

$\Gamma_{\mathbf{k}_1, \mathbf{k}_2}^{k_2-k_1} = \Gamma_{\mathbf{k}_1, \mathbf{k}_2}^0 = \Gamma_{\mathbf{k}_2, \mathbf{k}_1}^0$. The diagonal part of H^0 is therefore

$$H_{\text{diag}}^0 = E_0 + \sum_{\mathbf{k}} n_{\mathbf{k}} - 2 \sum_{\mathbf{k}_1, \mathbf{k}_2} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \Gamma_{\mathbf{k}_1, \mathbf{k}_2}^0, \quad (\text{A9})$$

and the average energy associated with the excitation of a spin wave is then¹⁸

$$\bar{\xi}_{\mathbf{k}} \approx \xi_{\mathbf{k}} - 4 \sum_{\mathbf{k}'} \langle n_{\mathbf{k}'} \rangle \Gamma_{\mathbf{k}, \mathbf{k}'}^0, \quad (\text{A10})$$

where $\langle \ \rangle$ denotes a thermal average and to this order the $\langle n_{\mathbf{k}} \rangle$ are uncorrelated. In the long-wavelength ap-

¹⁸ See, for example, M. Bloch, Phys. Rev. Letters 9, 286 (1962).

proximation for both \mathbf{k} and \mathbf{k}' , and making use of Eq. (17),

$$\bar{\xi}_{\mathbf{k}} = \xi_{\mathbf{k}} - (6/N) \sum_{\mathbf{k}'} \langle n_{\mathbf{k}'} \rangle \sum_1 E_1 \cos \mathbf{k} \cdot \mathbf{l}, \quad (\text{A11})$$

or, using Eq. (A7),

$$\bar{\xi}_{\mathbf{k}} = 2SJZ(1 - \gamma_{\mathbf{k}}) + 2S \sum_1 E_1 \cos \mathbf{k} \cdot \mathbf{l} [1 - (3/NS) \sum_{\mathbf{k}'} \langle n_{\mathbf{k}'} \rangle]. \quad (\text{A12})$$

Noting that $[M(T)/M(0)]^n \approx [1 - (n/NS) \sum_{\mathbf{k}'} \langle n_{\mathbf{k}'} \rangle]$, we have shown that the first-order pseudodipolar coupling energy varies as $[M(T)/M(0)]^3$.

Ultrasonic Attenuation in Metals in the Fluid-Dynamic Approximation*

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(Received 29 June 1964)

Ultrasonic attenuation in metals has been discussed from diverse points of view, and for the most part the derived attenuation formulas reduce in the first-order low-frequency limit to the viscous dissipative expression originally proposed by Mason. We undertake here a treatment for the low-frequency range which makes immediate contact with the transport-theory formalism of classical gases. "Fluid"-dynamic equations for the metal are formulated. For the electron-gas component a complete set of transport coefficients, including the several diffusion coefficients, is derived in a unified way on the simplifying assumption of a constant relaxation time. The acoustic attenuation coefficient for a longitudinal wave is deduced; the dominant term is of course the shear viscous one, but thermal and diffusion effects are also explicit.

INTRODUCTION

MOTIVATED by the problem of ultrasonic attenuation, we develop "fluid"-dynamic equations for a metal in a manner parallel to that employed for a mixture of classical gases.¹ We take as a two-component model a free-electron gas coupled to an elastic ion continuum by an electromagnetic field and by collision. From the Boltzmann transport equation, fluid-dynamic equations and a complete set of transport coefficients are derived for the electron gas in a unified way; thus the various diffusion coefficients are included. For the present purpose a constant-rate relaxation term suffices for the collision integral, and we are content simply to postulate isotropic continuum equations for the ions. The equations for the two components are combined into a simplified set for the system as a whole. This, in conjunction with the equation for the relative motion of the electrons and the Maxwell equations for the associated electromagnetic field, determines the behavior of

the model. Finally, we consider the propagation of a longitudinal-plane acoustic wave and compute the attenuation coefficient.

