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Three-Body Scattering Amplitude. II. Extension to Complex Angular Momentum*

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We have analyzed the problem of extending the three-body scattering amplitude to complex values of the total angular momentum J. We have found four main difficulties: (i) the disconnectedness of the collision matrix; (ii) the complexity of kinematics; (iii) the release of triangular inequalities or of inequalities like $|M|<J$, which, when J is complex, transforms finite sums into infinite sums which are most often divergent; and (iv) the presence of complex singularities in cosine angle variables in the full amplitude. This last difficulty is not examined in the present paper. We propose a generalization of the Froissart-Gribov formula for the three-body scattering amplitude. In the nonrelativistic problem, the use of the Fadeev equations takes care of difficulty (i), and difficulty (ii) is smoothed by the use of center-of-mass energies of the three particles and the total angular momentum as the only variables. Of the three natural techniques—using the Schrödinger equation, extending the Fadeev equations, and extending the Fredholm solution of the Fadeev equations to complex values of J—only the third one avoids difficulty (iii). We prove that the Fadeev equations cannot be extended because their kernel becomes unbounded and because they do not reduce to the physical equations when J reaches a physical value. However, the Fredholm solution for physical J can be formally extended to complex J, and the extended solution is expressed as the quotient of two Fredholm-type series, where each term of the series is analytic in J in a right half-plane. The Sommerfeld-Watson series never converges for the three-particle scattering amplitude, because of difficulties (iii) and (iv).

I. INTRODUCTION

THIS paper is devoted to an extension of the nonrelativistic three-body scattering amplitude to complex values of the total angular momentum J. It is admittedly of an exploratory character, without any attempt at mathematical rigor.

Regge has shown that the two-body scattering amplitude is a meromorphic function of the total angular momentum.[1] This result has been proved for the scattering of two nonrelativistic structureless particles interacting through a sufficiently well-behaved potential, for instance, a superposition of Yukawa potentials. It is then found that the poles in J of the partial-wave amplitudes, or Regge poles, control the asymptotic behavior of the scattering amplitude when the momentum transfer tends to infinity. Furthermore, the functions which give the positions of these poles in terms of the total energy interpolate the bound states and resonances of the system. In other words, a Regge trajectory corresponds to well-defined internal quantum numbers.

These two properties are so fundamental that it is necessary to know their degree of validity. Of particular importance is their possible extension to the relativistic case.[2,3] However, the relativistic problem contains so many new features, like inelasticity and crossing, that up to now it has yielded very few results.[4] The most interesting such result is the discovery by Mandelstam that there cannot be only Regge poles but there also must be cuts which, ultimately, govern the asymptotic behavior of the scattering amplitude when the energy tends to infinity.[5,6]

It is therefore interesting to investigate the nonrelativistic three-body problem for which, at least, we know a complete formulation. One can hope that it includes most of the inelasticity problems of the relativistic case, without the complications due to crossing. The interesting questions are, obviously:

(a) Is it possible to extend the scattering amplitude for three particles going into three particles as a function of complex total angular momentum J?

(b) Is this extended amplitude a meromorphic function of J?

(c) Do the singularities of the extended amplitude control the asymptotic behavior of the total three-body scattering amplitude when some momentum transfer tends to infinity? Or, in other words: is it possible to define a Sommerfeld-Watson transformation of the expansion in partial waves of the three-body scattering amplitude?[7] A tentative answer to these questions is the subject of this paper.

There are also other important questions, like:

(d) Are the Regge results valid for the scattering of a particle on a bound state?[8]

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[3] S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2206 (1962).

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[6] J. Polkinghorne, J. Math. Phys. **4**, 503 (1963); **5**, 431 (1964).

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[8] B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962). The cuts investigated by these authors appear in the angular momentum of a crossed channel, and we canot expect to find them by a completely nonrelativisitc approach.

(e) Is it possible to define either a partial-wave amplitude or a total amplitude for the scattering of a particle on a Regge pole? Or, in other words: does the interpolation property of Regge poles apply only to virtual particles or can it be extended to external particles?

(f) Is it possible to build up a general theory of the sense and nonsense channels for the three-particle systems?[9]

Although we feel that question (f) can be answered in the affirmative, we have not yet any definite result on this subject.

In order to investigate the two-particle problem, two methods have been used.

(a) Regge has extended the Schrödinger equation to complex values of J.[1,10]

(b) Brown, Fivel, Lee, and Sawyer have extended the Lippmann-Schwinger equation to complex J.[11] In fact, one could propose an alternative method, namely:

(c) Extend the Fredholm solution of the Lippmann-Schwinger equation. While the distinction between methods (b) and (c) is purely academic in the two-particle case, it will turn out to be significant in the three-particle case.

Before choosing to use any of these methods for the three-body problem, let us try to see what essentially new difficulties we shall encounter:

(i) The collision matrix consists of a connected and a disconnected part.[12] This is because two of the three particles can collide without suffering any interaction with the third one.

(ii) The kinematics is much more complicated.

(iii) Although these two difficulties are already met in actual physical problems, a third one will appear when J is made complex. The reason is that relations like the triangular inequalities between coupled angular momenta,

$$|l_1-l_2|\leq J\leq l_1+l_2,$$

or the inequalities $|M|<J$ between eigenvalues of J_z and J, are no more true when J becomes complex. Consequently, certain finite summations on M or on angular momenta involving Clebsch-Gordan coefficients become infinite sums and lead to convergence problems.[13,14]

(iv) The existence of several momentum transfers between the initial and the final state leads to complex singularities in any one of the cosine angles linked with one momentum transfer. This also leads to difficulties of convergence.

These four points seem, up to now, to be the only existing difficulties. Let us review the three possible methods (a), (b), and (c) in this light. (a) Newton[15] and Hartle[16] have tried to extend the three-particle Schrödinger equation to complex values of J. In fact, the differential-equation formulation of the problem completely conceals the difficulty of disconnectedness. Furthermore, their treatment of the kinematics leads them deeply into the mentioned problems of convergence. It seems very unlikely that one will be able to give any well-justified statement by using this approach.

(b) As was mentioned in the preceding paper, the difficulty of disconnectedness implies that the Lippmann-Schwinger equation for the three-particle problem is not of the Fredholm type.[17] However, the Watson equations for multiple scattering[18] are of the Fredholm type, as was shown by Fadeev, who rediscovered them independently.[19] This removes difficulty (i).[20]

In order to avoid the third difficulty, it seems advisable not to use any unnecessary partial angular momentum. This was made explicit in the preceding paper, where we proposed to use as a complete set of commuting variables the center-of-mass energies of the three particles ω_1, ω_2, and ω_3, the total angular momentum J, and its projections on a body-fixed axis M and on a space-fixed axis m.[21] In fact, m, being a trivial constant of the motion, will never enter into the equations. This was a well-defined answer to difficulty (ii). Of course, other choices are possible. For instance, one could replace ω_1, ω_2, and ω_3 by their sum $E=\omega_1+\omega_2+\omega_3$ and introduce a complete set of orthogonal functions of ω_1/E, ω_2/E, and ω_3/E.[22] Although this new choice seems to be advisable in order to discuss the problem of sense and nonsense channels, it will not be used here.

The Fadeev equations, considered as integral equa-

[9] M. Gell-Mann, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 533.

[10] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).

[11] L. Brown, D. I. Fivel, B. W. Lee, and R. F. Sawyer, Ann. Phys. (N. Y.) **23**, 187 (1963).

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[13] J. Gunson, University of Birmingham, 1962 (unpublished).

[14] R. L. Omnes, Institut des Hautes Etudes Scientifiques, 1963 (unpublished).

[15] R. G. Newton, Nuovo Cimento **29**, 400 (1963); Phys. Letters **4**, 11 (1963).

[16] J. B. Hartle, Phys. Rev. **134**, B620 (1964).

[17] For a discussion of these points, see: L. D. Fadeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; C. Lovelace, in *Lectures at the 1963 Edinburgh Summer School*, edited by R. G. Moorhouse, (Oliver and Boyd, London, to be published); also L. L. Foldy and W. Tobocman, Phys. Rev. **105**, 1099 (1957).

[18] K. M. Watson, Phys. Rev. **105**, 1388 (1957).

[19] L. D. Fadeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk. SSSR **138**, 565 (1961); **145**, 301 (1962) [English transls.: Soviet Phys.—Doklady **6**, 384 (1961); **7**, 600 (1962)]. Also Publications of the Stoklov Mathematical Institute No. 69 (1963). We thank Dr. J. B. Sykes for sending us his translation of the last paper.

[20] See also in this respect S. Weinberg, Phys. Rev. **133**, B232 (1964).

[21] R. L. Omnes, Phys. Rev. **134**, B1358 (1964), hereafter referred to as I.

[22] For a very interesting particular case of this tecnhique, see A. J. Dragt, Institute for Advanced Study, Princeton, 1964 (to be published).

tions for the three-particle scattering amplitude, imply a summation over the helicities M from $-J$ to $+J$. Therefore, when extended to complex values of J, the Fadeev equations will contain a summation on M from $-\infty$ to $+\infty$. It could therefore happen, and indeed we shall show it in the following, that the extended Fadeev equations make no sense according to difficulty (iii).

(c) It is clear that if we are able to find an extension to complex J of both the numerator and the denominator of the Fredholm solution of the Fadeev equations with a good choice of kinematics, we shall at least have got a sensible approach to the problem, free at least of the first three difficulties.

This paper is essentially devoted to defining such an extension of the Fredholm solution. No serious attempt has been made to investigate the analyticity properties of the extended Fredholm solution so that, from a mathematical standpoint, we can just claim to have stated the problem in a form amenable to analysis. However, there is no mention of difficulty (iv) in the present paper. It is our belief that this difficulty is the real problem and we expect to say more about it in a future paper.

In Sec. II, we give the expression of the complete three-body scattering amplitude $\langle \mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'|T|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\rangle$ as a series of partial-wave amplitudes $\langle \omega_1',\omega_2',\omega_3', M'|T^J|\omega_1,\omega_2,\omega_3,M\rangle$. This relation is closely analogous to the expansion of the two-body amplitude in terms of partial-wave amplitudes, except that rotation matrices[23] $\mathfrak{D}_{M'M}{}^J(R)$ replace the Legendre polynomials $P_J(x)$. In fact, we need to know the $\mathfrak{D}_{M'M}{}^J(R)$ as well as we know the $P_J(x)$. Therefore, in Sec. III and Appendix I, we investigate the properties of the $\mathfrak{D}_{M'M}{}^J(R)$ when J is complex, and we define second-kind rotation matrices $\mathcal{E}_{M'M}{}^J(R)$ which bear the same relation to the $\mathfrak{D}_{M'M}{}^J(R)$ as the Legendre functions of the second kind, $Q_J(x)$, bear to the Legendre functions of the first kind, $P_J(x)$. In particular, the well-known Neumann theorem, which allows one to express the Legendre coefficient of an analytic function either as an integral

$$\int_{-1}^{+1} f(x)P_J(x)dx$$

or a contour integral[24]

$$\frac{1}{2\pi i}\oint f(x)Q_J(x)dx,$$

is extended to express the "$\mathfrak{D}^J$ coefficient" of an analytic function on the rotation group either as

$$\int f(R)\mathfrak{D}_{M'M}{}^J(R)dR \quad \text{or} \quad \frac{1}{2\pi i}\oint f(R)\mathcal{E}_{M'M}{}^J(R)dR.$$

In Sec. IV, we show how this result could be used in the so-called axiomatic S-matrix theory[25] to define the extension of the three-body partial-wave amplitudes to complex values of J, and this leads us to a general formulation of the notion of signature.[3]

In Sec. V, we recall the fundamental facts about the Fadeev equations as stated in a preceding paper. In Sec. VI, the inhomogeneous term of these equations is extended to complex values of J, by use of the generalization of the Neumann theorem. For this we need to know the analytic and asymptotic properties of the off-the-energy-shell two-body scattering amplitude which are investigated in Appendix II and applied to the domain of definition of the inhomogeneous term. In Sec. VII we show that, while it is possible to extend the inhomogeneous term and the kernel of the Fadeev equations to complex values of J, it is impossible to extend the equations themselves. In Sec. VIII we show how to extend the Fredholm solution. Finally, in Sec. IX we show that no Sommerfeld-Watson transformation can be used in the three-body problem because of the infinite values assumed by the helicities.

