ although $\Gamma_0\approx k$. We might add that from our point of view the convergence of $\Gamma(\approx 1/\ln s)$ justifies the Goldberger-Treiman theory of $\pi\to l+\nu$ decay.

(c) Vector meson-lepton interactions: A = spin-1 meson, $B=C$ = lepton. $\Gamma\approx k^{-1}$ although $\Gamma_0\approx 1$.

(iii) Ward's identity is automatically satisfied

$$\partial\Delta_B^{-1}(p)/\partial p\approx\Gamma(p)\,,$$

if the numerical factors which characterize the asymptotic behavior of Δ and Γ are in the correct ratio.[13] We have let $B=C$ be the charged particle and made A the photon (a scalar particle from the point of view of high-energy properties, as its propagator $\approx k^{-2}$).

(iv) The kernel to the Bethe-Salpeter equation (see Fig. 5),

$$\begin{aligned}K=\Delta_C(p_B+k)\Gamma(p_B,\,p_B+k)\Delta_A(k)\\ \times\Gamma(p_C,\,p_C-k)\Delta_B(p_C-k)\approx k^{2c-2}k^{-(a+b+c-1)}\\ \times k^{2a-2}k^{-(a+b+c-1)}k^{2b-2}\approx k^{-4},\end{aligned}$$

is independent of the spins a, b, c. Consequently, the effective potential will never be highly singular so that a solution to the equation must exist and can in principle be obtained by directly summing the ladder diagrams—the extra logarithmic factors tend to improve convergence.[5]

5. EXPERIMENTAL CONSEQUENCES

Impossible though it is to say anything about the asymptotic behavior of Δ in experimental terms, there are reasons to suppose from the recently published data[6] that the asymptotic region of the electromagnetic form factors of the nucleon may have been reached, so that experimental verification of our predictions about Γ may be feasible. Noting that the Sachs form factors $G_{E,M}(s)$ are given in our notation essentially by $\Gamma(s)$, formula (30) states $\Gamma(s)\approx(\ln s)^{-1}$. For moderately large s one may hope to see the influence of some resonance poles, so that over a large part of the measured range we expect the behaviors

$$G_{E,M}(s)\approx\alpha_{E,M}s^{-1}+\beta_{E,M}(\ln s)^{-1}\,, \qquad (34)$$

where α and β are some constants.

Over the greater part of the range, the dependence s^{-1} is indeed clearly visible, but the presence of the second term, $(\ln s)^{-1}$, which is expected to dominate at sufficiently high s and to cause a flattening of the curves, is hard to detect. However, although the term $(\ln s)^{-1}$ could well exist, it is impossible to measure β with any precision, owing to the large experimental errors of the high-energy data. In any case, regardless of direct measurement of the α and β parameters, Eq. (34) predicts that both G_E and G_M vanish as $s\to\infty$, while their ratio tends to a constant. This is quite consistent with the present available data.

Turning to the strong interactions, there are indications that the vertex function is a strongly convergent quantity from an analysis of the differential cross section $d\sigma/dt$ at large energies ($s^{1/2}$) and large momentum transfers ($(-t)^{1/2}$). The experiments on high-energy pp scattering seem to follow the Serber law, $d\sigma/dt\approx O(t^{-5})$ for $|t|\to\infty$. If the Regge pole model has any validity at all, then in view of the absence of shrinkage in πp scattering,[14] the strong t dependence of $d\sigma/dt$ should be given by the residue function in this asymptotic region. Now on the Regge pole model $d\sigma/dt$ should be proportional to the fourth power of the residue function, and this residue function very probably corresponds to the vertex function $\Gamma(t)$ in terms of field theory. Since $\Gamma(t)\approx(t^{1/2}\ln t)^{-1}$ from (30), assuming the exchange of a vector meson [which would correspond to $\alpha(0)=1$], our prediction would be that[15] $d\sigma/dt\propto|\Gamma(t)|^4\approx t^{-2}(\ln t)^{-4}$ for large t. This lack of agreement with the experimental behavior should not be taken very seriously since most of the assumptions that have been fed into the theory to derive Eq. (30) are certainly oversimplified. However, what we wish to emphasize very strongly is that, naive though the theory is, it does indicate that Γ can be a highly convergent quantity, and we claim that this feature is probably retained is more exact theories.

ACKNOWLEDGMENTS

The author is indebted to Dr. B. R. Desai and Professor C. J. Goebel for discussions which served to clarify several points in this paper.

[13] The gauge properties of the photon amplitudes are powerful tools that can be manipulated to obtain explicit expressions for Δ and Γ satisfying Eq. (33). See Ref. 5.

[14] B. R. Desai, Phys. Rev. Letters 11, 512 (1963).

[15] Field theoretically the scattering of B and C by exchange of A (momentum $t^{1/2}$) at energy $s^{1/2}$ is represented at large t by

$$M(s,t)\approx t^{-(a+b+c-1)}(\ln t)^{-2}P_a(s,t)(\ln t)/t\approx t^{-(b+c)}(\ln t)^{-1}P_a(s,t),$$

where P_a is the Legendre polynomial of degree a in $\cos\theta_t=s/t$.

