Origin of Unitary Symmetry and Charge Conservation in Strong Interactions

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We discuss the relation of the existence of multiplets of (strongly) interacting particles and the possible unitary symmetry of their interactions. We present here a dynamical principle which concerns the one-particle propagators (two-point functions) but yielding the existence of a (unitary) symmetry group for their trilinear interactions. We derive, as a by-product, electric charge (and hypercharge) conservation in the interaction of these particles.

I. INTRODUCTION

THE isotopic spin invariance of nuclear interactions is now well established.[1] Recent developments in particle physics have lent credence to the hypothesis of (approximate) invariance of strong interaction phenomena under the special unitary group in three dimensions with the eightfold way realization of the multiplets.[2] Also recently suggestions have been made that the special unitary group in four dimensions offers a further systematization of the particle multiplets.[3] It thus appears that the special unitary groups are specially singled out from among the continuous groups. If such is the case, there must be a fundamental reason for the unitary groups to be associated with strong interactions. This paper presents a primitive dynamical principle which leads to unitary symmetry and at the same time provides for the conservation of electric charge (and other additive quantum numbers of the second kind).

It is a striking feature of the strongly interacting particles that they occur in multiplets, with each member having the same spin and parity and (approximately) equal masses. If the particles can decay by virtue of strong interactions, their (reduced) widths for decay are also equal. Of course, if one assumes the existence of some basic symmetry group, these properties are immediate consequences of the requirement that the particle multiplet furnishes an irreducible representation of the symmetry group of strong interactions. Also, in such a case, the equality of reduced widths could be "analytically continued" to relate suitable sums of squared vertex operators. Such a technique forms the basis of the so-called Smushkevich method of determining the consequences of a symmetry group for decay widths, scattering cross sections, and electromagnetic properties.[4]

We might now ask ourselves the following question: Suppose that we do not assume the existence of a symmetry group *a priori*, but we assert that not only are the masses and spins of the various members of a multiplet equal, but also the total squared transition matrix elements into members of other multiplets. Then the propagators of each of the particles belonging to a multiplet are the same. Does this imply invariance of the interactions between the particles under a suitable continuous symmetry group?

We shall see below that within a suitable dynamical framework the question can be answered, and the answer is "yes." In view of the fundamental role played in this framework by the postulated equality of the propagators for members of a particle multiplet, we propose to elevate this postulate to the status of a dynamical principle, to be called the Smushkevich principle. We can formulate it more precisely as follows: If the members of a boson multiplet have the associated fields $\phi^\alpha(x)$, the Smushkevich principle asserts

$$\langle 0|T(\phi^\alpha(x)\phi^{+\beta}(y))|0\rangle=\delta^{\alpha\beta}\Delta_F{}^R(x-y). \qquad (1.1)$$

Similarly, if the members of a fermion multiplet have the associated fields $\psi^\alpha(x)$, then

$$\langle 0|T(\psi^\alpha(x),\bar{\psi}^\beta(y))|0\rangle=\delta^{\alpha\beta}S_F{}^R(x-y). \qquad (1.2)$$

Because of the well-known relations connecting the spectral function of these two-point functions with the mass renormalization constant and with the physical mass, it follows that the masses and self-masses of the various members of a multiplet are equal.

In discussing the additive conservation laws for strong interactions, we encounter two kinds of additive quantum numbers. An additive quantum number of the first kind has the same value for each member of an irreducible multiplet; each multiplet is associated with a fixed value for each of these quantum numbers. The

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[1] See, for example, K. Nishijima, *Fundamental Particles* (W. A. Benjamin, Inc., New York, 1963), p. 66.

[2] For a recent review, Y. Néeman, Proceedings of the International Conference on Nucleon Form Factors, Stanford University, June 1963 (to be published).

[3] P. Roman, Boston University (unpublished); P. Tarjanne and V. L. Teplitz, Phys. Rev. Letters **11**, 441 (1963).

[4] E. C. G. Sudarshan, in *Proceedings of the Athens Topical Conference on Newly Discovered Resonant Particles, Athens, Ohio, 1963* (University of Ohio, Athens, Ohio, 1963).

most relevant example is the baryon number. On the other hand, an additive quantum number of the second kind, like electric charge or hypercharge, has different values for different members of a multiplet. We shall see below that we can *derive* the conservation law for additive quantum numbers of the second kind within our dynamical framework.