II. PARTIAL-WAVE EXPANSIONS

Let us define the partial-wave expansion of the three-particle scattering amplitude. To the initial and final sets of momenta in the total center-of-mass system, $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ and $(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$, we attach a well-defined set of body-fixed axes, say S and S'. According to the relation (valid in the total c.m. system)

$$\begin{aligned}\langle \mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3|\mathbf{P}\omega_1\omega_2\omega_3 JMm\rangle &= [(2J+1)(2\pi)^6/m_1m_2m_3 8\pi^2]^{1/2}\\ &\quad\times\delta(\omega_1-p_1{}^2/2m_1)(\delta(\omega_2-p_2{}^2/2m_2)\\ &\quad\times\delta(\omega_3-p_3{}^2/2m_3)\delta(\mathbf{P})(2\pi)^3\mathfrak{D}_{mM}{}^{J*}(\hat{S}),\end{aligned}\quad (2.1)$$

where $\hat{S}$ represents the rotation which carries the space-fixed set of axes into S, one gets

$$\begin{aligned}\langle \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'|T|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle &= \int \langle \mathbf{p}'|\mathbf{P}\omega' JM'\rangle d^3P'd^3\omega'\langle\omega' JM'|T|\omega JM\rangle\\ &\qquad\times d^3Pd^3\omega\langle\omega JM|\mathbf{p}\rangle\\ &= \sum_{JMM'} (1/8\pi^2 m_1m_2m_3)(2J+1)\\ &\qquad\times\langle\omega' JM'|T|\omega JM\rangle\mathfrak{D}_{M'M}{}^J(R),\end{aligned}\quad (2.2)$$

where R is the rotation $\hat{S}\hat{S}'^{-1}$ which carries S into S'.

Equation (2.1) is the generalization of the expansion of the two-body scattering amplitude in partial-wave

[23] Here we follow the notations of A. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), except for a change of sign in the exponentials which relate the $d_{MM'}{}^J(\beta)$ to the $\mathfrak{D}_{MM'}{}^J(\alpha\beta\gamma)$ functions.

[24] G. Szego, *Orthogonal Polynomials* (American Mathematical Society Colloquium Publications, New York, 1959).

[25] G. F. Chew, Address to the Washington Meeting of the American Physical Society, April 1964 (to be published); H. P. Stapp, Phys. Rev. **125**, 2139 (1962); and Lawrence Radiation Laboratory Report UCRL-10843, August 1963 (to be published).

amplitudes,

$$\langle \mathbf{p}_1'\mathbf{p}_2'|T|\mathbf{p}_1\mathbf{p}_2\rangle = \sum_J (2J+1)a^J P_J(\cos\theta), \quad (2.3)$$

where θ is the scattering angle.

Using the orthogonality property of the rotation matrices, one can invert Eq. (2.2) in order to get an explicit form of the three-body partial-wave amplitude;

$$\langle \omega' JM'|\mathcal{T}|\omega JM\rangle$$
$$= m_1m_2m_3 \int \langle \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'|T|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle dR\, \mathfrak{D}_{MM'}{}^J(R), \quad (2.4)$$

where dR is the invariant measure on the rotation group (in terms of the Euler angles α, β, γ, $dR = d\alpha d\cos\beta d\gamma$). Equation (2.4) generalizes the relation

$$a^J = \frac{1}{2}\int_{-1}^{+1} \langle \mathbf{p}_1'\mathbf{p}_2'|T|\mathbf{p}_1\mathbf{p}_2\rangle d\cos\theta P_J(\cos\theta). \quad (2.5)$$

The similarity between the two-body and the three-body partial-wave expansions suggests immediately a common approach to the definition of their extension to complex angular momenta.

Let us pause to recall how Eqs. (2.3) and (2.5) are used in that respect.[25] One first notices that Eq. (2.5), when extended to any value of J, increases too rapidly, when J tends to infinity with complex values, to be of any use. In fact, the Sommerfeld-Watson transform of Eq. (2.3),

$$\frac{1}{2i}\oint \frac{(2J+1)}{\sin\pi J} dJ a^J P_J(\cos\theta), \quad (2.6)$$

would not converge with the definition (2.5) of a_J. However, according to a theorem by Neumann,[24] Eq. (2.5) can also be written for J an integer as

$$a^J = \frac{1}{2\pi i}\int_C T(E,x) Q_J(x) dx, \quad (2.7)$$

where $x = \cos\theta$, and $Q_J(x)$ is the Legendre function of the second kind. The amplitude $\langle \mathbf{p}\phi'\mathbf{p}_2'|T|\mathbf{p}_1\mathbf{p}_2\rangle = T(E,x)$ is supposed to be analytic in x in a neighborhood of the segment $(-1,+1)$. The contour C encloses this segment.

When the analytic function of x, $T(E,x)$, has only a cut going from $x = x_0 > 1$ to $+\infty$ along which its discontinuity is $2iA(E,x)$,[26] Eq. (2.7) can be replaced by

$$a^J = \frac{1}{\pi}\int_{x_0}^{\infty} A(E,x) Q_J(x) dx, \quad (2.8)$$

which is the Froissart-Gribov formula.[27] As $Q_J(x)$ decreases when J tends to infinity, the Sommerfeld-Watson integral (2.6) converges when a^J, for any J, is replaced by its expression (2.7). Moreover, according to a theorem by Carlson,[28] (2.7) is the unique interpolation of the physical values of a^J which is analytic in the right-hand half-plane $\mathrm{Re}J > -1$ and which does not increase as rapidly as $e^{\pi|J|}$.

When there is an exchange potential in the two-body interaction, $T(E,x)$ has two cuts: one going from $-\infty$ to $-x_2 < -1$, say, and the other from $x_1 > 1$ to $+\infty$. Then the contour C in Eq. (2.7) cannot be applied without care along the left-hand cut, since $Q_J(x)$ itself has also a cut going from $-\infty$ to $+1$. This difficulty is avoided by introducing the even and odd parts of $T(E,x)$

$$T^{(+)}(E,x) = \tfrac{1}{2}[T(E,x) + T(E,-x)],$$
$$T^{(-)}(E,x) = \tfrac{1}{2}[T(E,x) - T(E,-x)], \quad (2.9)$$

and putting

$$a^{J(+)} = \frac{2}{\pi}\int_{x_0}^{\infty} A^{(+)}(E,x) Q_J(x) dx,$$
$$a^{J(-)} = \frac{2}{\pi}\int_{x_0}^{\infty} A^{(-)}(E,x) Q_J(x) dx, \quad (2.10)$$

where $x_0 = \min(x_1,x_2)$. The $+$ or $-$ signs in Eq. (2.9) are called the signature.[3]

III. SOME PROPERTIES OF THE ROTATION MATRICES

We shall try to follow the essential steps of the preceding analysis while extending it to the three-body case. However, that means that we shall have to substantiate some well-known properties of the Legendre functions, which were tacitly assumed, by corresponding properties of the rotation matrices.

The generalization of the Legendre equation is given by the set of differential relations

$$(J_x{}^2 + J_y{}^2 + J_z{}^2)\mathfrak{D}_{MM'}{}^J(R) = J(J+1)\mathfrak{D}_{MM'}{}^J(R),$$
$$J_z \mathfrak{D}_{MM'}{}^J(R) = M\mathfrak{D}_{MM'}{}^J(R), \quad (3.1)$$
$$J_z{}' \mathfrak{D}_{MM'}{}^J(R) = M'\mathfrak{D}_{MM'}{}^J(R),$$

where J_x, J_y, J_z are the angular momentum operators of the symmetric top, i.e., differential operators with respect to the Euler angles (α,β,γ).[29] J_z (J_z') is the projection of the total angular momentum upon the body-fixed (space-fixed) z axis. The last two Eqs. (3.1) allow us to write

$$\mathfrak{D}_{MM'}{}^J(R) = e^{-iM\alpha} d_{MM'}{}^J(\beta) e^{-iM'\gamma}, \quad (3.2)$$

while the first of Eqs. (3.1) gives a differential equation analogous to the Legendre equation [Eq. (I.2) of Appendix I]. Just as the Legendre functions are defined by

[26] This is the case if $T(E,x)$ satisfies the Mandelstam representation; S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[27] M. Froissart, Report to La Jolla Conference on Weak and Strong Interactions, 1961 (unpublished); V. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 (1961) [English transl.: Soviet Phys.—JETP **14**, 478 (1962)].

[28] R. Boas, *Entire Functions* (Academic Press Inc., New York, 1954), p. 153.

[29] For explicit expressions for these operators see Ref. 23.

the Legendre equation, we shall define the $d_{MM'}{}^J(\beta)$ as the regular solution at $\cos\beta=1$ of this differential equation for any value of J.

In Appendix I it is shown that $d_{MM'}{}^J(\beta)$ has essentially the same properties as $P_J(x)$, namely:

(a) as a function of $x=\cos\beta$ it is an analytic function with a cut going from $-\infty$ to -1 if $M+M'$ is even, or $(1-x^2)^{1/2}$ times such an analytic function when $M+M'$ is odd;

(b) when J tends to infinity, it increases as does $e^{\mathrm{Im}J\beta}$.

We shall define a second solution $e_{MM'}{}^J(x)$ of the differential equation for $d_{MM'}{}^J(x)$, regular at infinity. The precise normalization is given in Appendix I. Its properties are closely analogous to the properties of the $Q_J(x)$, namely:

(a) as a function of x it is an analytic function with a cut going from $-\infty$ to $+1$;

(b) when J tends to infinity, it behaves like $Q_J(x)$.

However, while the $Q_J(x)$ is a meromorphic function of J with poles at $J=-1, -2, \cdots$, $e^J{}_{MM'}(x)$ has also singularities at $J=0, 1, \cdots\{\max|M|-1,|M'|-1\}$. According to the values of M and M', these singularities can be poles or branch-point singularities. They are indicated in detail in Appendix I.

When J is an integer, $e^J{}_{MM'}(x)$ has only a cut going from -1 to $+1$ alongside which its discontinuity is proportional to $d_{MM'}{}^J(x)$

$$e_{MM'}{}^J(x+i0)-e_{MM'}{}^J(x-i0)=i\pi d_{MM'}{}^J(x)\,. \quad (3.3)$$

However, this is not enough to insure that an integral like

$$\int_{-1}^{+1} f(x)d_{MM'}{}^J(x)dx \quad (3.4)$$

can be written as

$$\frac{1}{i\pi}\int_C f(x)e_{MM'}{}^J(x)dx\,, \quad (3.5)$$

since $d_{MM'}{}^J(x)$ is not always an analytic function of x in a neighborhood of the segment $(-1,+1)$. This is not surprising, since the properties of the $d_{MM'}{}^J(x)$ are simple only when interpreted on the rotation group. Therefore we go back to the full set of variables (α,β,γ) by defining, in analogy with Eq. (3.2),

$$\mathcal{E}_{MM'}{}^J(\alpha,\beta,\gamma)=e^{-iM\alpha}e_{MM'}{}^J(\beta)e^{-iM'\gamma}\,. \quad (3.6)$$

Then, as proved in Appendix I, one has the following theorem:

Theorem: *Let $f(\alpha,\beta,\gamma)$ be a function defined over the rotation group, analytic in $\cos\beta$ and $\sin\beta$ in a neighborhood of the segment $\cos\beta=-1$ to $\cos\beta=+1$. Then*

$$\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_{-1}^{+1}d\cos\beta\, f(\alpha,\beta,\gamma)\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma)$$
$$=\frac{1}{i\pi}\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\oint_C d\cos\beta$$
$$\times f(\alpha,\beta,\gamma)\,\mathcal{E}_{MM'}{}^J(\alpha,\beta,\gamma)\,, \quad (3.7)$$

where C is a contour enclosing the segment $\cos\beta=-1$ to $+1$.

If the contour C can be displaced around the singularities of $f(\alpha,\beta,\gamma)$ in such a way as to go to infinity, and if the asymptotic behavior of $f(\alpha,\beta,\gamma)$ when $\cos\beta$ tends to infinity allows one to neglect the contour at infinity, Eq. (3.7) is a natural generalization of the Froissart-Gribov formula.

In order to continue the right-hand member of Eq. (3.7) to complex values of J, one must take care that, generally, the singularities of the function $f(\alpha,\beta,\gamma)$ depend upon α and γ. On the other hand, as $e_{MM'}{}^J(\beta)$ has a left-hand cut in $\cos\beta$ when J is not an integer, it could happen that the singularities of $f(\alpha,\beta,\gamma)$ encounter that cut. This is in fact the problem of introducing the signature into the many-body kinematics.