FORMULATION OF MACROSCOPIC EQUATIONS

The electron-distribution function $f(\mathbf{r}, \mathbf{v}, t)$ obeys the Boltzmann transport equation

$$\mathcal{D}f \equiv \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_1} \cdot \frac{\partial}{\partial \mathbf{v}} \right] f = \frac{\partial f}{\partial t} \Big|_{\text{coll}}, \quad (1)$$

where symbols have their common meanings. When distinction is needed, indices 1 and 2 will be used to denote electrons and ions, respectively. Let $\psi(\mathbf{v})$ be a particle property which is a function of velocity. According to standard transport theory,¹ (1) is multiplied by $\psi(\mathbf{v})$ and integrated over velocity to obtain the equation of change for the mean value $\langle \psi \rangle$:

$$\frac{\partial}{\partial t} n_1 \langle \psi \rangle + \frac{\partial}{\partial \mathbf{r}} \cdot n_1 \langle \mathbf{v} \psi \rangle - n_1 \left\langle \frac{\mathbf{F}}{m_1} \cdot \frac{\partial \psi}{\partial \mathbf{v}} \right\rangle = \Delta_c \langle \psi \rangle. \quad (2)$$

Here n_1 is the electron-particle density. We assume the electron and ion densities are equal. $\Delta_c \langle \psi \rangle$ is the rate of change of ψ per unit volume due to collisions; \mathbf{F} has been

* Part of this work was performed at The University of California, Los Angeles, with support by the U. S. Office of Naval Research Contract N6 onr 233 (48), Project NR014-302.

¹ J. O. Hirschfelder, C. F. Curtiss, R. B. Bird, *Molecular Theory of Gases and Liquids* (John Wiley & Sons, New York, 1954), Chap. 7; S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press, Cambridge, England, 1952).

limited to velocity-independent or magnetic forces. Let \mathbf{u} be the center-of-mass velocity. ψ is then set successively equal to m_1 , and $m_1\mathbf{v}$ or $m_1(\mathbf{v}-\mathbf{u})$, and $\frac{1}{2}m_1(\mathbf{v}-\mathbf{u})^2$ in (2) to obtain equations of change for mass, momentum, and relative kinetic or internal energy. To state these equations we shall need the following additional definitions: Let $\rho_1 = n_1 m_1$ = electron-mass density; ϵ = electron "internal" energy; $\langle \epsilon_1 \rangle$ = mean electron internal energy; $\hat{\epsilon}_1 = \langle \epsilon_1 \rangle / m_1$ = electron-specific internal energy; $\mathbf{u}_1 = n_1^{-1} \int d^3v f \mathbf{v}$ = mean electron velocity. We note that $\mathbf{u} = \rho_1 \mathbf{u}_1 + \rho_2 \mathbf{u}_2$. Let $\mathbf{V} = \mathbf{v} - \mathbf{u}$ = relative electron velocity. Define the kinetic-stress tensor by $\mathbf{P}_1 = \rho_1 \langle \mathbf{V}\mathbf{V} \rangle$; the electron relative- or diffusion-current density by $\mathbf{J}_1 = n_1 \langle \mathbf{V} \rangle$; the internal-energy current density or "heat" flux by $J_{\epsilon_1} = \frac{1}{2} \rho_1 \langle V^2 \rangle$. Finally, let $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ be the material derivative. Then the equations of change deducible¹ from (2) are: equation of continuity,

$$D\rho_1/Dt + \rho_1 \nabla \cdot \mathbf{u}_1 = 0$$

or

$$D\rho_1/Dt + \rho_1 \nabla \cdot \mathbf{u} + m_1 \nabla \cdot \mathbf{J}_1 = 0; \quad (3)$$

momentum equation,

$$\rho_1 \frac{D\mathbf{u}_1}{Dt} + \nabla \cdot \mathbf{P}_1 - \frac{\rho_1}{m_1} \mathbf{F}_1 = \Delta_c [m_1(\mathbf{v}-\mathbf{u})]; \quad (4)$$

internal-energy equation,

$$\rho_1 D\hat{\epsilon}_1/Dt + \nabla \cdot \mathbf{J}_{\epsilon_1} + \mathbf{P}_1 : \nabla \mathbf{u} = \Delta_c [\frac{1}{2} m_1(\mathbf{v}-\mathbf{u})^2]. \quad (5)$$

We now assume analogous macroscopic equations for the positive ions. Their stress tensor, however, is predominantly of elastic rather than kinetic origin. Upon addition of the corresponding equations for electrons and ions, the collision terms cancel as internal transfers, and convenient equations for the system as a whole result:

$$D\rho/Dt + \rho \nabla \cdot \mathbf{u} = 0, \quad (6)$$

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla \cdot \mathbf{P} - \frac{\rho_1}{m_1} \mathbf{F}_1 - \frac{\rho_2}{m_2} \mathbf{F}_2 = 0, \quad (7)$$

$$\rho D\hat{\epsilon}/Dt + \nabla \cdot \mathbf{J}_\epsilon + \mathbf{P} : \nabla \mathbf{u} = 0. \quad (8)$$