It is to be noted that for both boson and fermion multiplets, we may make an arbitrary unitary transformation of the particles belonging to a multiplet. This is tantamount to a redefinition of the "particles" constituting the multiplet. Under such a transformation, the additive quantum numbers of the first kind are unaltered; and the Smushkevich equations are unaltered, which is as it should be. Explicit use is made of this circumstance in the sequel.

In the following section we illustrate the general method by considering the pion-nucleon system. Here, as well as in the general case, we shall assume a trilinear interaction involving two multiplets with n members each, and one multiplet with n^2-1 members. The pion-nucleon system corresponds to the choice $n=2$, and we then deduce the invariance of the interaction under SU_2. In the following section we generalize this proof to deduce invariance under SU_n. The paper concludes with some comments on the primitive entities in the eightfold way realization of the SU_3 symmetry of strong interactions, and on the connection of the Smushkevich principle with the Smushkevich method in strong interaction physics.[5,6]

II. CHARGE INDEPENDENCE OF STRONG INTERACTIONS

In this section we wish to derive the charge independence (SU_2 invariance) of the pion-nucleon interaction (of the Yukawa type) from the Smushkevich principle *without assuming charge conservation.* We write the trilinear interaction in the form[7] (suppressing gamma matrices):

$$H_{\text{int}}=f_{rs}{}^{\alpha}N_r{}^{\dagger}N_s\pi^{\alpha}, \tag{2.1}$$

where summation over the repeated indices r, s, α is implied; r and s take on two values and α takes on three values. No generality is lost by taking the pion field to be Hermitian. Hermiticity of the Hamiltonian (2.1)

FIG. 1. Pion diagrams.

then requires

$$(f_{rs}{}^{\alpha})^{*}=f_{sr}{}^{\alpha}. \tag{2.2}$$

We can now introduce a great deal of simplification in the formalism by considering the quantities $f_{rs}{}^{\alpha}$ as a set of $n\times n$ matrices f^{α}. By Eq. (2.2), these matrices are Hermitian. Then, for the meson propagators computed in perturbation theory, the Smushkevich principle yields a series of relations of the form:

$$\operatorname{tr}(f^{\alpha}f^{\beta})=A_1\delta^{\alpha\beta}, \tag{2.3a}$$

$$\operatorname{tr}(f^{\alpha}f^{\gamma}f^{\beta}f^{\gamma})=A_2\delta^{\alpha\beta}, \tag{2.3b}$$

$$\operatorname{tr}(f^{\alpha}f^{\gamma}f^{\delta}f^{\beta}f^{\gamma}f^{\delta})=A_3\delta^{\alpha\beta},\ \text{etc.} \tag{2.3c}$$

As before, the summation over repeated indices is understood. These terms correspond to the propagator contributions from the diagrams indicated in Fig. 1. Similarly, by considering the nucleon propagators, we obtain relations of the type:

$$f^{\alpha}f^{\alpha}=B_1I, \tag{2.4a}$$

$$f^{\alpha}f^{\beta}f^{\alpha}f^{\beta}=B_2I, \tag{2.4b}$$

$$f^{\alpha}f^{\beta}f^{\gamma}f^{\alpha}f^{\beta}f^{\gamma}=B_3I,\ \text{etc.} \tag{2.4c}$$

(where I is the $n\times n$ unit matrix), corresponding to the propagator contributions from the diagrams shown in Fig. 2.

Before embarking on the solution of these equations, we note that in any case the f^{α} will be undetermined up to the following two types of transformations:

(i) A unitary transformation

$$f^{\alpha}\to f'^{\alpha}=Uf^{\alpha}U^{-1} \tag{2.5}$$

in the space of the N_r.

(ii) A real unitary (orthogonal) transformation

$$f^{\alpha}\to f''^{\alpha}=V^{\alpha\beta}f^{\beta} \tag{2.6}$$

in the space of the π^{α}. In each case the corresponding linear transformations on the boson and fermion fields preserve Eqs. (1.1) and (1.2), as discussed in the introduction, as well as Eqs. (2.3) and (2.4).

Our aim will be to combine this freedom with the Smushkevich equations (2.3) and (2.4) to deduce that the f^{α} are proportional to the isotopic spin matrices τ^{α}.

We begin by using the transformation (2.6) to make

$$\operatorname{tr}(f^2)=\operatorname{tr}(f^3)=0,$$

and the transformation (2.5) to diagonalize the traceless

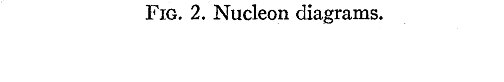

FIG. 2. Nucleon diagrams.