Let us start from the symmetry relation of the $d_{MM'}{}^J(\beta)$, valid when J is physical,[23]

$$d_{MM'}{}^J(\beta+\pi)=(-1)^{J+M'}d_{M,-M'}{}^J(\beta)\,, \quad (3.8)$$

and let us introduce two new functions

$$a_{M,M'}{}^{J(\pm)}=\frac{1}{2}\int f(\alpha,\beta,\gamma)[\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma)\pm(-1)^{J+M'}$$
$$\times\mathfrak{D}_{M,-M'}{}^J(\alpha,\beta,\gamma)]d\alpha d\cos\beta d\gamma \quad (3.9)$$
$$=\frac{1}{2}\int[f(\alpha,\beta,\gamma)\pm f(\alpha,\beta+\pi,-\gamma)]$$
$$\times\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma)d\alpha d\cos\beta d\gamma \quad (3.10)$$
$$=\int F^{(\pm)}(\alpha,\beta,\gamma)\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma)d\alpha d\cos\beta d\gamma\,,$$

where

$$F^{(\pm)}(\alpha,\beta,\gamma)=\tfrac{1}{2}[f(\alpha,\beta,\gamma)\pm f(\alpha,\beta+\pi,-\gamma)]\,, \quad (3.11)$$

and

$$F^{(\pm)}(\alpha,\beta,\gamma)=\pm F^{(\pm)}(\alpha,\beta+\pi,-\gamma)\,. \quad (3.12)$$

The continuation of $a_{MM'}{}^{J(\pm)}$ to complex values of J will be given by

$$a_{MM'}{}^{J(\pm)}=\frac{2}{i\pi}\int_C d\alpha d\cos\beta d\gamma$$
$$\times F^{(\pm)}(\alpha,\beta,\gamma)\,\mathcal{E}_{MM'}{}^J(\alpha,\beta,\gamma)\,, \quad (3.13)$$

where the symmetry properties of $F^{(\pm)}(\alpha,\beta,\gamma)$ are used to insure that the contour C encloses only the singularities of $F(\alpha,\beta,\gamma)$ which are in the right-hand half-plane,

$$\mathrm{Re}\,\cos\beta \geq 0. \tag{3.14}$$

Clearly, $a_{MM'}{}^{J(+)}$ $(a_{MM'}{}^{J(-)})$ coincides with the physical value $a_{MM'}{}^{J}$ when J is an integer and $J+M'$ is even (odd).

IV. CONTINUATION TO COMPLEX VALUES OF J IN S-MATRIX THEORY

It is interesting, for the sake of orientation, to define the extension of the three-body partial-wave amplitudes when one assumes the validity of the axiomatic S-matrix theory.[25] In this theory, one assumes that $\langle \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'|T|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle$ can be extended to complex values of the momenta with some well-defined, although not too simple, singularities. If one replaces the momenta by the variables ω_1, ω_2, ω_3; ω_1', ω_2', ω_3' and the rotation R, which defines the orientation of the final triangle $(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$, with respect to the initial triangle $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$, the connected part of the three-body scattering amplitude can then be written as a function $f(\omega_1,\omega_2,\omega_3;\omega_1',\omega_2',\omega_3';\alpha,\beta,\gamma)$ with well-defined singularities in α, β, and γ.

The preceding analysis can then be applied to this function $f(\omega,\omega';\alpha,\beta,\gamma)$, so that Eq. (3.13) becomes a generalization of the Froissart-Gribov formula. Some special care should be exercised because of the existence of singularities within the physical region.

In this form the physical meaning of signature is obvious: changing (α,β,γ) into $(\alpha,\beta+\pi,-\gamma)$ means replacing the rotation R by a rotation R', which leads to a new reference system linked to the final state. The old and new systems of axes differ by a rotation around the x axis. This is also obviously the meaning of signature in the two-body case if one chooses the a axis along the final momentum in the center-of-mass system. Let us note that, according to Eq. (1.17) of the Appendix, the extended matrix element is not bounded when J tends to infinite complex values, because of the presence of complex singularities in the cos-β plane.

V. THE NONRELATIVISTIC PROBLEM

Let us start from the Fadeev equations [Eq. (44) of paper I], which we shall write in an abbreviated form as

$$\mathcal{T}^J(i)=\mathcal{T}_{kl}{}^J-\sum_{(j)} K_J{}^{(i,j)}\mathcal{T}^J(j)\,, \tag{5.1}$$

where

$$i,\,j,\,k,\,l=1,\,2,\,3\colon\quad i\neq j,\; i\neq k,\; i\neq l$$

and

$$k\neq l.$$

Let us recall that $\mathcal{T}_{12}{}^J$, for instance, is the two-body amplitude for the scattering of particles 1 and 2 and the total three-body amplitude is $\mathcal{T}^J=\mathcal{T}^J(1)+\mathcal{T}^J(2)+\mathcal{T}^J(3)$. Equation (5.1) cannot be used directly to define an extension of the collision matrix to complex values of the total angular momentum because its kernel is not completely continuous.[30] We therefore iterate Eq. (5.1),

$$\mathcal{T}^J(i)=\mathcal{T}_{kl}{}^J-\sum_{(j)} K_J{}^{(i,j)}\mathcal{T}_{lm}{}^J+\sum_j K_J{}^{2(i,j)}\mathcal{T}^J(j)\,, \tag{5.2}$$

where $l\neq j$, $m\neq j$. We shall introduce the connected part $U_{(j)}{}^J=\mathcal{T}_{(i)}{}^J-\mathcal{T}_{kl}{}^J$, which satisfies

$$U^J=-K_J\mathcal{G}^J+K_J{}^2\mathcal{G}^J+K_J{}^2U^J\,, \tag{5.3}$$

where we have written $\mathcal{G}^J=(\mathcal{T}_{23}{}^J,\mathcal{T}_{13}{}^J,\mathcal{T}_{12}{}^J)$. According to Eq. (47) of paper I, the kernel $K_J{}^2$ has the form

$$K_J{}^2=\begin{bmatrix} H_{12}+H_{13} & H_{13} & H_{12} \\ H_{23} & H_{21}+H_{23} & H_{21} \\ H_{32} & H_{31} & H_{31}+H_{32} \end{bmatrix}, \tag{5.4}$$

where a typical term is, for instance, H_{12}:[21,31]

$$\begin{aligned} &H_{12M'M}{}^J(\omega',\omega,z) \\ &=\int \frac{1}{p_1'p_2}F_{23}(\omega',\,\omega'',\,u,\,z-\omega_1')F_{13}(\omega'',\,\omega,\,v,\,z-\omega_2) \\ &\quad\times d_{M'M_1}{}^J(-\alpha_1')e^{iM_1u}d_{M_1M_2}{}^J(\theta_{21}'')e^{iM_2v}d_{M_2M}{}^J(\alpha_2) \\ &\quad\times[(\omega_1'+\omega_2+\omega_3''-z)(\omega_1+\omega_2+\omega_3-z)]^{-1} \\ &\qquad\times du\,dv\,d\omega_3''. \end{aligned} \tag{5.5}$$

The inhomogeneous term $K\mathcal{G}$ is of the form (5.4) and (5.5) except for the absence of $(\sum\omega_i-z)^{-1}$. When we wrote Eq. (5.5), we took into account the equality $\theta_{21}''=\alpha_2''-\alpha_1''$, and we have integrated ω_1'' and ω_2'', which—according to the delta functions in Eq. (47) of Paper I—are given by

$$\omega_1''=\omega_1' \quad\text{and}\quad \omega_2''=\omega_2. \tag{5.6}$$

The integrations over u and v in Eq. (5.5) go from 0 to 2π. The integration over ω_3'' is over a limited range such that, according to Eq. (5.6),

$$|p_1'-p_2|\leq p_3''\leq p_1'+p_2. \tag{5.7}$$

According to the relation (16) of Paper I between ω_3'' and θ_{21}'', one can also write

$$d\omega_3''=-2(m_1m_2\omega_1'\omega_2)/m_3\,d\cos\theta_{12}''\,, \tag{5.8}$$

the integration upon $\cos\theta_{12}''$ going from -1 to $+1$.

Let us now consider the extension of the first term $K_J\mathcal{G}^J$ in Eq. (5.3) to complex values of J. A slight change in kinematics is necessary in order to put the right-hand side of Eq. (5.5) into the form (3.7). In place of u, v, and ω_3'', we must use as variables the three Euler angles, say (α,β,γ), of the rotation which applies the set of axes linked to $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ to the set of axes linked

[30] According to Ref. 19, the iterated kernel is completely continuous when the total energy z is not a real positive number. In that case the fifth iterated kernel is completely continuous.

[31] Here z means the total energy. Remember that these equations are defined off the energy shell.

to $(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$. Let us call R that rotation. Let us also call R_0 the rotation with Euler angles $(u,\ \theta_{12}'',v)$, and r_1' and r_2 the rotations of angles α_1' and α_2 around the y axis. Then, according to the relation

$$R=r_1'^{-1}R_0r_2\,, \tag{5.9}$$

one has, owing to the invariance of the group-theoretic measure,[32]

$$d\alpha d\ \cos\beta d\gamma=dud\ \cos\theta_{12}''dv\,. \tag{5.10}$$

Accordingly, the inhomogeneous term $K_J\mathcal{G}^J$ has the form $(B_{12}+B_{13},\ B_{21}+B_{23},\ B_{31}+B_{32})$ where, for instance,

$$B_{12M'M}{}^J(\omega',\omega,z)=\int f^{12}(\omega',\omega,z,\alpha,\beta,\gamma)\times\mathfrak{D}_{M'M}{}^J(\alpha,\beta,\gamma)d\alpha d\ \cos\beta d\gamma \tag{5.11}$$

and

$$f^{12}(\omega',\omega,z;\alpha,\beta,\gamma)=2\frac{m_1m_2\omega_1'\omega_2}{p_1'p_2m_3(\omega_1'+\omega_2+\omega_3''-z)}F_{23}(\omega',\ \omega,\ \alpha,\ \beta,\ \gamma;\ z-\omega_1')\times F_{13}(\omega',\ \omega,\ \alpha,\ \beta,\ \gamma;z-\omega_2)\,. \tag{5.12}$$

Here F_{23} is the two-body scattering amplitude $\langle\mathbf{p}_2'\mathbf{p}_3'|T_{23}(z-\omega_1')|\mathbf{p}_2''\mathbf{p}_3''\rangle$ expressed in terms of the variables ω', ω, α, β, and γ. The kinematical situation is illustrated in Fig. 1.

When considering the second term $K_J{}^2\mathcal{G}^J$ of Eq. (5.3) we shall again introduce the Euler angles (α,β,γ) of the rotation which applies the initial reference system linked to $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ to the final system linked to $(\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3')$. However, when writing explicitly $K_J{}^2\mathcal{G}^J$ it will be necessary to introduce two sets of intermediate states, let us say $(\mathbf{p}_1'',\mathbf{p}_2'',\mathbf{p}_3'')$ and $(\mathbf{p}_1''',\mathbf{p}_2''',\mathbf{p}_3''')$. We shall take as variables, in addition to (α,β,γ), the angle between the $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ triangle and the $(\mathbf{p}_1'',\mathbf{p}_2'',\mathbf{p}_3'')$ triangle as well as the two undetermined sides of this last triangle. Then $K_J{}^2\mathcal{G}^J$ will assume the form (5.11) while f will contain three two-body scattering amplitudes and three more integrations.

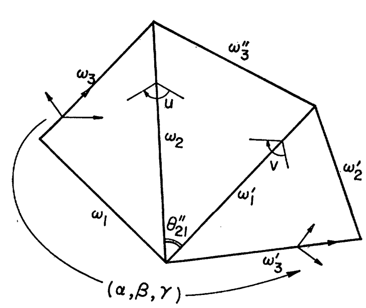

FIG. 1. The variables used for evaluating the contribution $K\mathcal{G}$ to the inhomogeneous term.

[32] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962).

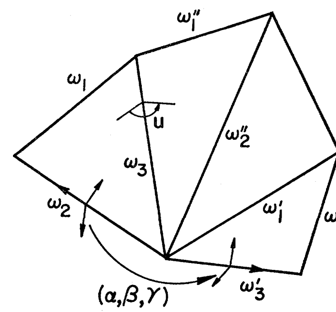

FIG. 2. The variables used for evaluating the contribution $K^2\mathcal{G}$ to the inhomogeneous term. The variables are ω_1'', ω_2'', u, and the Euler angles $(\alpha,\ \beta,\ \gamma)$.