Here $\hat{\epsilon} = \hat{\epsilon}_1 + \hat{\epsilon}_2$ and $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ are the total specific internal energy and stress tensor. We remark also that

$$\mathbf{P} = \hat{p} \mathbf{I} + \mathbf{\Pi}, \quad (9)$$

where \mathbf{I} is the unit tensor, \hat{p} the scalar pressure, and $\mathbf{\Pi}$ is the viscous stress tensor. Since we shall be concerned with electric forces, (7) may be written as

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla \cdot \mathbf{P} + \left(\frac{\rho_1}{m_1} - \frac{\rho}{m_2} \right) e \mathbf{E} = 0, \quad (7')$$

where \mathbf{E} is the electric-field intensity, e is the magnitude of the electron charge, m_1 and m_2 are the electron and ion masses, $\rho_2 \approx \rho$. To the above equations we should

adjoin those of the electromagnetic field, which is activated by the slight phase lag in the electron and ion motions; the system is really a three-component one. However, we shall limit our application to longitudinal wave propagation, so we list here only Poisson's equation,

$$\nabla \cdot \mathbf{E} + 4\pi e(\rho_1/m_1 - \rho/m_2) = 0. \quad (10)$$

In the energy equation (8) it is convenient to have the temperature rather than the internal energy present as the thermodynamic variable. With the aid of standard thermodynamic identities and (3), (6), and (9), Eq. (8) can be transformed to²

$$\rho \hat{c}_v \frac{DT}{Dt} + \nabla \cdot \mathbf{J}_\epsilon + \mathbf{\Pi} : \nabla \mathbf{u} - e \mathbf{J}_1 \cdot \mathbf{E} + T \left(\frac{\partial \hat{p}}{\partial T} \right)_{\rho, \mathbf{P}_1} \nabla \cdot \mathbf{u} - \hat{\epsilon}_1 \nabla \cdot \mathbf{J}_1 = 0. \quad (11)$$

In arriving at (11) we have put $J_2 = 0$ and neglected $m_1 \hat{\epsilon}_2 / m_2$ relative to $\hat{\epsilon}_1$. The set (6), (7'), (11), (3), (10) are the continuum dynamic equations for the metal. However, we have still to provide constitutive equations for \mathbf{P} , \mathbf{J}_1 , \mathbf{J}_ϵ . For the electron gas, our model admits a microscopic derivation of such equations.

CONSTITUTIVE EQUATIONS FOR ELECTRON GAS

To this end we require the solution of the Boltzmann transport equation for the electron-distribution function. We obtain this in first order. As frequently done in the transport theory of classical gases, we take the non-equilibrium distribution to be a small perturbation of local (rather than absolute) equilibrium; i.e.,

$$f(\mathbf{r}, \mathbf{v}, t) = f^0(\mathbf{v}-\mathbf{u}, \zeta, T) + f^{(1)}(\mathbf{v}-\mathbf{u}, \zeta, T). \quad (12)$$

Here f^0 is the Fermi-Dirac distribution function; $f^{(1)}$ is the perturbation function, and the parameters \mathbf{u} , ζ , and T are functions of \mathbf{r} and t .

The customary assumption in ultrasonic attenuation literature, that lattice scattering of electrons relaxes the distribution toward local equilibrium, is concordant with this general assumption of a small perturbation of local equilibrium. We write for the electron-lattice scattering term, in the constant relaxation-time approximation

$$\left. \frac{\partial f}{\partial t} \right|_{el} = - \frac{f(\mathbf{v}) - f^0(\mathbf{v}-\mathbf{u})}{\tau_{el}} = - \frac{f - f^0(\mathbf{V})}{\tau_{el}} = - \frac{f^{(1)}}{\tau_{el}}.$$

Although usually ignored as small, we may allow for electron-electron scattering by a term

$$\left. \frac{\partial f}{\partial t} \right|_{ee} = - \frac{f - f^0(\mathbf{v}-\mathbf{u}_1)}{\tau_{ee}} = - \frac{f - f^0(\mathbf{V})}{\tau_{ee}} - \frac{\partial f^0(\mathbf{V})}{\partial \mathbf{V}} \cdot \frac{(\mathbf{u}_1 - \mathbf{u})}{\tau_{ee}}.$$

² R. B. Bird, W. E. Stewart, E. N. Lightfoot, *Transport Phenomena* (John Wiley & Sons, New York, 1960), p. 562.