[5] I. M. Smushkevich, Doklady Akad. Nauk SSSR. **103**, 235 (1955); A. J. Macfarlane, G. Pinski, and E. C. G. Sudarshan, (to be published); see also, P. Roman, *Elementary Particles* (North-Holland Publishing Company, Amsterdam, 1960); R. E. Marshak and E. C. G. Sudarshan, *Introduction to Elementary Particle Physics* (Interscience Publishers, Inc., New York, 1961).

[6] The extension of the Smushkevich method to invariance under arbitrary groups has been discussed in C. Dullemond, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. Letters **10**, 423 (1963); A. J. Macfarlane, N. Mukunda, and E. C. G. Sudarshan, Phys. Rev. **133**, B475 (1964); J. Math. Phys. **5**, 576 (1964); M. E. Mayer, *Lectures on Strong and Electromagnetic Interactions* (Brandeis University Press, Waltham, Massachusetts, 1963), Vol. 1.

[7] H. Yukawa, Proc. Phys. Math. Soc. Japan **17**, 48 (1935).

Hermitian matrix f^3 in the form

$$f^3=g\tau^3. \tag{2.7}$$

At this point we make our first use of the Smushkevich equation (2.3a) and the tracelessness of f^2 to obtain

$$f^2=g_1\tau^1+g_2\tau^2;\quad g_1^2+g_2^2=g^2.$$

By suitable transformation of the kind (2.5), we can retain Eq. (2.7), but cast f^2 in the form

$$f^2=g\tau^2. \tag{2.8}$$

A further use of (2.3a), together with (2.7) and (2.8), gives

$$f^1=g_1'\tau^1+g_0I;\quad g_1'^2+g_0^2=g^2.$$

Here the term containing the unit matrix I appears because f^1 is not necessarily traceless. We may now use Eq. (2.4a) to deduce that

$$g_0g_1'=0,$$

so that either g_0 or g_1' must vanish. Use of the Smushkevich equation (2.3b) eliminates the possibility that g_1' can vanish, so that we have

$$f^1=\pm g\tau^1. \tag{2.9}$$

If we now consider the real orthogonal transformation

$$\pi^1\rightarrow\pm\pi^1;\quad \pi^2\rightarrow\pi^2;\quad \pi^3\rightarrow\pi^3,$$

we finally obtain

$$f^\alpha=g\tau^\alpha,$$

as required; i.e., the interaction Hamiltonian now assumes the familiar charge-independent form[8]:

$$H_{\text{int}}=gN^\dagger\tau^\alpha\phi^\alpha N. \tag{2.10}$$

We have thus established the charge independence of the pion-nucleon interaction.[9] It is important to note that we have *not assumed* charge conservation in this derivation. We may now *deduce* the conservation of electric charge if it is defined as a linear sum of the "third" component of isotopic spin and half the baryon number.

We might now ask whether the strange-particle interactions are also charge-independent. Clearly, the cascade hyperon-pion system behaves in just the same way. The nucleon–kaon–Σ-hyperon system behaves in essentially the same way, except that the triplet of Σ fields may not be taken Hermitian. But what about Σ-hyperon–pion system for which all indices take on three values? It turns out that for this system the method fails, since a coupling scheme satisfying the Smushkevich equations (2.3) and (2.4) can be devised, which violates charge independence. For the nucleon–kaon–Σ-hyperon system, the preceding analysis does not apply directly, but it can be adapted to deduce charge independence (see Sec. III below).

We are then led to suggest that in a theory where charge independence is the highest symmetry of strong interactions, only the nucleon-pion, cascade hyperon-pion, nucleon–kaon–Σ-hyperon and cascade hyperon–kaon–Σ-hyperon trilinear couplings are fundamental, the other couplings being induced effects. It is interesting to note that the singlet Λ hyperon does not enter any of these reactions. Of course, if charge independence is a consequence of a larger symmetry group, these restrictions do not apply; they are replaced by other conditions.

In concluding this section, we point out that once charge independence is deduced, all the equations (2.3) and (2.4) are automatically satisfied.