A typical kinematic situation for $K^2\mathcal{G}$ is depicted in Fig. 2.

The generalization of this approach to any term of the Born-series expansion of U^J is obvious. Unfortunately, it becomes more and more involved as one considers higher order Born terms, and this seems to be the price to pay for the three-body kinematics. In order to say something about the extension of these terms, and particularly (5.11), we need to know more about the analytical properties of the off-the-energy-shell two-body scattering amplitude.

VI. CONTINUATION OF THE INHOMOGENEOUS TERMS

The off-the-energy-shell two-body scattering amplitude $T(\mathbf{p},\mathbf{p}',\xi)$ (where ξ is the complex total energy as it appears in the Lippmann-Schwinger equation[33]) can be considered as a function of ξ and of the three invariants p^2, p'^2, and $t=(\mathbf{p}-\mathbf{p}')^2$.

A sketchy analysis given in Appendix II indicates that $T(p^2,p'^2,t,\xi)$ is an analytic function of p^2, p'^2, t, and ξ with the following singularities[34]:

(i) a cut in p^2 from 0 to infinity,
(ii) a cut in p'^2 from 0 to infinity,
(iii) a cut in t from $\mu_0{}^2$ to infinity, $\mu_0{}^{-1}$ being the range of the potential,
(iv) a cut in ξ from 0 to infinity,
(v) poles in ξ for values of ξ which are the energies of the bound states.

These singularities give rise to singularities of $f^{12}(\omega',\omega,z;\alpha,\beta,\gamma)$ in Eq. (5.11) which, for fixed values of α and γ, are singularities in $\cos\beta$. The singularities of the two-body scattering amplitudes always take place outside of the integration domain in (α,β,γ). The singularity in $q_{23}''^2$ of $F_{23}(\omega',\ \omega'',\ z-\omega_1')$, for instance, can touch this physical region. In fact, $q_{23}''^2=0$ corresponds to a kinematical situation where $\mathbf{p}_2$ and $\mathbf{p}_3''$ are equal, and this can happen only when $\omega_1'=0$ or $\omega_1'=2\omega_2$. (see Fig. 1). Therefore, except for a set of zero measure of the

[33] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[34] The analytic properties of the off-the-energy shell two-body scattering amplitude have been previously studied by A. Grossman and T. T. Wu, J. Math. Phys. **2**, 710 (1961); A. Grossman, *ibid.* **2**, 714 (1961), also presents some results for the partial-wave amplitude.

initial and final variables, the theorem (3.7) can be applied without special care, and allows us to define the continuation of the first inhomogeneous term $K_J\mathcal{G}^J$ to complex values of J.

When the second inhomogeneous term $K_J{}^2\mathcal{G}^J$ is being extended more care must be exercised, because the second triangle (see Fig. 2) is variable and there is always a kinematic situation where, for instance, $\mathbf{p}_1'=\mathbf{p}_2''$. However, since the integrand is also analytic in ω_1'', ω_3'', and u, we shall assume that it is possible to displace slightly the contour of integration of these variables in order to avoid that singularity. As this kind of singularity only touches the physical region without crossing it, such a displacement is always possible.

We have up to now replaced Eq. (5.11) by

$$B_{12M'M}(\omega',\omega,z)=\frac{1}{\pi}\oint d\alpha d\cos\beta d\gamma \times f^{12}(\omega',\omega,z;\alpha,\beta,\gamma)\mathcal{E}_{M'M}{}^J(\alpha,\beta,\gamma), \quad (6.1)$$

where α and γ are integrated from 0 to 2π, and $\cos\beta$ is integrated along a contour which encloses the singularities of the integrand and goes therefore to infinity. The equivalence of Eqs. (5.11) and (6.1) as well as the existence of the integral in Eq. (6.1) depend upon the asymptotic behavior of $f^{12}(\omega',\omega,z;\alpha,\beta,\gamma)$ when $\cos\beta$ tends to infinity.

When $\cos\beta$ tends to infinity the components of the vectors $\mathbf{p}_1'$, $\mathbf{p}_2'$, $\mathbf{p}_3'$ in Fig. 1 become infinite (and complex) while ω_1', ω_2', and ω_3', the squares of their moduli, remain finite. Accordingly, $\mathbf{p}_3''=\mathbf{p}_1'-\mathbf{p}_2$ becomes infinite as well as the c.m. momenta

$$\begin{aligned}\mathbf{q}_{23}'&=(m_3\mathbf{p}_2'-m_2\mathbf{p}_3')(m_2+m_3)^{-1},\\ \mathbf{q}_{23}''&=(m_3\mathbf{p}_2'-m_2\mathbf{p}_3'')(m_2+m_3)^{-1},\\ \mathbf{q}_{13}''&=(m_1\mathbf{p}_2''-m_3\mathbf{p}_1)(m_1+m_3)^{-1}.\end{aligned} \quad (6.2)$$

Since the components of $\mathbf{p}_3''$ increase linearly with $\cos\beta$ and since

$$\begin{aligned}q_{23}''^2=&[2m_2m_3/(m_2+m_3)](\omega_1+\omega_2'+\omega_3'')\\ &-[2m_2m_3(m_1+m_2+m_3)/(m_2+m_3)^2]\omega_1\end{aligned} \quad (6.3)$$

and ω_3'' increases linearly with $\cos\beta$, the c.m. momenta q_{23}'' and q_{13}'' as well as the scattering angles $\cos\phi_{2'3''}$ and $\cos\phi_{3''1}$ tend to infinity linearly in $(\cos\beta)^{1/2}$.

The fact that the scattering angles tend to infinity could lead one to believe that the Regge poles of the two-body scattering amplitude determine the asymptotic behavior of the integrand in (6.1). However, it is shown in Appendix III that the fact that the center-of-mass momenta also tend to infinity compensates that effect, and that the over-all behavior of F_{23}, for instance, is

$$F_{23}(\omega',\omega;z-\omega_1';\alpha,\beta,\gamma)\sim(\cos\beta)^{-1/2}. \quad (6.4)$$

According to Eq. (I.15) of Appendix I, $\mathcal{E}_{MM'}{}^J(x)$ behaves like x^{-J-1} when x tends to infinity and, therefore, the integrand in Eq. (6.1) behaves like $(\cos\beta)^{-J-3}$ when $\cos\beta$ tends to infinity, so that it converges when $\mathrm{Re}J$ is larger than -2.

Accordingly, the domain of analyticity of the inhomogeneous terms $K_J\mathcal{G}^J$ and $K_J{}^2\mathcal{G}^J$ of the Fadeev equations can be extended to complex values of J in the half-plane[35]

$$\mathrm{Re}J\geq\max\{|M|-1;\,|M'|-1\}. \quad (6.5)$$

This result holds true for any term of the Neumann expansion

$$K\mathcal{G}+K^2\mathcal{G}+K^3\mathcal{G}+\cdots \quad (6.6)$$

of the solution of the Fadeev equations.

VII. CONTINUATION OF THE FADEEV EQUATIONS

We have found that the inhomogeneous terms of the reduced Fadeev equations[36] can be extended to complex values of J into analytic functions (of J) within the half-plane (6.5).

Let us now consider the problem of extending the reduced iterated Fadeev equations themselves to complex values of J. To do so, let us first write them explicitly for physical values of J:

$$U_{M'M}{}^{(i)J}(\omega',\omega) =V_{M'M}{}^{(i)J}(\omega',\omega)+\sum_{M''=-J}^{+J}\sum_{j=1}^{3}\int K_{M'M}{}^{2(i,j)J}(\omega',\omega'') U_{M''M}{}^{(j)J}(\omega'',\omega)d\omega''; \quad (7.1)$$

for simplicity we have written V in place of $K\mathcal{G}$ and $K^2\mathcal{G}$, and the kernel K^2 in Eq. (7.1) is given by Eqs. (5.4) and (5.5). Both V and K^2 have the form (5.11).

We know that, in order to extend the partial-wave expansion (2.2) into a Sommerfeld-Watson formula, the extension of U^J must not increase as rapidly as $\sin\pi J$ when J tends to infinity. This is not a very compelling reason in the problem we are considering, because we shall see later that the Sommerfeld-Watson series never converges. However, the present choice is the one which reduces to the customary Froissart-Gribov formula in the case of the scattering of a particle on a bound state of the other two. This property leads us to choose the extension of V^J to complex values of J described in Sec. VI, since the inhomogeneous term must have the same properties as the full solution.

When we come to the kernel K^{2J}, there is no direct condition upon it, but indirect conditions, namely:

(i) the extended equation must have the physical solution for physical values of J,

(ii) the extended solution must not increase as rapidly as $\sin\pi J$ when J tends to infinity.

Condition (i) is not trivial because when J becomes

[35] This domain arises from the singularities of the $\mathcal{E}_{MM'}{}^J$ function.

[36] As the results of this section are essentially negative, it can be skipped in a first reading.

a complex parameter, there are no more limitations upon M'' and Eq. (7.1) becomes

$$U_{M'M}{}^{(i)J}(\omega',\omega) = V_{M'M}{}^{(i)J}(\omega',\omega)+\sum_{M''=-\infty}^{+\infty}\sum_{j=1}^{3}\int K_{M'M''}{}^{2(i,j)J}(\omega',\omega'') \times U_{M''M}{}^{(j)J}(\omega'',\omega)d\omega'', \quad (7.2)$$

and we should make sure that when J is an integer and $|M|$, $|M'|$ two integers smaller than J, the summation upon $|M''|$ in Eq. (7.2) from J to infinity gives identically zero.

There are *a priori* two simple ways of extending K^{2J}, namely

(a) keep it in the form of Eq. (5.5) as an integral upon a $\mathfrak{D}^J$;

(b) transform it into a contour integral of the form of Eq. (5.11) upon an $\mathcal{E}^J$.

Let us discuss the consistency between these two extensions and condition (i).

We shall first discuss extension (a). In order to simplify the argument, we shall restrict ourselves to the case in which J passes from a complex value to zero and, therefore, we shall consider $M=M'=0$ in Eq. (7.2).

We shall use[23]

$$d_{0M}{}^J(\beta)=\left[\frac{\Gamma(J-M+1)}{\Gamma(J+M+1)}\right]^{1/2}(-1)^M P_J{}^M(\cos\beta); \quad M>0, \quad (7.3)$$

$$e_{M0}{}^J(\beta)=\left[\frac{\Gamma(J-M+1)}{\Gamma(J+M+1)}\right]^{1/2} Q_J{}^M(\cos\beta); \quad M>0. \quad (7.4)$$

Let us now consider the integral in Eq. (7.2). If we assume that, as in S-matrix theory, $U_{M''0}{}^J$ has the form (3.13) or that it has the same properties as $V_{M''0}{}^J$, it will behave like $e_{M''0}{}^J$, i.e., become infinite like $J^{-1/2}$ when $M''>0$ and J tends to zero. On the other hand, $K_{0M''}{}^{2J}$ behaves like $d_{0M''}{}^J$, i.e., vanishes as $J^{1/2}$, since the Legendre function in Eq. (7.3) vanishes as J and the normalization coefficient behaves like $J^{-1/2}$. Therefore we must inquire about the result of the summation of Eq. (7.2).

The quantity we want to keep is

$$(K^2)_{00}{}^0 U_{00}{}^0, \quad (7.5)$$

and the quantity which should vanish is

$$\sum_1^\infty (K^2)_{0M''}{}^0 U_{M''0}{}^0 \quad (7.6)$$

(taking into account the symmetry properties (I.5) of the d and e functions with respect to M'' allows one to sum only upon positive integer values of M'').