The total collision term is then

$$\left. \frac{\partial f}{\partial t} \right|_{\text{coll}} = -\frac{f^{(1)}}{\tau'} \frac{\partial f^0}{\partial \mathbf{V}} \cdot \frac{\langle \mathbf{V} \rangle}{\tau_{ee}}, \quad (13)$$

where $1/\tau' = 1/\tau_{el} + 1/\tau_{ee}$. In the Boltzmann equation (1) we now set $f = f^0 + f^{(1)}$, use (13) and retain only zero-order terms on the left side. The resulting equation for $f^{(1)}$ is

$$f^{(1)} = -\tau' \mathfrak{D} f^0 - \frac{\tau'}{\tau_{ee}} \frac{\partial f^0}{\partial \mathbf{V}} \cdot \langle \mathbf{V} \rangle. \quad (14)$$

To evaluate $\mathfrak{D} f^0$ we note that zero order refers to local equilibrium; hence $\mathbf{u}_1^0 = \mathbf{u}_2^0 = \mathbf{u}$, $\mathbf{V}^0 = \mathbf{J}_1^0 = \mathbf{J}_{e1}^0 = \mathbf{\Pi} = 0$, $\mathbf{P}^0 = p\mathbf{I}$; and the equations of change (7') and (11) in zero order reduce to

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p - \left(\frac{\rho_1}{m_1} - \frac{\rho}{m_2} \right) e\mathbf{E}, \quad (15)$$

$$\frac{DT}{Dt} = -aT \nabla \cdot \mathbf{u}, \quad (16)$$

with $a = (\partial p / \partial T)_{\rho p_1} / \rho c_v$. In \mathfrak{D} we write the operator $[\partial / \partial t + \mathbf{v} \cdot \partial / \partial \mathbf{r}]$ in the form $[\partial / \partial t + \mathbf{u} \cdot \nabla + (\mathbf{v} - \mathbf{u}) \cdot \nabla] = [D / Dt + \mathbf{V} \cdot \nabla]$ so that

$$\mathfrak{D} f^0 = \left[\frac{D}{Dt} + \mathbf{V} \cdot \nabla \right] f^0 + \frac{\mathbf{F}_1}{m_1} \cdot \frac{\partial f^0}{\partial \mathbf{v}},$$

or

$$\begin{aligned} \mathfrak{D} f^0 = & \frac{\partial f^0}{\partial T} \left[\frac{D}{Dt} + \mathbf{V} \cdot \nabla \right] T + \frac{\partial f^0}{\partial \zeta} \left[\frac{D}{Dt} + \mathbf{V} \cdot \nabla \right] \zeta \\ & + \frac{\partial f^0}{\partial \mathbf{u}} \left[\frac{D}{Dt} + \mathbf{V} \cdot \nabla \right] \mathbf{u} + \frac{\mathbf{F}_1}{m_1} \cdot \frac{\partial f^0}{\partial \mathbf{v}}. \end{aligned} \quad (17)$$

The time derivatives in Eq. (17) for $\mathfrak{D} f^0$ can now be eliminated. $(D\mathbf{u}/Dt)$ and (DT/Dt) are replaced by their zero-order values in (15) and (16); $(D\zeta/Dt)$, by recourse to the second-order expression for ζ and its derivative

$$\zeta = \zeta_0 \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\zeta_0} \right)^2 \right];$$

$$\frac{D\zeta}{Dt} = \left(\frac{\partial \zeta}{\partial T} \right)_{v_1} \frac{DT}{Dt} + \left(\frac{\partial \zeta}{\partial \rho_1} \right)_{T, v_1} \frac{D\rho_1}{Dt}.$$