III. UNITARY SYMMETRY

Consider the derivation of SU_n invariance for a system consisting of two multiplets E and F containing n particles, each coupled trilinearly to a multiplet ϕ containing n^2-1 particles. We may write the effective interaction in the form

$$H_{\text{int}}=C_{rs}{}^\alpha E_r{}^\dagger F_s\phi^\alpha+(C_{sr}{}^\alpha)^*F_r{}^\dagger E_s\phi^{\dagger\alpha}. \tag{3.1}$$

Once again, summation over repeated indices is implied, and we regard $C_{rs}{}^\alpha$ as elements of matrices C^α. Note, however, that the matrices C^α are in general not Hermitian, since E and F are distinct. If the interaction is invariant under SU_n, it could be cast in the form

$$H_{\text{int}}=gX_{rs}{}^\alpha\{E_r{}^\dagger F_s\phi^\alpha+F_r{}^\dagger E_s\phi^{\dagger\alpha}\},$$

where X^α are the (normalized) Hermitian generators of SU_n. Without loss of generality, we may normalize X^α by the relation

$$\operatorname{tr}(X^\alpha X^\beta)=n\delta^{\alpha\beta}. \tag{3.2}$$

In the case $E\equiv F$, the Smushkevich equations satisfied by the C^α may be written in the form

$$\operatorname{tr}(C^\alpha C^\beta)=A_1\delta^{\alpha\beta}=nG^2\delta^{\alpha\beta}, \tag{3.3a}$$

$$\operatorname{tr}(C^\alpha C^\gamma C^\beta C^\gamma)=A_2\delta^{\alpha\beta}, \tag{3.3b}$$

$$\operatorname{tr}(C^\alpha C^\gamma C^\delta C^\beta C^\gamma C^\delta)=A_3\delta^{\alpha\beta},\ \text{etc.} \tag{3.3c}$$

and

$$C^\alpha C^\alpha=B_1I=(n^2-1)G^2I, \tag{3.4a}$$

$$C^\alpha C^\beta C^\alpha C^\beta=B_2I, \tag{3.4b}$$

$$C^\alpha C^\beta C^\gamma C^\alpha C^\beta C^\gamma=B_3I,\ \text{etc.} \tag{3.4c}$$

As in the pion-nucleon case, we have the possibility of making the transformations

$$C^\alpha\rightarrow C'^\alpha=UC^\alpha U^{-1}, \tag{3.5}$$

$$C^\alpha\rightarrow C''^\alpha=V^{\alpha\beta}C^\beta, \tag{3.6}$$

[8] H. Fröhlich, W. Heitler, and N. Kemmer, Proc. Roy. Soc. (London) **166**, 154 (1938); N. Kemmer, Proc. Cambridge Phil. Soc. **34**, 354 (1938).

[9] M. Grisaru has shown that it is possible to derive charge independence for the $NN\pi\pi$ coupling using the Smushkevich principle. We thank Professor Grisaru for communicating this result to us prior to publication.

where U and V are unitary $n\times n$ and $(n^2-1)\times(n^2-1)$ dimensional matrices, respectively. Our aim is now to use Eqs. (3.3), (3.4), (3.5), and (3.6) to show that

$$C^\alpha = GX^\alpha.$$

We can make a transformation of the type (3.6) to make the traces of all the C^α vanish, except (possibly) that of C'. The coupling matrices now take the form

$$C^\alpha = a^{\alpha\mu}X^\mu + (t/n)\delta^{\alpha 1}I, \tag{3.7}$$

where t is the trace of C^1.

Substituting (3.7) in (3.3a), and taking account of the tracelessness of X^μ, we get

$$nG^2\delta^{\alpha\beta} = na^{\alpha\mu}(a^{\beta\nu})\delta^{\mu\nu} + (t^2/n)\delta^{\alpha\beta}\delta^{\alpha 1},$$

so that

$$a^{\alpha\mu}(a^{\beta\mu}) = [G^2-(t/n)^2\delta^{\alpha 1}]\delta^{\alpha\beta}.$$

Let us define

$$b^{\alpha\mu} = \{G^2-(t/n)^2\delta^{\alpha 1}\}^{-\frac{1}{2}}a^{\alpha\mu},$$

so that

$$b^{\alpha\mu}(b^{\beta\mu}) = \delta^{\alpha\beta} \tag{3.8}$$

and

$$c^\alpha = \{G^2-(t/n)^2\delta^{\alpha 1}\}^{\frac{1}{2}}Y^\alpha + (t/n)\delta^{\alpha 1}I, \tag{3.9}$$

with

$$Y^\alpha = b^{\alpha\mu}X^\mu. \tag{3.10}$$

The Y^α so defined satisfy Eqs. (3.3) and (3.4) by virtue of the properties of X^α. We have, in particular,

$$\mathrm{tr}(Y^\alpha Y^\beta) = n\delta^{\alpha\beta};$$
$$\mathrm{tr}(Y^\alpha Y^\gamma Y^\beta Y^\gamma) = n'\delta^{\alpha\beta};$$
$$Y^\gamma Y^\gamma = (n^2-1)I.$$