Now, let us shift the normalization factor

$$[\Gamma(J-M''+1)/\Gamma(J+M''+1)]^{1/2}$$

of $U_{M''0}{}^J$ upon $(K^2)_{0M''}{}^0$, defining in this way $(K'^2)_{0M''}{}^0$, so that the summation in Eq. (7.2) will be replaced by

$$\sum_{M''=-\infty}^{\infty}\int (K'^2)_{0M''}{}^0(\omega',\omega'')U_{M''0}{}^0(\omega'',\omega)d\omega'', \quad (7.7)$$

where

$$K_{0M''}{}'^{2J}(\omega',\omega'')=\int F(\omega',\omega'';\alpha,\beta,\gamma)e^{-iM''\gamma} \times\frac{\Gamma(J-M''+1)}{\Gamma(J+M''+1)}(-1)^{M''}P_J{}^{M''}(\cos\beta)d\alpha d\cos\beta d\gamma, \quad (7.8)$$

$$U_{M''0}{}^J(\omega'',\omega)=\oint G(\omega'',\omega,\alpha,\beta,\gamma)e^{-iM''\alpha} \times Q_J{}^{M''}(\cos\beta)d\alpha d\cos\beta d\gamma. \quad (7.9)$$

We can see more clearly the properties of Eq. (7.8) if we notice that[37]

$$\frac{\Gamma(J-M+1)}{\Gamma(J+M+1)}P_J{}^M(x) = P_J{}^{-M}(x)=\frac{(-1)^M}{\cos J\pi\Gamma(J+M+1)\Gamma(M-J)} \times[Q_{-J-1}{}^M(x)-Q_{+J}{}^M(x)], \quad (7.10)$$

so that, for $M>0$, the integrand in Eq. (7.8) will behave like

$$[(-1)^M/\Gamma(M)][Q_{-1}{}^M(x)-Q_0{}^M(x)]. \quad (7.11)$$

If we perform now the symmetrization of Eq. (7.8) in order to introduce the signature, we shall keep all terms which have the parity of $P_0(x)$, i.e., $Q_{-1}{}^M(x)$, and we shall cancel the terms of opposite parity, i.e., $Q_0{}^M(x)$. We can see more easily the nature of the remaining term by using[38]

$$[(-1)^M/\Gamma(M)]Q_{-1}{}^M(x)=(\pi/2)^{1/2}(x^2-1)^{1/4} \times P_{M-1/2}{}^{1/2}[x(x^2-1)^{-1/2}]. \quad (7.12)$$

To conclude, we see that extension (a) does not satisfy condition (i).

If we use extension (b), the argument is the same as above up to Eq. (7.8), where $P_J(\cos\beta)$ has to be replaced by $Q_J(\cos\beta)$. Therefore, both the terms of the kernel and of the solution tend to infinity like $J^{-1/2}$ when J tends to zero and the sum (7.6) has no meaning.

Therefore, we cannot find an extension of the Fadeev equations to complex values of J which has the physical solution when J is an integer. The reason must be traced to the existence of nonsense channels for which $|M|>|J|$.

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[37] *Bateman Manuscript Project, Higher Transcendental Functions I*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1954), p. 140.

[38] Reference 37, p. 141.

The preceding negative results would be very strange if the extended Fadeev equations had a solution, while not satisfying condition (i). In fact, the asymptotic behaviors of $d_{MM'}{}^J(x)$ and $e_{MM'}{}^J(x)$ when M and M' tend simultaneously to infinity is exponentially increasing [see Eqs. (I.20a) and (I.21) of Appendix I] so that Eq. (7.2) has an unbounded kernel with both extensions (a) and (b). Furthermore, if we assume that the solution $U_{M'M}{}^J$ has the same behavior in M and M' as in S-matrix theory or as $V_{M'M}{}^J$, the asymptotic behaviors of $d_{M'M}{}^J(x)$ and $e_{MM'}{}^J(x)$ when M' tends to infinity [as given by Eqs. (I.18) and (I.19) of Appendix I] show that the summation upon M'' in Eq. (7.2) will never converge.

To conclude: the removal of the constraint $|M|<|J|$ when J is complex, which has no counterpart in the two-body problem, is enough to render an extension of the Fadeev equations devoid of all meaning.

VIII. EXTENSION OF THE FREDHOLM SOLUTION

Let us rewrite the Fadeev equations for physical J [(7.1)] as

$$U_J=V_J+K_J{}^2U_J. \tag{8.1}$$

We know that the kernel $K_J{}^2$ is completely continuous since, according to Fadeev, the full kernel K^2 has that property, and also satisfies

$$\text{Trace } K^2K^{\dagger 2}=\sum_J(2J+1)\text{ Trace } K_J{}^2\,K_J{}^{\dagger 2}<\infty\,. \tag{8.2}$$

Therefore we can solve Eq. (8.1) by the Fredholm method, which gives

$$U_J=N_J/D_J\,, \tag{8.3}$$

where[39]

$$D_J=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{vmatrix}1 & 0 & 0 & \cdots & 0 & K_1\\ K_1 & 2 & 0 & \cdots & 0 & K_2\\ K_2 & K_1 & 3 & \cdots & 0 & K_3\\ \vdots & \vdots & \vdots & \cdots & n-1 & \vdots\\ K_{n-1} & K_{n-2} & \vdots & \cdots & K_1 & K_n\end{vmatrix}\,, \tag{8.4}$$

$$N_J=\sum_{n=0}^{\infty}\frac{1}{n!}\times\begin{vmatrix}1 & 0 & 0 & \cdots & K_1-K_J{}^2\\ K_1-K_J{}^2 & 2 & 0 & \cdots & K_2-K_J{}^4\\ K_2-K_J{}^6 & K_1-K_J{}^2 & 3 & \cdots & K_3-K_J{}^6\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ K_{n-1}K_J{}^{2n-2} & \vdots & \vdots & K_1-K_J{}^2 & K_n-K_J{}^{2n}\end{vmatrix}\,. \tag{8.5}$$

[39] This form of the solution is taken from Morse and Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 1024.

We have introduced the notation

$$K_n=\text{Trace } K_J{}^{2n}. \tag{8.6}$$

We have already shown how to extend terms like $K_J{}^nV_J$ to complex values of J. Let us show now how to extend the quantities K_n.

In order to express $K_J{}^{2n}$ for physical values of J, we shall use the same kind of variables as in Sec. V, i.e., a rotation $R(\alpha,\beta,\gamma)$ giving the relative positions of the initial and final reference systems plus other momentum and angle variables labeled collectively y which determine the relative position of the 2nd, 3rd, $\cdots$, $2n-2$ intermediate states when $K_J{}^{2n}$ is expanded in terms of matrix element of two-body scattering amplitudes. Then one has, according to Eq. (2.4),

$$\langle\omega'M'|K_J{}^{2n}|\omega M\rangle=\int\mathfrak{D}_{M'M}{}^J(\alpha,\beta,\gamma)\times f^{(n)}(\omega',\omega;y;\alpha,\beta,\gamma)dyd\alpha d\cos\beta d\gamma\,, \tag{8.7}$$

so that

$$K_n=\int\sum_{M=-J}^{+J}\mathfrak{D}_{MM}{}^J(\alpha,\beta,\gamma)\times f^{(n)}(\omega,\omega;y;\alpha,\beta,\gamma)dyd\alpha d\cos\beta d\gamma d\omega\,. \tag{8.8}$$

In order to evaluate the trace of the rotation matrix, it is convenient to define the rotation R by a unitary vector $\hat{n}$ along its axis (or, equivalently, the spherical angles θ and ϕ of $\hat{n}$) and the angle ρ of the rotation. Then, one has

$$\sum_{M=-J}^{J}\mathfrak{D}_{MM}{}^J(\alpha,\beta,\gamma)=\sum_M[\exp(i\mathbf{J}\cdot\hat{n}\rho)]_{MM}=\sum_{M=-J}^{J}e^{iM\rho}=\frac{\sin(2J+1)\frac{1}{2}\rho}{\sin\frac{1}{2}\rho}\,. \tag{8.9}$$

A routine computation shows

$$d\alpha d\cos\beta d\gamma=2\sin\tfrac{1}{2}\rho(d\cos\tfrac{1}{2}\rho)(d\cos\theta)d\phi\,, \tag{8.10}$$

so that Eq. (8.8) takes the form

$$K_n=\int dy\int_0^{2\pi}d\phi\int_0^{\pi}d\cos\theta\int_{-1}^{+1}2f^{(n)}(\omega,\omega;y;\theta,\phi,\rho)\times\frac{\sin(2J+1)\frac{1}{2}\rho}{\sin\frac{1}{2}\rho}\sin\tfrac{1}{2}\rho d\cos\tfrac{1}{2}\rho\,. \tag{8.11}$$

Let us now use the fact that two rotations with opposite axes and angles are in fact identical (i.e., R_3 and not SU_2) so that

$$f^{(n)}(\omega,\omega;y;\pi-\theta,\phi+\pi,-\rho)=f^{(n)}(\omega,\omega;y;\theta,\phi,\rho)\,, \tag{8.12}$$

and introduce

$$F^{(n)}(\omega;\omega;y;\theta,\phi,\cos\tfrac{1}{2}\rho)=\tfrac{1}{2}[f^{(n)}(\omega,\omega;y;\theta,\phi,\rho)+f^{(n)}(\omega,\omega;y,\pi-\theta,\phi+\pi,-\rho)]. \quad (8.13)$$

According to Eq. (8.12), and the analyticity of the two-body scattering amplitudes, $F^{(n)}(\omega,\omega;y;\theta,\phi,\rho)$, is an analytic function of $\cos\frac{1}{2}\rho$, and one has

$$K_n=\int dy\int_0^{2\pi}d\phi\int_0^{\pi}d\cos\theta \times\int_{-1}^{+1}2F^{(n)}(\omega,\omega;y;\theta,\phi,x)(1-x^2)^{1/2}T_{2J-1}(x)dx. \quad (8.14)$$

We can now use the results of Appendix III, where it is shown that

$$T_n(x)=(\sin n\tfrac{1}{2}\rho)/(\sin\tfrac{1}{2}\rho),\quad x=\cos\tfrac{1}{2}\rho, \quad (8.15)$$

is a Tchebysheff polynomial,[40] so that we can introduce a Tchebysheff function of the second kind $S_n(x)$ such that (8.14) takes the form

$$K_n=\int dyd\phi d\cos\theta \times\oint_C 2F^{(n)}(\omega,\omega;y,\theta,\phi,x)(1-x^2)^{1/2}S_{2J-1}(x)dx, \quad (8.16)$$

where C is a contour enclosing the segment $(-1,+1)$.

We can now proceed as before: deform the contour C along the singularities of $F^{(n)}(\omega,\omega;y,\theta,\phi,x)$. As the Tchebysheff functions of the second kind have exactly the same asymptotic behavior in x and in J as the Jacobi functions,[41] all the discussion of Sec. VI, about the convergence of Eq. (6.1) can be carried out without change, and shown that K_n is an analytic function of J in the half-plane

$$\mathrm{Re}J\geq-\tfrac{1}{2}. \quad (8.17)$$

Unfortunately, the presence of complex singularities does not allow one to satisfy the conditions of the Carlson theorem.

Putting these results together, we see that we are able to extend all the terms in the Fredholm solution (8.3) to complex values of J. In this extension, there does not remain any summation upon indices M, which was the essential difficulty met in the extension of the Fadeev equation.

While the solution (8.3) consists of the quotient of two series where each term is analytic in the domain[35]

$$\mathrm{Re}J\geq\max\{|M|-1;|M'|-1\}, \quad (8.18)$$

as long as nothing can be said about the convergence of these series, we cannot find the analytic properties in J of the solution. This problem seems to be extremely difficult, and we have noting to present about it.

Of course, the simplest situation would be that the series converge uniformly, as they do for physical values of J, so that the zeros of D_J would define the Regge poles of the three-body scattering amplitude. As D_J depends only upon z, the trajectories of these poles are given by

$$D_J(z)=0. \quad (8.19)$$

These Regge poles interpolate the three-body bound states and resonances. However, difficulty (iv) could very well invalidate such a simple result.

IX. THREE-PARTICLE SOMMERFELD-WATSON EQUATION

Let us start from the partial-wave expansion (2.2). By use of the symmetry property

$$d_{MM'}{}^J(\pi+\beta)=(-1)^{J+M'}d_{M,-M'}{}^J(\beta), \quad (9.1)$$

Eq. (2.2) can be transformed into a contour integral along a contour C which encloses the physical values $J=0,1,2,\cdots$, i.e.,[42]

$$\langle \mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'|U|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle =\frac{1}{8\pi^2m_1m_2m_3}\sum_{M,M'}\frac{1}{2i}\int_C\frac{2J+1}{\sin\pi J}U_{M'M}{}^J(\omega',\omega) \times\mathfrak{D}_{M,-M'}{}^J(\alpha,\pi-\beta,\gamma)dJ. \quad (9.2)$$

However, a function $\mathcal{T}_{M'M}{}^J$ as given by the extension of the Fredholm series (8.3) or by axiomatic S-matrix theory will behave like $e_{MM'}{}^J$ when M and M' tend to infinity. A few special cases of the asymptotic behavior of $e_{MM'}{}^J$ and $d_{M-M'}{}^J$ as given in Appendix I, Eqs. (I.18), (I.20a), (I.20b), and (I.21), show that the summation upon M and M' in Eq. (9.2) does not converge.