Finally, employing the various Fermi-Dirac derivatives,

$$\begin{aligned} \frac{\partial f^0}{\partial T} &= -\frac{\partial f^0}{\partial \epsilon} \frac{(\epsilon - \zeta)}{kT}, & \frac{\partial f^0}{\partial \zeta} &= -\frac{\partial f^0}{\partial \epsilon}, \\ \frac{\partial f^0}{\partial \mathbf{u}} &= -\frac{\partial f^0}{\partial \epsilon} m_1 \mathbf{V}, & \frac{\partial f^0}{\partial \mathbf{v}} &= -\frac{\partial f^0}{\partial \epsilon} m_1 \mathbf{V}, \end{aligned}$$

and some algebraic manipulation, one arrives at an expression for $f^{(1)}$ without explicit time dependence:

$$\begin{aligned} f^{(1)} = & \tau' \frac{\partial f^0}{\partial \epsilon} \left\{ \left[\frac{1}{2} m_1 V^2 - \zeta \right] \mathbf{V} \cdot \frac{1}{T} \nabla T \right. \\ & + \mathbf{V} \cdot \mathbf{d} - \mathbf{V} \cdot \frac{m_1}{n_1 \tau_{ee}} \mathbf{J}_1 + \left[m_1 \mathbf{V} \mathbf{V} - \frac{1}{3} m_1 V^2 \mathbf{I} \right] : \nabla \mathbf{u} \\ & \left. + \left[\frac{1}{2} m_1 V^2 - \zeta \right] \left(\frac{2}{3} - a \right) \nabla \cdot \mathbf{u} \right\}. \end{aligned} \quad (18)$$

For brevity we have set

$$\mathbf{d} = (\nabla \zeta - m_1 \nabla p / \rho + e\mathbf{E}), \quad (19)$$

which constitutes a "diffusion force." The presence of a term in $(\nabla \cdot \mathbf{u})$ provides a bulk viscosity. For perfect gases, quantal as well as classical, $a = \frac{2}{3}$, and the latter vanishes. While it is questionable, of course, that the ion assembly behaves like a perfect gas, computation shows that the ratio of bulk to shear viscosity contains the factor $(kT/\zeta)^2$ and is thus negligible in any event. We therefore drop the term. We can now compute the particle "diffusion" current density, which is given by

$$\begin{aligned} \mathbf{J}_1 &= \frac{2m_1^3}{h^3} \int d^3 V f^{(1)}(\mathbf{V}) \mathbf{V} \\ &= \frac{2m_1^3}{h^3} \int d^3 V \frac{\partial f^0}{\partial \epsilon}(\mathbf{V}) \mathbf{V} \mathbf{V} \left\{ \left[\frac{1}{2} m_1 V^2 - \zeta \right] \right. \\ & \quad \left. \times \frac{1}{T} \nabla T + \mathbf{d} - \frac{m_1}{n_1 \tau_{ee}} \mathbf{J}_1 \right\} \end{aligned} \quad (20)$$

since only these terms yield even integrands which do not vanish upon integration. The integrals in this section are readily evaluated with use of such identities as

$$\begin{aligned} \int d^3 V \partial f^0 / \partial \epsilon V^n \mathbf{V} \mathbf{V} &= \frac{1}{3} \int d^3 V \partial f^0 / \partial \epsilon V^n V^2 \mathbf{I} \\ &= \frac{4}{3} \pi \int dV \partial f^0 / \partial \epsilon V^{n+4} \mathbf{I} \end{aligned}$$

and the standard approximation

$$-\int_0^\infty \varphi(\epsilon) \partial f^0 / \partial \epsilon d\epsilon = \varphi(\zeta) + \frac{1}{6} \pi^2 (kT)^2 \partial^2 \varphi(\zeta) / \partial \epsilon^2.$$

The result is

$$\mathbf{J}_1 = -(n_1 \tau' / m_1) \mathbf{d} - (n_1 \tau' / m_1) c_{v1} \nabla T - (\tau' / \tau_{ee}) \mathbf{J}_1,$$

which may be written

$$\mathbf{J}_1 = -(n_1 \tau / m_1) \mathbf{d} - (n_1 \tau c_{v1} / m_1) \nabla T, \quad (21)$$

where $1/\tau = 1/\tau_{el} + 2/\tau_{ee}$ and c_{v1} is the constant-volume heat capacity per electron. Similarly, the internal-

energy current density is determined by the same terms in $f^{(1)}$:

$$\mathbf{J}_{\epsilon 1} = \frac{m_1^4}{h^3} \int d^3V \frac{\partial f^0}{\partial \epsilon} V^2 \mathbf{V} \mathbf{V} \times \left\{ \left[\frac{1}{2} m_1 V^2 - \zeta \right] \frac{1}{T} \nabla T + \mathbf{d} - \frac{m_1}{n_1 \tau_{ee}} \mathbf{J}_1 \right\}. \quad (22)$$