From these equations we can show that

$$\mathrm{tr}([Y^\alpha,Y^\gamma][Y^\beta,Y^\gamma]) = -k^2\delta^{\alpha\beta}$$

where k^2 is a non-negative constant. Putting successively $\alpha=\beta=1$ and $\alpha=\beta=2$, we get

$$\sum_{\gamma=3}^{n^2-1}\mathrm{tr}([Y^1,Y^\gamma][Y^1,Y^\gamma]) = \sum_{\gamma=3}^{n^2-1}\mathrm{tr}([Y^2,Y^\gamma][Y^2,Y^\gamma]). \tag{3.11}$$

On the other hand, from Eqs. (3.3) and (3.4), we can deduce

$$\sum_{\gamma=3}^{n^2-1}\mathrm{tr}([C^1,C^\gamma][C^1,C^\gamma]) = \sum_{\gamma=3}^{n^2-1}\mathrm{tr}([C^2,C^\gamma][C^2,C^\gamma]).$$

Since C^1 enters only in the commutators on the left-hand side of this equation, the multiple of the unit matrix in (3.9) does not contribute. We can thus rewrite the above equation in the form

$$G^2\{G^2-(t/n)^2\}\sum_{\gamma=3}^{n^2-1}\mathrm{tr}([Y^1,Y^\gamma][Y^1,Y^\gamma]) = G^4\sum_{\gamma=3}^{n^2-1}\mathrm{tr}([Y^2,Y^\gamma][Y^2,Y^\gamma]). \tag{3.12}$$

Now, the traces occurring in Eqs. (3.11) and (3.12) are all negative definite. Hence, on comparing Eqs. (3.11) and (3.12), we deduce that

$$t=0.$$

Consequently, Eq. (3.9) becomes

$$C^\alpha = GY^\alpha.$$

If we now use the real unitary transformation

$$\phi^\alpha = b^{\alpha\mu}\chi^\mu,$$

we may rewrite the interaction (3.1) in the form

$$H_{\mathrm{int}} = GX_{rs}{}^\alpha E_r{}^\dagger E_s\chi^\alpha + \mathrm{H.c.},$$

which is the required SU_n invariant form. We have thus deduced unitary symmetry for trilinear interactions from the Smushkevich principle. The extension to the case $E\not\equiv F$ will be given in a forthcoming publication by H. Leutwyler.

IV. DISCUSSION

It is gratifying to see that among symmetry groups of rank two, the dynamical framework considered above singles out SU_3. However, there are two circumstances that ought to be considered. First, none of the triplet representations of SU_3 has been discovered experimentally to date; secondly, the SU_3 symmetry is not exact, but is only approximate. The apparent nonexistence of the triplets may be accounted for by assuming that they are very heavy in mass. The eight-component multiplets may be taken to be the pseudoscalar meson octet comprising pions, kaons, antikaons, and the eta; or the corresponding vector-meson octet.

If we choose a theory with only one fundamental triplet (and its distinct antiparticle triplet), the quanta of these fields will have to have nonintegral values of baryon number and electric charge. On the other hand, we may choose two triplets, one with baryon number zero and one with baryon number one. Even in this case (unless new conservation laws are postulated), the electric charge would have fractional values. These entities would then obey an associated production rule, and could not decay into ordinary particles (with integral electric charges).

While such entities have been discussed recently in related contexts,[10] in the present framework there is a

[10] M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig, CERN (unpublished); C. R. Hagen and A. J. Macfarlane, University of Rochester (unpublished). Some work of Biedenharn and Fowler and of Baird and Biedenharn are referred to in the paper of Hagen and Macfarlane.

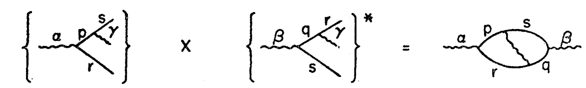

FIG. 3. Illustrating the relation of Smushkevich's method and the Smushkevich principle.

primitive eightfold multiplet which participates in the primitive trilinear interaction. This entails introducing a larger number of primitive entities than in the formulation in which the symmetry group is postulated; but on the other hand, the present work derives the symmetry from "first" principles. Note that one of the triplets may be a baryon triplet, and the other one a meson triplet together with a baryon octet; for example, in the SU_2 case, we could consider the nucleon–kaon–Σ-hyperon coupling.