Therefore, here again, the summation upon all helicities M and M' forbids convergence. Accordingly, one cannot use Eq. (9.2) in order to investigate the asymptotic behavior of the connected part of the three-body scattering amplitude when one momentum transfer tends to infinity.

X. CONCLUSIONS

We have found essential differences between the three-body and the two-body scattering amplitudes when extended to complex angular momenta. In fact, the three-body problem is much more difficult than the two-body problem from the analytical standpoint, and we have not even touched any question of convergence, except to show up some obvious divergences.

The fact that no integral equation can be written for this problem must certainly be traced to the existence of infinitely many nonsense channels, and a careful

[40] See Morse and Feshbach, Ref. 39; Ref. 37, or Ref. 24.

[41] G. Szego, Ref. 24; Ref. 37.

[42] This equation was first written by J. B. Hartle, Phys. Rev. 134, B162 (1964).

examination of the sense and nonsense channels in the three-body case is certainly the next step to make before contemplating the formidable problem of investigating the convergence of Eqs. (8.4) and (8.5).

Although we are yet very far from being able to draw any conclusion, we have now come to the point where all difficulties of an essentially kinematical nature have been removed. It remains now the much more difficult task of investigating the convergence properties of the solution. The presence of complex singularities, which have very strong implications on the asymptotic properties in J will be certainly a determining feature for convergence. We expect to investigate this effect in a forthcoming paper.

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APPENDIX I: PROPERTIES OF THE ROTATION MATRICES

1. Definition of the $d_{MM}{}^J(x)$

The rotation matrix[43] corresponding to a rotation with Euler angles (α,β,γ) can be written[23]

$$\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma)=e^{-iM\alpha}d_{MM'}{}^J(\cos\beta)e^{-iM'\gamma}. \quad \text{(I.1)}$$

The function $d_{MM'}{}^J(x)$ is the regular solution near $x=1$ of the differential equation.

$$(x^2-1)y''+2xy'+\{(M^2+M'^2-2MM'x)/(1-x^2)] -J(J+1)\}y=0. \quad \text{(I.2)}$$

It can be related to the Jacobi polynomials $P_n{}^{(\alpha,\beta)}(x)$ by[41]

$$d_{MM'}{}^J(x)=\left[\frac{(J+M)!(J-M)!}{(J+M')!(J-M')!}\right]^{1/2} \times\left(\frac{1+x}{2}\right)^{(M+M')/2}\left(\frac{1-x}{2}\right)^{(M-M')/2} \times P_{J-M}{}^{(M-M',M+M')}(x) \quad \text{(I.3)}$$

[43] After the completion of this work, we received a preprint by Andrews and Gunson [M. Andrews and J. Gunson, University of Birmingham, 1964 (to be published)] which contains many of the results given in this Appendix. We have used their notations and sometimes their presentations of the results in this paper. We also received a preprint by M. H. Choudhury of Tait Institute of Mathematical Physics, University of Edinburgh; where the author considers essentially the same problem, and concludes the Hilbert-Schmidt character of the kernel. That is, unfortunately, an overlooking of difficulty (iii) to which this section is devoted.

or to the hypergeometric functions by

$$d_{MM'}{}^J(x)=\left[\frac{\Gamma(J+M+1)\Gamma(J-M'+1)}{\Gamma(J+M'+1)\Gamma(J-M+1)}\right]^{1/2} \times\left(\frac{1+x}{2}\right)^{(M+M')/2}\left(\frac{1-x}{2}\right)^{(M-M')/2} \times\frac{F(-J+M,\,J+M+1,\,1+M-M',\,\frac{1}{2}-\frac{1}{2}x)}{\Gamma(1+M-M')}. \quad \text{(I.4)}$$

Equations (I.3) and (I.4) are true only when both $M-M'$ and $M+M'$ are nonnegative. The other cases are given by the symmetry relations

$$\begin{aligned} d_{MM'}{}^J(x)&=(-1)^{M-M'}d_{M',M}{}^J(x)\\ &=(-1)^{M-M'}d_{-M,-M'}{}^J(x) \quad \text{(I.5)}\\ &=d_{-M',-M}{}^J(x). \end{aligned}$$

2. Extension to Complex J

$d_{MM'}{}^J(x)$ can be defined as the correctly normalized solution of Eq. (I.2) regular at $x=1$, or as given by Eq. (I.4). It is an analytic function of x with a cut going from $-\infty$ to -1 when $M+M'$ is even, to $+1$ when $M+M'$ is odd.

As a function of J it is the product of an entire function by the normalization factor

$$[\Gamma(J+M+1)\Gamma(J-M'+1)/\Gamma(J+M'+1)\Gamma(J-M+1)]^{1/2}$$

(for $M-M'>0$, $M+M'>0$).

The corresponding singularities when J is an integer are as follows:

(a) $|M|,|M'|\leq|J+\frac{1}{2}|-\frac{1}{2}$; no singularity, $d_{MM'}{}^J(x)$ finite (physical case);

(b) $|M|,|M'|>|J+\frac{1}{2}|-\frac{1}{2}$; $MM>0$; no singularity, $d_{MM'}{}^J(x)$ finite (but unphysical);

(c) $|M|,|M'|>|J+\frac{1}{2}|-\frac{1}{2}$; $MM'<0$; no singularity, $d_{MM'}{}^J(x)=0$;

(d) $|M|\leq|J+\frac{1}{2}|-\frac{1}{2}'\,M'>|J+\frac{1}{2}|-\frac{1}{2}$, a square-root branch point where $d_{MM'}{}^J(x)$ vanishes.

For $M=M'=0$, one has $d_{MM'}{}^J(x)=P_J(x)$. (I.6)

3. Definition of the $E_{MM'}{}^J(\alpha,\beta,\gamma)$

We shall define the function $e_{MM'}{}^J(x)$ as the solution of Eq. (I.2) which is regular when x is infinite. Its normalization is fixed by its relation with the Jacobi function of the second kind[41]

$$e_{MM'}{}^J(x)=(-1)^{M-M'}\left[\frac{\Gamma(J+M+1)\Gamma(J-M+1)}{\Gamma(J+M'+1)\Gamma(J-M'+1)}\right]^{1/2} \times\left(\frac{1+x}{2}\right)^{(M+M')/2}\left(\frac{x-1}{2}\right)^{(M-M')/2} \times Q_{J-M}{}^{M-M',M+M'}(x), \quad \text{(I.7)}$$

or, in terms of the hypergeometric function,

$$e_{MM'}{}^J(x)=\tfrac{1}{2}[\Gamma(J+M+1)\Gamma(J-M+1)\Gamma(J+M'+1)\times\Gamma(J-M'+1)]^{1/2}[\tfrac{1}{2}(1+x)]^{-(M+M')/2}\times\left(\frac{1-x}{2}\right)^{-(M-M')/2}\left(\frac{x-1}{2}\right)^{-J+M-1}\times\frac{F\left(J-M+1;J-M'+1;2J+2;\frac{2}{1-x}\right)}{\Gamma(2J+2)}. \quad (I.8)$$

We shall define

$$\mathcal{E}_{MM'}{}^J(\alpha,\beta,\gamma)=e^{-iM\alpha}e_{MM'}{}^J(\cos\beta)e^{-iM'\gamma}. \quad (I.9)$$

4. Properties of the $e_{MM'}{}^J(x)$

$e_{MM'}{}^J(x)$ is an analytic function of x with a cut going from $-\infty$ to $+1$. As a function of J it has singularities when J is an integer, as follows:

(a) when $|M|$, $|M'|\leq|J+\frac{1}{2}|-\frac{1}{2}$, $e_{MM'}{}^J(x)$ is finite for $J\geq 0$, has a pole of residue $d_{MM'}{}^J(x)$ for $J<-1$;

(b) when $|M|$, $|M'|>|J+\frac{1}{2}|-\frac{1}{2}$ and $MM'>0$; pole of residue $\frac{1}{2}d_{MM'}{}^J(x)$;

(c) $|M|$, $|M'|>|J+\frac{1}{2}|-\frac{1}{2}$ and $MM'<0$; pole of residue $(-1)^{J-M+1}d_{M,-M'}{}^J(x)$;

(d) $|M|\leq|J+\frac{1}{2}|-\frac{1}{2}$ and $|M'|>|J+\frac{1}{2}|-\frac{1}{2}$; square-root branch point where $e_{MM'}{}^J(x)$ is infinite.

One has[41]

$$e_{MM'}{}^J(-x)=-e^{\pm i\pi(J-M)}e_{MM'}{}^J(x)\quad(\mathrm{Im}x\gtrless 0). \quad (I.10)$$

The discontinuity of $e_{MM'}{}^J(x)$ along the cut going from -1 to $+1$ is[41]

$$e_{MM'}{}^J(x+i0)-e_{MM'}{}^J(x-i0)=-i\pi d_{MM'}{}^J(x). \quad (I.11)$$

Furthermore, one has

$$e_{MM'}{}^J(x)=[\pi/2\sin\pi(J-M)]\times[e^{\mp i\pi(J-M)}d_{MM'}{}^J(x)-d_{M,-M'}{}^J(-x)]\quad(\mathrm{Im}x\gtrless 0). \quad (I.12)$$

Equation (I.12) is derived from Eq. (2.6) of Bateman, Ref. 37, Sec. 29 taking into account the correct drawing of the cuts.

5. Extension of the Neumann Theorem

Equation (I.11) suggests relating the integral

$$\int_{-1}^{+1}f(x)d_{MM'}{}^J(x)dx,$$

when $f(x)$ is an analytic function of x in a neighborhood of the segment $(-1,+1)$ and J is an integer, to the contour integral

$$+\frac{1}{i\pi}\oint_C f(x)e_{MM'}{}^J(x)dx,$$

where C is a contour enclosing $(-1,+1)$. However, as $d_{MM'}{}^J(x)$ can have a singularity at $x=1$, some care must be exercised.

Proof of Equation (3.7)

Let us replace the integrand by

$$\tfrac{1}{2}[f(\alpha,\beta,\gamma)\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma)+f(\alpha+\pi,\beta,\gamma+\pi)\mathfrak{D}_{MM'}{}^J(\alpha+\pi,\beta,\gamma+\pi)]=\tfrac{1}{2}[f(\alpha,\beta,\gamma)+(-1)^{M+M'}f(\alpha+\pi,\beta,\gamma+\pi)]\times\mathfrak{D}_{MM'}{}^J(\alpha,\beta,\gamma).$$

This does not change the integral. Since $f(\alpha,\beta,\gamma)$ is defined on the rotation group (and not SU2) one has

$$f(\alpha+\pi,\beta,\gamma+\pi)=f(\alpha,-\beta,\gamma); \quad (I.13)$$

therefore the integrand contains $\frac{1}{2}[f(\alpha,\beta,\gamma)+(-1)^{M+M'}\times f(\alpha,-\beta,\gamma)]\equiv F(\alpha,\beta,\gamma)$. Since $f(\alpha,\beta,\gamma)$ is an analytic function of $\cos\beta$ and $\sin\beta$, $F(\alpha,\beta,\gamma)$ is an analytic function of $\cos\beta$ when $M+M'$ is even and $\sin\beta$ times an analytic function of $\cos\beta$ when $M+M'$ is odd. As $\sin\beta^{1+(-1)^{M+M'+1}}d_{MM'}{}^J(\beta)$ is always an analytic function of $\cos\beta$, Eq. (3.7) follows from Eq. (I.11).

6. Asymptotics

All the asymptotic properties given below can be obtained by using the asymptotic properties of the hypergeometric functions as given by Bateman as well as the identities satisfied by those functions.