Integration yields

$$\mathbf{J}_{\epsilon 1} = - \left(\frac{5 \langle \epsilon_1 \rangle n_1 \tau}{3 m_1} \right) \mathbf{d} - \left(\frac{5 c_{v1} n_1 \tau}{3 m_1} \right) \nabla T, \quad (23)$$

which may be written in the form

$$\mathbf{J}_{\epsilon 1} = h_1 \mathbf{J}_1 - \kappa_1 \nabla T, \quad (24)$$

where $\kappa_1 = \pi^2 n_1 \tau k^2 T / 3 m_1$ is the conventionally defined thermal conductivity at zero-particle flow, and $h_1 = 5 \langle \epsilon_1 \rangle / 3 =$ mean enthalpy per electron (to first order).

The electron-stress tensor is given by

$$\mathbf{P}_1 = (2 m_1^3 / h^3) \int d^3V (f^0 + f^{(1)}) m_1 \mathbf{V} \mathbf{V} = p_1 \mathbf{I} + \mathbf{\Pi}. \quad (25)$$

The first integral leads to the familiar equilibrium pressure relation, $p_1 = \frac{2}{3} \rho_1 \langle \epsilon_1 \rangle / m_1$. The nonvanishing viscous contribution emerging from $f^{(1)}$ is

$$\mathbf{\Pi}_1 = \frac{2 m_1^4}{h^3} \int d^3V \frac{\partial f^0}{\partial \epsilon} \mathbf{V} \mathbf{V} [\mathbf{V} \mathbf{V} - \frac{1}{3} V^2 \mathbf{I}]: \nabla \mathbf{u}. \quad (26)$$

This integral provides a shear-viscous stress

$$\mathbf{\Pi}_1 = -2\eta (\nabla \mathbf{u} - \frac{1}{3} \nabla \cdot \mathbf{u} \mathbf{I}), \quad (27)$$

where the coefficient of shear viscosity is $\eta = \frac{2}{5} n_1 \tau \zeta$.

It is convenient to eliminate p and ζ from Eq. (19) for \mathbf{d} in favor of the variables ρ, ρ_1, T for which equations of change have already been formulated. As a consequence \mathbf{J}_1 takes the form

$$\mathbf{J}_1 = - (D/m_1) \nabla \rho_1 + (D_\rho/m_1) \nabla \rho - (D_T/m_1) \nabla T - (\sigma/e) \mathbf{E}, \quad (28)$$

where the various diffusion coefficients are given by the following formulas:
ordinary diffusion coefficient,

$$D = \left[\frac{2}{3} \tau \zeta_0 / m_1 - \rho_1 \rho (\partial p / \partial \rho_1)_{\rho T} \right];$$

$$D_\rho = \rho_1 \rho (\partial p / \partial \rho)_{\rho T};$$

thermodiffusion coefficient,

$$D_T = \left[\frac{2}{3} \rho_1 \tau \hat{c}_{v1} - \rho_1 \tau \rho (\partial p / \partial T)_{\rho \rho'} \right]; \quad (29)$$

electrical (particle) conductivity,

$$\sigma/e = n_1 \tau e / m_1.$$

To make numerical assessments, we express the derivatives of p in terms of those of p_1 and p_2 with the aid of the relation $\rho_2 = \rho - \rho_1$:

$$(\partial p / \partial \rho_1)_{T \rho} = (\partial p_1 / \partial \rho_1)_{T \rho} - (\partial p_2 / \partial \rho_2)_{T \rho};$$

$$(\partial p / \partial \rho)_{T \rho_1} = (\partial p / \partial \rho_2)_{T \rho};$$

$$(\partial p / \partial T)_{\rho \rho_1} = (\partial p_1 / \partial T)_{\rho_1} + (\partial p_2 / \partial T)_{\rho_2}.$$