We must also take into account the breaking of the unitary symmetry. A clue to the possible violation of the symmetry in the Smushkevich framework is provided by the structure of the one-particle propagator which is susceptible to spontaneous symmetry violation, either from mass differences or from the lack of symmetry of the ground state.[11] But a quantitative study of these effects requires dynamical calculations going beyond the algebraic techniques used here.

The Smushkevich principle used here is rather intimately related to the Smushkevich method in strong interaction physics.[4,6] Consider, for example, the amplitude for the (virtual) process

$$\pi^\alpha \to N_r + \bar{N}_s + \pi^\beta .$$

Smushkevich equations for the π^α include the statement

$$\sum_{r\bar{s}\beta} M_{\alpha\to r\bar{s}\beta} M^*{}_{\alpha'\to r\bar{s}\beta} = \Gamma\delta_{\alpha\alpha'} ;$$

but in the framework of trilinear interactions (and use of perturbation theory) we have the additional result that

$$M_{\alpha\to r\bar{s}\beta} \sim M_{\alpha\to r\bar{t}} M_{\beta\to s\bar{t}}^* ,$$

so that

$$M_{\alpha\to r\bar{s}\beta} = \zeta C_{rt}{}^{\alpha} C_{ts}{}^{\dagger\beta} .$$

The Smushkevich equation for the production process now coincides with Eq. (3.3b), as illustrated in Fig. 3. Similar comments apply to the other propagator diagrams as well.

We must also discuss the relation of the present work with a more limited application[12] in which charge (and hypercharge) conservation is imposed at the start. In this case the number of coupling constants are smaller, but so are the number of useful equations, since most of the Smushkevich equations become identities. The previous demonstrations of charge independence of pion-nucleon system required the postulate of charge conservation. For the Σ-hyperon–pion system, for which, as mentioned above, the Smushkevich principle fails to yield SU_2 invariance if used alone, the Smushkevich principle is successful if we use charge conservation as well. But with sufficiently high multiplets, either method would fail; and the reason is simple. If charge conservation is imposed, for trilinear interactions, the number of coupling constants increases as the second power of the multiplicity, but the number of useful Smushkevich equations increase linearly with the number of components of the multiplets. Without any such constraints, the number of coupling constants increases as the third power of the number of components, while the number of useful Smushkevich equations increase as the square. In view of this, it is curious to observe that usually only the lower-lying multiplets are in practice realized.

Some other comments are in order. With strong interactions, one may be skeptical about the relevance of using algebraic relations deduced by considering perturbation diagrams. However, it is to be noted that we do not use the perturbation-theoretic estimates for the actual amplitudes, but only their dependence on the "internal" labels. What is even more to the point is that similar equations are obtained as self-consistency relations in the strong coupling limit. We may think of the Smushkevich equations as reflecting the self-consistency of the trilinear vertex and the orthogonality and completeness of "wave functions" of members of a multiplet considered as bound states of members of the other two multiplets. In the same spirit, we may also think of the trilinear interactions between the three multiplets with n, n, and n^2-1 members as itself being caused by the direct coupling of four multiplets with n members each, which leads to n^2-1 bound states. Perhaps these considerations are of relevance to the theory of strongly interacting particles.

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[11] The question of broken symmetries is discussed by several authors in the *Proceedings of the Seminar on Unified Theories of Elementary Particles*, edited by D. Lurié and N. Mukunda (University of Rochester Press, Rochester, 1963).

[12] J. J. Sakurai, Phys. Rev. Letters **10**, 446 (1963). In Sakurai's demonstration of charge independence of the pion-nucleon Yukawa interaction, he makes use of the conservation of electric charge explicitly. In his demonstration of SU_3 invariance, he imposes charge independence (and charge conjugation invariance) for the isotopic multiplets. But in such a framework, where the meson-octet components are taken to be degenerate in masses, we cannot *derive* charge independence from "first" principles using his method. In the present work on the interaction of two triplets and an octet, we do *not impose* charge independence, but *derive* it as a consequence of SU_3 invariance. See also the derivation of charge independence for pion-nucleon interaction by Fröhlich, Heitler, and Kemmer, Ref. 8.