APPENDIX

Given that for $x>\bar{s}$ (real)

$$\text{(I)}\quad |\text{Im}\Delta(x)|<Ax^{-\alpha},\quad \alpha>0\,,$$
$$\text{(II)}\quad |\text{Im}\Delta'(x)|<g(x)\quad\text{where}\quad g(x)\geqslant Bx^{-\alpha-1}\,,$$

then if $\text{Im}\Delta(x)$ is integrable, it has been rigorously proved by Lanz and Prosperi[16] that

$$\Delta(s)=\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\text{Im}\Delta(x)dx}{x-s-i\epsilon}$$

is bounded in the following manner when $|s|>s_1>\bar{s}$:

(i) If $\alpha<1$, $|\Delta(s)|<C|s|^{-\alpha}[\ln|s^{\alpha+1}\bar{g}(s)|+c]$;
(ii) if $\alpha=1$, $|\Delta(s)|<Cs^{-1}[2\ln|s^2\bar{g}(s)|+\ln s+c]$;
(iii) if $\alpha>1$, certain moments of the spectral function may vanish if it oscillates suitably. Putting $\alpha=n+\alpha'$ with n, a nonzero positive integer, and $0<\alpha'\leqslant 1$, and allowing for

$$\int\text{Im}\Delta(x)dx=\int x\,\text{Im}\Delta(x)dx=\cdots=\int x^{m-1}\,\text{Im}\Delta(x)dx=0\,,$$

then when $m<n$, $|\Delta(s)|<$ the greater of

$$C|s|^{-m-1}\quad\text{and}\quad C|s|^{-\alpha}[2\ln|s^{-\alpha-1}\bar{g}(s)|+\delta_{1\alpha'}\ln s+c]\,,$$

while when

$$m=n,\quad |\Delta(s)|<C|s|^{-\alpha}[2\ln|s^{-\alpha-1}\bar{g}(s)|+\delta_{1\alpha'}\ln s+c]\,.$$

In these expressions C represents suitable positive numbers and $\bar{g}(s)$ is the maximum of $g(s)$ in the interval

$$(s-\epsilon[s^\alpha g(s)]^{-1},\quad s+\epsilon[s^\alpha g(s)]^{-1})\,.$$

Of special interest to us is the case where the spectral function exhibits the simple behavior

$$\text{Im}\Delta(x)\approx Ax^{-\alpha}\quad\text{and}\quad \text{Im}\Delta'(x)\approx Ax^{-\alpha-1},\quad x>\bar{s}.$$

The above theorem states that all sufficiently large $|s|$:
(i) If $\alpha>1$, and there are no vanishing moments,

$$\Delta(s)\approx O(|s|^{-1})+is^{-\alpha}\theta(s)\,;$$

is there are m vanishing moments,

$$\Delta(s)=O(|s|^{-m-1})+is^{-\alpha}\theta(s)\quad(m<n)\,,$$
$$\Delta(s)\approx O(|s|^{-\alpha})+is^{-\alpha}\theta(s)\quad(m=n)\,.$$

Here n is the nearest positive integer to α.

(ii) If $\alpha=1$, $\Delta(s)\approx O(|s|^{-1}\ln|s|)+is^{-1}\theta(s)$.
(iii) If $0<\alpha<1$, $\Delta(s)\approx O(|s|^{-\alpha})+is^{-\alpha}\theta(s)$.

Note that for $\alpha=\frac{1}{2}$, $\Delta(s)\approx is^{-1/2}\theta(s)+O(|s|^{-1})$.

(iv) If $\alpha\leqslant 0$, put $\alpha=\alpha'-n$, where $0<\alpha'\leqslant 1$ and n is integral. Making n subtractions at $\tilde{s}<s_0$, we obtain

$$\Delta(s)=\sum_{r=0}^{n-1}A_r(s-s)^r+\frac{(s-\tilde{s})^n}{\pi}\int\frac{\text{Im}\Delta(x)dx}{(x-\tilde{s})^n(x-s)}\,,$$

and using (ii) and (iii) for the finite integral on the right-hand side,

$$\Delta(s)=O(|s|^{-\alpha}\ln|s|)+is^{-\alpha}\theta(s)\quad(-\alpha\text{ integral})\,,$$
$$\Delta(s)=O(|s|^{-\alpha-\frac{1}{2}})+is^{-\alpha}\theta(s)\quad(-\alpha\ \tfrac{1}{2}\text{ odd integral})\,,$$
$$\Delta(s)=O(|s|^{-\alpha})+is^{-\alpha}\theta(s)\quad(\text{all other }-\alpha)\,.$$

Thus, except for very pathological cases [when $\bar{g}(s)\neq g(s)$ and $\ln|s^{-\alpha-1}g(s)|$ is unbounded], and providing $\alpha+\frac{1}{2}$ is not a positive integer, $\Delta(s)$ cannot become purely absorptive at large s when it possesses a single cut. This is especially true for the propagator where $\text{Im}\Delta(x)$ cannot be negative. Only when $\Delta(s)$ possesses two cuts is it a simple matter for $\text{Im}\Delta(s)$ to predominate over $\text{Re}\Delta(s)$. For example, the high-energy scattering amplitude $M(s,0)$ becomes purely imaginary for large s because of the inclusion of the crossed channel so that

$$M(s,0)\approx\int\frac{dx}{x^\alpha}\left(\frac{1}{x-s}+\frac{1}{x+s}\right)\approx s^{-\alpha}\left(\frac{1+e^{i\pi\alpha}}{\sin\pi\alpha}\right).$$

This equality of spectral functions for *both cuts*, the Pomeranchuk result, is of course what is needed to get dominant absorption.

Frye and Warnock have investigated the possibility that the spectral function is bounded by powers of logarithms, viz.,

$$|\text{Im}\Delta(x)|\langle C(\ln x)^{-\beta}\quad\text{for}\quad x>\bar{s}.$$

They find that $\Delta(s)=O(\ln^{1-\beta}|s|)+i\theta(s)\ln^{-\beta}s$. If, instead, $|\text{Im}\Delta(x)|<Cx^{-\alpha}\ln^{-\beta}x$, it is true that when $\alpha<0$, the powers of x by and large dominate the high-energy behavior of $\Delta(s)$.

[16] L. Lanz and G. M. Prosperi (to be published).