Since lattice and electron bulk moduli are of comparable magnitude ($\sim 10^{12}$ dyn cm $^{-2}$), $(\partial p / \partial \rho_1)_{T \rho} \approx (\partial p_1 / \partial \rho_1)_{T \rho} \sim 10^{16}$ dyn cm g $^{-1}$, and $(\partial p_2 / \partial \rho)_{T \rho} \sim 10^{11}$ cm 2 sec $^{-2}$. On the other hand the derivative $(\partial p / \partial T)_{\rho \rho_1}$ is at constant composition, so ordinary static relations and values may be used:

$$\left(\frac{\partial p}{\partial T} \right)_\rho = - \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_\rho / \frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T,$$

which for silver is $\sim 10^6$ dyn cm $^{-2}$ °K $^{-1}$. We take also $n_1 \sim 10^{23}$ c m $^{-3}$, $\rho_1 \sim 10^{-4}$ g cm $^{-3}$, $\rho \sim 10$ g cm $^{-3}$, $\tau \sim 10^{-14}$ sec $^{-1}$, $\zeta_0 \sim 10^{-11}$ erg, $m_1 \sim 10^{-27}$ g, $e \sim 5 \times 10^{-10}$ esu, $\hat{c}_{v1} \sim 10^6$ erg g $^{-1}$. With these values $D \sim 10^2$ cm 2 g $^{-1}$ sec $^{-1}$, $D_\rho \sim 10^{-8}$ cm 2 g $^{-1}$ sec $^{-1}$, $D_T \sim 10^{-12}$ cm $^{-1}$ °K $^{-1}$, $\sigma \sim 10^{17}$ sec $^{-1}$.

EQUATIONS FOR THE SYSTEM AS A WHOLE

So far as the ions are concerned, we have no need for a constitutive equation for J_2 . The linear equation for $\mathbf{J}_{\epsilon 2}$ is then simply $\mathbf{J}_{\epsilon 2} = \kappa_2 \nabla T$, and the total internal-energy current is

$$\mathbf{J}_\epsilon = h_1 \mathbf{J}_1 - \kappa \nabla T, \quad (30)$$

where $\kappa = \kappa_1 + \kappa_2 =$ total thermal conductivity. κ_1 and κ_2 should be of the same order of magnitude. Similarly the scalar pressures p_1 and p_2 should be comparable. The ion lattice or phonon viscosity, on the other hand, has been estimated to be small compared to the electron viscosity.³

The five equations of change (6), (7'), (11), (3), (10), the various constitutive relations (27), (28), (29), (30), and the electromagnetic equations comprise a set which determine the nonsteady-state behavior of the simple two- (or three-) component model of a metal in the continuum dynamic approximation.

ULTRASONIC ATTENUATION

We now consider the propagation of a longitudinal, damped harmonic wave in first order. We write $\rho = \rho_0 + \rho'$, $\rho_1 = \rho_{10} + \rho_1'$, $T = T_0 + T'$, where primes indicate deviations from equilibrium. All such variations are taken proportional to $e^{i(\omega t - \chi x)}$, with $\chi = k - i\alpha$, α being the amplitude attenuation coefficient. The linear-

³ W. P. Mason, Phys. Rev. **97**, 557 (1955); W. P. Mason and H. E. Bommel, J. Acoust. Soc. Am. **28**, 930 (1956).

ized equations of change are

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u}{\partial x} = 0, \quad (31)$$

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} - \frac{4}{3} \eta \frac{\partial^2 u}{\partial x^2} = 0, \quad (32)$$

$$\rho_0 \hat{c}_v \frac{\partial T}{\partial t} + \frac{\partial J_\epsilon}{\partial x} + T_0 \left(\frac{\partial p}{\partial T} \right)_{\rho, \rho_1} - \epsilon_1 \frac{\partial J_1}{\partial x} = 0, \quad (33)$$

$$\frac{4\pi e}{m_1} \rho_1' - \frac{4\pi e}{m_2} \rho' + \frac{\partial E}{\partial x} = 0, \quad (34)$$

$$\frac{\partial \rho_1'}{\partial t} + \rho_{10} \frac{\partial u}{\partial x} + m_1 \frac{\partial J_1}{\partial x} = 0. \quad (35)$$

With the usual replacements $\partial/\partial x = -i\chi \partial^2/\partial x^2 = -\chi^2$, $\partial/\partial t = i\omega$, we obtain five homogeneous linear equations and set their determinant equal to zero to determine the nontrivial solutions. The resulting secular equation, the dispersion relation for the longitudinal wave, reduces to a cubic in χ^2 . It is awkwardly long, so we do not display it. The ratio of the imaginary to the real part of χ being very small ($\alpha/k \sim 10^{-8}$), the dispersion equation is easily solved by approximation. For "low" frequencies ($\omega \leq 10^6 \text{ sec}^{-1}$) the velocity has the familiar value in first order

$$v^2 \approx v_0^2 = \gamma \left(\frac{\partial p}{\partial \rho} \right)_{T, c}, \quad (36)$$