(a) When x tends to infinity,

$$d_{MM'}{}^J(x)=\Gamma(2J+2)[\Gamma(J+M+1)\Gamma(J+M'+1)\Gamma(J-M+1)\Gamma(J-M'+1)]^{-1/2}(x/2)^{+J}, \quad (I.14)$$

$$e_{MM'}{}^J(x)=\tfrac{1}{2}e^{\pm i\frac{1}{2}\pi(M-M')}\frac{[\Gamma(J+M+1)\Gamma(J-M+1)\Gamma(J+M'+1)\Gamma(J-M'+1)]^{1/2}}{\Gamma(2J+2)}(x/2)^{-J-1};\quad \mathrm{Im}x\gtrless 0. \quad (I.15)$$

(b) When J tends to infinity ($x=\cos\beta$),

$$d_{MM'}{}^J(\cos\beta)\sim(\cos\tfrac{1}{2}\beta)^{M+M'}(\sin\tfrac{1}{2}\beta)^{M-M'}2^{1/2}(\pi J\sin\beta)^{-1/2}\cos\{|J-M+\tfrac{1}{2}|\beta-\tfrac{1}{4}\pi\}, \quad (I.16)$$

$$e_{MM'}{}^J(x)\sim(\tfrac{1}{2}\pi)^{1/2}e^{\pm i\frac{1}{2}\pi(M-M')}J^{-1/2}[x-(x^2-1)^{1/2}]^{J+1/2}(x^2-1)^{-1/4};\quad \mathrm{Im}x\gtrless 0\quad\text{for}\quad -\pi<\arg J<\pi. \quad (I.17)$$

(c) When M tends to infinity, M' fixed,

$$d_{MM'}{}^J(x)\sim\text{constant}\times\left[\frac{\Gamma(J-M'+1)}{\Gamma(J+M'+1)}\right]^{1/2}[\sin\pi(J-M)]^{1/2}M^{2M'-1}\left(\frac{1-x}{1+x}\right)^{M/2}\left[\frac{4}{(1-x)(1+x)}\right]^{M'/2}, \tag{I.18}$$

$$e_{MM'}{}^J(x)\sim\text{constant}\times[\sin\pi(J-M)]^{1/2}\left\{e^{\pm i(J-M)}\left[\frac{\Gamma(J-M'+1)}{\Gamma(J+M'+1)}\right]^{1/2}M^{2M'-1}\left(\frac{1-x}{1+x}\right)^{M/2}\left[\frac{4}{(1-x)(1+x)}\right]^{M/2}\right.$$
$$\left.-\left[\frac{\Gamma(J+M'+1)}{\Gamma(J-M+1)}\right]^{1/2}M^{-2M'-1}\left(\frac{1+x}{1-x}\right)^{M/2}\left[\frac{4}{(1-x)(1+x)}\right]^{-M'/2}\right\}. \tag{I.19}$$

(d) When $M=M'$ tends to infinity,

$$d_{MM}{}^J(x)\sim[(1+x)/2]^J\,2^{-(2J+1)}(\pi M)^{-1/2}[1-\xi+(\xi^2-1)^{1/2}]^{-1/2}$$
$$\times[1+\xi-(\xi^2-1)^{1/2}]^{2J+1/2}[\xi+(\xi^2-1)^{1/2}]^{M+J};\ \xi=1-2(x-1)/(x+1). \tag{I.20a}$$

(e) When $M=-M'$ tends to infinity

$$d_{M,-M}{}^J(x)\sim\frac{\sin\pi(J-M)\Gamma(\frac{1}{2})}{\pi\Gamma(M-J)\Gamma(M+J)}\left(\frac{1-x}{2}\right)^M 2^{-2J}\left(\frac{\xi-1}{2}\right)^{-J+1}M^{-1/2}$$
$$\times[\xi-(\xi^2-1)^{1/2}]^{-J+M}[1-\xi+(\xi^2-1)^{1/2}]^{2J+1/2}[1-\xi-(\xi^2-1)^{1/2}]^{-1/2};\ \xi=1-4(1-x)^{-1}. \tag{I.20b}$$

(f) Using Eqs. (I.12), (I.20a), and (I.20b), we find the asymptotic behavior of $e_{MM}{}^J(x)$ when $M=M'$ tends to infinity:

$$e_{MM}{}^J(x)=\frac{\pi e^{\mp i\pi(J-M)}}{2\sin\pi(J-M)}\left(\frac{1+x}{2}\right)^J 2^{-(2J+1)}(\pi M)^{-1/2}$$
$$\times[1-\xi+(\xi^2-1)^{1/2}]^{-1/2}[1+\xi-(\xi^2-1)^{1/2}]^{2J+1/2}[\xi+(\xi^2-1)^{1/2}]^{J+M}, \tag{I.21}$$

$\xi=1-2(x-1)/(x+1)$.

7. Grouplike Properties

A fundamental property of the rotation matrices is their group property

$$\mathfrak{D}_{MM'}{}^J(R'R)=\sum_{M''=-J}^{M''=J}\mathfrak{D}_{MM''}{}^J(R')\mathfrak{D}_{M''M'}{}^J(R),$$
(J an integer),

where R and R' are two rotations characterized by their Euler angles (α,β,γ) and (α,β',γ') and $R'R$ is the product of R' and R. It can be extended to any value of J by giving up the conditions on M'' [13,14]

$$\mathfrak{D}_{MM'}{}^J(R'R)=\sum_{M''=-\infty}^{M''=+\infty}\mathfrak{D}_{MM''}{}^J(R)\mathfrak{D}_{M''M'}{}^J(R). \tag{I.22}$$

A particular case of (I.22) is the well-known addition property of Legendre functions,[44,45]

$$P_j(\cos\beta\cos\beta'+\sin\beta\sin\beta'\cos\phi)$$
$$=P_j(\cos\beta)P_j(\cos\beta')+2\sum_{m=1}^{\infty}\frac{\Gamma(j-m+1)}{\Gamma(j+m+1)}(-1)^m$$
$$\times P_j{}^m(\cos\beta)P_j{}^m(\cos\beta')\cos m\phi,$$

[44] Reference 37, pp. 76–77.
[45] Reference 37, p. 169.

which is a particular case of Eq. (I.22) for $M=M'=0$, where use has been made of Eqs. (I.1) and (7.3) and where we have defined $\phi=\gamma'+\alpha$.

One can prove Eq. (I.22) in several ways. For simplest proof, consider the Fourier expansion of $d_{MM'}{}^J(\cos\beta\cos\beta'+\sin\beta\sin\beta'\cos\phi)$ in a series of terms proportional to $e^{M''\phi}$. Then, except for convergence considerations, Eq. (I.22) is equivalent to the set of equalities

$$2\pi^{-1}\int_0^{2\pi}d_{MM'}{}^J(\cos\beta\cos\beta'+\sin\beta\sin\beta'\cos\phi)e^{-iM''\phi}d\phi$$
$$=d_{MM''}{}^J(\cos\beta)d_{M''M'}{}^J(\cos\beta'). \tag{I.23}$$

Equation (I.22) can be proved by applying the differential operator that appears in (I.2) to the left-hand side in order to prove that the right-hand side is proportional to $d_{MM'}{}^J(\cos\beta)$. The calculation is in fact rather tricky.

Another proof consists in considering real values of β and β', such that $|\cos\beta|$, $|\cos\beta'|$, and $|\cos\beta\cos\beta'+\sin\beta\sin\beta'\cos\phi|$ stay less than 1 when $\cos\phi$ runs from -1 to $+1$. Then, for M, M', and M'' fixed, Eq. (I.23) is true for any positive integral value of J. Furthermore, Eq. (I.16) shows that both members do not increase as rapidly as $e^{i\pi j}$ when j tends to infinity. Both members can be then extended through to complex values of j in a way that satisfies the conditions of Carlson's theorem. Therefore, Eq. (I.23) is true for any value of J. The conditions on $\cos\beta$ and $\cos\beta'$ can be removed by noticing that both members are analytic functions of $\cos\beta$ and $\cos\beta'$. Finally, Eq. (I.23) is true as long as $\cos\beta\neq-1$, $\cos\beta'\neq-1$, and $\cos\beta'+\sin\beta\sin\beta'\cos\phi$ does not pass through -1 when $\cos\phi$ varies from -1 to $+1$. In fact, even that last condition could be removed by displacing the integration contour.

The domain of convergence of Eq. (I.22) is easily deduced from Eq. (I.18). Equation (I.22) converges inside a corona (set closed by two circles of radii R and R', $RR'=1$) passing through the point $\cot\beta''=-\cot\beta\cot\beta'$.

Equation (I.22) can be extended to the rotation matrices of the second kind as follows for $1<\cos\beta<\cos\beta'$:

$$\mathcal{E}_{MM'}{}^J(R'R)=\sum_{M''=-\infty}^{M''=+\infty}\mathcal{E}_{MM''}{}^J(R')\mathfrak{D}_{M''M'}{}^J(R). \quad \text{(I.24)}$$

Equation (I.24) follows immediately from application of Eqs. (I.12) to (I.13).

APPENDIX II: THE TWO-BODY SCATTERING AMPLITUDE

1. Analyticity Properties

We need to know the analytic properties[34] of the off-the-energy shell two-body scattering amplitude $T(\mathbf{p},\mathbf{p}',\xi)$ solution of the Lippmann-Schwinger equation

$$T(\mathbf{p},\mathbf{p}',\xi)=\tilde{V}(\mathbf{p}-\mathbf{p}')+\int\frac{d^3q}{(2\pi)^3}\tilde{V}(\mathbf{p}-\mathbf{q})\frac{1}{q^2-\xi}T(\mathbf{q},\mathbf{p}',\xi), \quad \text{(II.1)}$$

where $T(\mathbf{p},\mathbf{p}',\xi)$ can be considered as a function of the invariants p^2, p'^2, $=t=-(\mathbf{p}-\mathbf{p}')^2$ and of ξ. While all the techniques which have been applied for investigating the on-the-energy-shell amplitude give some information on the off-the-energy-shell amplitude, this new problem is more complicated, since it involves a function of several complex variables and not only of two variables. Therefore our approach will rather have a trial-and-error character than the form of an exact proof.

We shall restrict ourselves to the case in which the potential $V(r)$ is a superposition of Yukawa potentials and, for simplicity, we shall write explicitly only the case of an isolated Yukawa potential $V(r)=ge^{-\mu r}/r$, so that its Fourier transform is $\tilde{V}(\mathbf{p})=g(p^2+\mu^2)^{-1}$. It would be easy to extend our results to the more general case.

Let us first consider the properties of T as a function of ξ. It is well known that the Lippmann-Schwinger kernel is of the Hilbert-Schmidt type.[46] As it is also bounded and operator-analytic as a function of ξ, T is an analytic function of ξ except for a cut going from 0 to infinity and poles which are located at the energies of the bound states. The position of these singularities is independent of p^2, p'^2, and t.

Let us now go to the analytic properties in t. Let us write[47]

$$T(\mathbf{p},\mathbf{p}',\xi)=\int\exp\left[-i\frac{\mathbf{p}'-\mathbf{p}}{2}(\mathbf{x}+\mathbf{y})-i\frac{\mathbf{p}'+\mathbf{p}}{2}(\mathbf{x}-\mathbf{y})\right]\times V^{1/2}(x)K(\mathbf{x},\mathbf{y};\xi)V^{1/2}(y)d^3xd^3y, \quad \text{(II.2)}$$

where $K(\mathbf{x},\mathbf{y},\xi)$ is the resolvent of the kernel $V^{1/2}[1/(H_0-\xi)]V^{1/2}$, which is L^2. Keeping $\mathbf{p}+\mathbf{p}'$ fixed, and using the exponential decrease of the potentials when x or y tend to infinity, it is easy to show that T is analytic for

$$|\mathrm{Im}(\mathbf{p}-\mathbf{p}')|<\mu/2, \quad |\mathrm{Im}(\mathbf{p}+\mathbf{p}')|<\mu/2. \quad \text{(II.3)}$$

Since $T(\mathbf{p},\mathbf{p}',\xi)$ is rotational-invariant, it is an analytic function of the invariants p^2, p'^2, and t in the image Δ of the domain (II.3). The form of Δ is rather complicated, however; when p^2 and p'^2 are real and positive the projection of Δ upon the t-complex plane is the interior of the parabola[48]

$$\mu^2\mathrm{Im}t=2(\mathrm{Re}t+\mu^2)^2. \quad \text{(II.4)}$$

Our only need for this domain is to make sure that the Legendre series expansion of T as a function of the scattering angle $\cos\theta=\mathbf{p}\cdot\mathbf{p}'/pp'$ converges, at least for some values of p and p'

$$T(\mathbf{p},\mathbf{p}',\xi)=\frac{\pi}{p}\sum_l(2l+1)T_l(p,p',\xi)P_l(\cos\theta). \quad \text{(II.5)}$$

The partial-wave amplitude $T_l(p,p',\xi)$ satisfies the equation[11]

$$T_l(p,p',\xi)=\frac{g}{2p'}Q_l\left(\frac{p^2+p'^2+\mu^2}{2pp'}\right)+\frac{g}{\pi}\int_0^\infty\frac{dq}{q^2-\xi}Q_l\left(\frac{p^2+q^2+\mu^2}{2pq}\right)T_l(q,p',\xi). \quad \text{(II.6)}$$

The inhomogeneous term has a singularity at $p=\pm p'\pm i\mu$. Letting p become complex in the integral, we see that the integral is convergent and defines an analytic function of p within the strip D_1 defined by

$$D_1:\ |\mathrm{Im}p|<\mu. \quad \text{(II.7)}$$

We can now deform the contour of integration of q in a new contour, Γ_1, also going from 0 up to an infinite real value. Γ_1 must stay within D_1 and avoid the singularity of the Born term at $\pm p'\pm i\mu$ and the singularity of the

[46] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[47] S. Weinberg, M. Scadron, and J. Wright, Phys. Rev. **135**, B202 (1964). Also F. Coester, *ibid.* **133**, B1516 (1964), and K. Meetz, J. Math. Phys. **3**, 690 (1961).

[48] This domain of analyticity can be extended by a trick due to Bottino, Longoni, and Regge, Ref. 10.

integrand at $q=\pm\xi$. Now letting p become complex in the integral along Γ_1, we see that $T_l(p,p',\xi)$ is analytic within the strip

$$D_2:\ |\mathrm{Im}p|<2\mu, \tag{II.8}$$

indented by cuts, parallel to the imaginary axis and ending at $\pm p'\pm i\mu$ and $\pm\xi$. This procedure can be iterated to show that $T_l(p,p',\xi)$ is an analytic function of p, except for these cuts. The analytic properties of $T_l(p,p',\xi)$ as a function of p' could be obtained in the same way by using the Lippmann-Schwinger equation for the initial state. The results are of course the same as for p.

Let us now notice that the cut at $p=\pm\sqrt{\xi}$ does not in fact exist for the full amplitude. Indeed, it would mean that there is a singularity in ξ which depends upon the value of p, which we have shown not to be true.

The singularities at $p=-p'\pm i\mu$ do not exist in the full amplitude. Indeed, if we put $p^2=p'^2=\xi$, we know that this singularity is absent. It represents the left-hand cut of the partial-wave amplitude associated with the singularity at $t=+\mu^2$ of the full amplitude. The same argument holds for the singularities at $p=p'\pm\mu$.

This analysis, whose results are given in Sec. VI, is admittedly sketchy and nonrigorous.

2. Asymptotic Properties

We need to know the asymptotic behavior of $T(\mathbf{p},\mathbf{p}',\xi)$ when both p' and $\cos\theta$ tend to infinity like some constant multiple of a complex parameter $\omega^{1/2}$

$$p'\sim A\omega^{1/2},\quad \cos\theta\sim B\omega^{1/2}. \tag{II.9}$$

It is difficult to find the exact behavior by starting from Eq. (II.1), because $\tilde{V}(\mathbf{p}-\mathbf{q})$ can become infinite upon some part of the q domain of integration. We shall therefore use the results of the analysis of Eq. (II.6) by Brown, Fivel, Lee, and Sawyer.[11] They have shown that this equation is of the Fredholm type when the solution is sought within the Hilbert space with metric $dq(q^2-\xi)^{-1}$, when $\mathrm{Im}\xi\neq0$. Therefore, the solution $T_l(p,p',\xi)$ is a meromorphic function of ξ for fixed l or of l for fixed ξ. Its poles are the zeros of the Fredholm determinant

$$D(l,\xi)=0. \tag{II.10}$$

One can use a Watson-Sommerfeld transformation in order to show that, for infinite values of $\cos\theta$, the asymptotic behavior of the full amplitude is given by

$$T(\mathbf{p},\mathbf{p}',\xi)\sim\frac{1}{p}\frac{2\alpha(\xi)+1}{\sin\pi\alpha(\xi)}P_{\alpha(\xi)}(-\cos\theta)\beta(p,p',\xi), \tag{II.11}$$

where $\beta(p,p',\xi)$ is the residue of $T_l(p,p',\xi)$ at $l=\alpha(\xi)$ and $\alpha(\xi)$ is the leading Regge pole.

In order to find the behavior of $\beta(p,p',\xi)$ when p' tends to infinity, we shall notice that it is equal, up to constant factors depending only upon ξ, to the scalar product of the inhomogeneous term of Eq. (II.6) and the solution $\psi_l(q)$ of the homogeneous equation

$$\beta(p,p',\xi)\propto\int_0^\infty\frac{dq}{q^2-\xi}\frac{1}{p'}Q_l\left(\frac{q^2+p'^2+\mu^2}{2qp'}\right)\psi_l(q), \tag{II.12}$$

where $\psi(q)$ satisfies

$$\psi_l(p)=\frac{g}{\pi}\int_0^\infty\frac{dq}{q^2-\xi}Q_l\left(\frac{p^2+q^2+\mu^2}{2pq}\right)\psi_l(q);\ l=\alpha(\xi). \tag{II.13}$$

When l is an integer, we know the asymptotic behavior of $\psi_l(p)$. In fact we know that $\psi(p)$ must be equal to the wave function divided by p^2 [a factor p for the one in Eq. (II.5) and another one for the metric dq in place of q^2dq in Eq. (II.6)]. As the configuration-space wave function $\psi_l(\mathbf{x})$ behaves like r^l when $r\to0$, $\psi_l(p)$ behaves like p^{-l-1} when $p\to\infty$.

One can show, using Eq. (II.13), that this behavior is still consistent with complex values of l. In fact, let us assume that $\psi(q)\sim q^{-l-1}$ within the integrand of Eq. (II.13). Let us now split the integration domain into two parts going from 0 to ρp and from ρp to ∞, where ρ is a fixed number $\rho\ll1$, say $\rho=\frac{1}{10}$. Then

$$\begin{aligned}\int_0^{\rho p}&Q_l\left(\frac{q^2+p^2+\mu^2}{2qp}\right)\psi_l(q)\frac{dq}{q^2-\xi}\\&=\int_0^{\rho p}\left(\frac{2q}{p}\right)^{l+1}[1+O(1)]\psi_l(q)\frac{dq}{q^2-\xi}\\&<\frac{1}{p^{l+1}}\int_0^\infty(2q)^{l+1}[1+O(1)]\psi_l(q)\frac{dq}{q^2-\xi}\\&<\frac{\text{constant}}{p^{l+1}},\end{aligned} \tag{II.14}$$

and

$$\begin{aligned}\int_{\rho p}^\infty&Q_l\left(\frac{q^2+p^2+\mu^2}{2qp}\right)\psi_l(q)\frac{dq}{q^2-\xi}\\&<\int_{\rho p}^\infty Q_l\left(\frac{q^2+p^2+\mu^2}{2qp}\right)q^{-l-1}[1+O(1)]\frac{dq}{q^2}\\&<(\rho p)^{-l-1-\epsilon}\int_{\rho p}^\infty Q_l\left(\frac{q^2+p^2+\mu^2}{2qp}\right)\frac{dq}{q^{2-\epsilon}}\\&<\frac{\text{constant}\times\ln p}{p^{l+1+\epsilon}}.\end{aligned} \tag{II.15}$$

Although this is not a direct proof, it shows that, by continuity, the limit $\psi_l(q)q^{-l-1}$ extrapolated from integral values of l is consistently defined and that

$$|\psi_l(q)|\sim q^{-\lambda-2},\quad\text{for}\quad q\to\infty,\quad \lambda=\mathrm{Re}l. \tag{II.16}$$

Now exactly the same technique can be used in

Eq. (II.12) to show that

$$|\beta(p,p',\xi)|\sim p'^{-\lambda-2}, \quad \text{for} \quad p'\to\infty. \tag{II.17}$$

So, using the asymptotic form of P_α and Eq. (II.18), we see that the leading term in Eq. (II.11) becomes proportional to $1/\omega$ when p' and $\cos\theta$ are given by Eq. (II.9). Notice the cancellation of the dependence on the Regge Pole.[49]

In fact, this analysis is inconclusive insofar as we should take into accout the contributions of all the Regge poles which all behave as ω^{-1}. We should therefore consider it as a proof that the leading Regge pole does not determine the asymptotic behavior of $T(\mathbf{p},\mathbf{p}',\xi)$.

Let us now write the complete expression of Eq. (II.11)

$$T(\mathbf{p},\mathbf{p}',\xi)=\sum_{\alpha_j(\xi)}\frac{2\alpha_j(\xi)+1}{\sin\pi\alpha_j(\xi)}P_{\alpha_j(\xi)}(\cos\theta)\beta_j(p,p',\xi)$$
$$+\frac{1}{\pi ip}\int_{-i\infty}^{+i\infty}\frac{2l+1}{\sin\pi l}P_l(-\cos\theta)T_l(p,p',\xi)dl. \tag{II.18}$$

The same kind of analysis shows that, when $\mathrm{Re}l=0$, $|T_l(p,p',\xi)|$ behaves like $1/p'$ when p' tends to infinity, therefore $T(\mathbf{p},\mathbf{p}',\xi)$ behaves like $\omega^{-1/2}$, the precise asymptotic behavior, (i.e., the coefficient of $\omega^{-1/2}$) depending upon A and B in Eq. (II.9).

An alternative approach would have been to use the Lippmann-Schwinger equation

$$T(\mathbf{p},\mathbf{p}',\xi)=\tilde{V}(\mathbf{p}-\mathbf{p}')$$
$$+\int\frac{d^3q}{(2\pi)^3}T(\mathbf{p},\mathbf{q},\xi)\frac{1}{q^2-\xi}\tilde{V}(\mathbf{q}-\mathbf{p}'), \tag{II.19}$$

together with Fadeev's bound,

$$|T(\mathbf{p},\mathbf{q},\xi)|<\text{constant}(1+q)^{-1-\epsilon}, \tag{II.20}$$

valid for physical values of $\mathbf{p}$ and $\mathbf{q}$. This shows that for p physical and within the domain of p' and $\cos\theta$ where (II.19) converges, $|T(\mathbf{p},\mathbf{p}',\xi)|$ is bounded by a constant. But this limitation to the domain where (II.19) is well defined is not enough for our purposes.

[49] The cancellation between the Regge asymptotic behavior and the behavior of the residue is the mechanism responsible for the cancellation of the Amati-Fubini and Stanghellini cuts (see Ref. 5). These results could be extended to show this cancellation in the sum of all three-body ladder diagrams; in fact, it is easy to write generalized Fadeev equations which bear the same relation to the Bethe-Salpeter equation as the Fadeev equations bear to the Lippmann-Schwinger equations.

APPENDIX III: TCHEBYSHEFF FUNCTIONS

The polynomials[41]

$$T_n(x)=\sin\tfrac{1}{2}n\beta/\sin\tfrac{1}{2}\beta, \quad x=\cos\beta \tag{III.1}$$

are orthogonal on the segment $(-1,+1)$ with the weight function $(1-x^2)^{1/2}$. They are a special case of Jacobi polynomials[42]

$$T_n(x)=\pi^{1/2}[\Gamma(n-1)/\Gamma(n-\tfrac{1}{2})]P_{n-1}^{(-1/2,-1/2)}(x). \tag{III.2}$$

They satisfy the differential equation

$$(x^2-1)(d^2T_n/dx^2)+x(dT_n/dx)-(n-1)^2T_n=0. \tag{III.3}$$

Another solution of Eq. (III.3), regular for $x=\infty$, can be defined as

$$S_n(x)=\frac{\pi}{2^{n-1}n(n-1)}(x-1)^{-n+1/2}(x+1)^{1/2}$$
$$\times F\left(n-\tfrac{1}{2},n;2n-1;\frac{2}{1-x}\right) \tag{III.4}$$

for $n=2, 3, \cdots$, and

$$S_1(x)=\pi^{1/2}\frac{\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}\left[\ln(1+x)-\frac{1}{\pi}(x^2-1)^{1/2}\right.$$
$$\left.\times\int_{-1}^{+1}\frac{(1-t^2)^{-1/2}}{x-t}\ln(1+t)dt\right]. \tag{III.5}$$

The Neuman theorem for the Jacobi functions gives[42]

$$\int_{-1}^{+1}(1-x^2)^{-1/2}T_n(x)f(x)dx$$
$$=\frac{1}{i\pi}\int_C(y^2-1)^{-1/2}S_n(y)f(y)dy, \tag{III.6}$$

where $f(x)$ is an analytic function in a neighborhood of the segment $(-1,+1)$ and C is a contour enclosing that segment. Formula (III.6) is valid for $n=1, 2, 3, \cdots$.