where γ is the specific-heat ratio. In arriving at (36), we have passed from derivatives of the function $p = p(\rho, \rho_1, T)$ to those of $p = p(\rho, c, T)$, where $c = \rho_1/\rho$ measures the composition, by the use of the relations

$$\begin{aligned} (\partial p / \partial \rho)_{\rho_1 T} + \rho_1 (\partial p / \partial \rho_1)_{\rho T} / \rho &= (\partial p / \partial \rho)_{c T}; \\ (\partial p / \partial T)_{\rho, \rho_1} &= (\partial p / \partial T)_{\rho c} \end{aligned}$$

and the thermodynamic identity

$$(\partial p / \partial T)_{\rho c} / (\partial p / \partial \rho)_{T, c} \rho^2 \hat{c}_v = (\gamma - 1).$$

In stating the amplitude attenuation coefficient we use the following definitions for brevity: the kinematic viscosity $\eta' = \eta/\rho$, thermometric conductivity $\kappa' = \kappa/\rho \hat{c}_v$, "rationalized" resistivity $R = 4\pi/\sigma$ and $H_1 \equiv \hat{h}_1 - \hat{\epsilon}_1 = 2\hat{\epsilon}_1/3$. The attenuation coefficient is then

$$\begin{aligned} \alpha &= \frac{\omega^2}{v_0^3} \left\{ \eta' + \kappa' \frac{(\gamma - 1)}{\gamma} \right\} + \frac{\omega^4}{v_0^5} \\ &\times \{ \eta' \kappa' R + \eta' D R + \kappa' D R + \kappa' D_\rho R + \eta' D_T H_1 R \}. \quad (37) \end{aligned}$$

For silver, in cgs units, $\eta' \sim 10^{-4}$, $\kappa' \sim 10^{-4}$, $\gamma - 1 \sim 10^{-3}$, $R \sim 10^{-16}$, $D \sim 10^2$, $D_\rho \sim 10^{-8}$, $D_T \sim 10^{-12}$, $H_1 \sim 10^{16}$. The viscous term of course is the only significant one; the thermal term is negligible because of the factor $(\gamma - 1)$. The various diffusion coefficients, including resistivity, enter in products of three and hence also contribute negligibly. For a neutral gas mixture on the other hand the dissipation due to diffusion enters in first order and may be comparable with viscous absorption.

Mason³ first proposed the shear viscosity of the free-electron gas as the source of low-temperature ultrasonic attenuation in metals and employed the standard attenuation formula for a one component system. The present analysis supports this procedure in the low-frequency limit, as did of course the more comprehensive microscopic treatments⁴ which followed that of Mason.

The relative phase of the motions of ions and electrons is such that the electron gas occupies the role of an electrically driven system and the ions that of the driving system. The method most frequently used to assess the acoustic attenuation has been, in fact, to compute the average rate at which electrical (and collisional) work is done on the electron gas, namely $\langle e \mathbf{J} \cdot \mathbf{E} \rangle$ [and $\langle \mathbf{u} \cdot n_1 m_1 (\mathbf{u}_1 - \mathbf{u}) / \tau \rangle$]. In the presence of diverse spatial gradients, this work is not, however, a simple joule dissipation, at least in the ordinary sense. To display the relation of such computation (in the low-frequency limit) to the viscosity calculation, we simply write a representative equation of motion of the driven electron gas (neglecting thermal conduction, etc., for simplicity):

$$\rho_1 \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial x} - \frac{4}{3} \eta \frac{\partial^2 u_1}{\partial x^2} = \rho_0 F_0 e^{i(\omega t - \chi x)}. \quad (38)$$

This is analogous to the equation of a driven oscillator, and of course it is easily verified that the work done on the electron gas by the driving force is equal in magnitude to the negative work done by the viscous force.

ACKNOWLEDGMENT

The author has had the pleasure of stimulating conversation with Professor T. Holstein and computational assistance from Charles Huff.

⁴ A. B. Pippard, *Phil. Mag.* **46**, 1104 (1955); R. W. Morse, *Phys. Rev.* **97**, 1716 (1955); M. S. Steinberg, *Phys. Rev.* **111**, 425 (1958); T. Holstein, Westinghouse Research Memo. 60-94698-3-m17, 1956 (unpublished); E. I. Blount, *Phys. Rev.* **114**, 418 (1959). For a more complete list of references, cf. the review article by H. Stoltz, *Phys. Status Solidi* **3**, 1153 (1963).