Singular Logarithmic Potentials and Peratization*

H. H. ALY,‡ RIAZUDDIN, AND A. H. ZIMERMAN†
Department of Physics and Astronomy, University of Rochester, Rochester, New York
(Received 29 June 1964)

Logarithmic singular potentials are considered to test the applicability of the peratization technique. It is shown that if one sums up the leading singular terms in each order of the perturbation series no finite results can be obtained.

RECENTLY, there have been some attempts to obtain meaningful finite results in nonrenormalizable field theories. One of the techniques adopted by Feinberg and Pais[1] for this purpose is called peratization, which may be summarized as follows: From the series of the perturbation expansion in terms of the coupling constant, the most singular parts in terms of the cutoff parameter are isolated in each order of the perturbation. One then tries to sum up the series of the most singular terms in order to see whether the sum is finite when the cutoff parameter tends to *infinity*. Terms which are less singular than the first set are then considered and a similar treatment is applied to them. This process of isolating singular terms by their degree of singularity and summation continues till one is left with only finite parts of the perturbation series.

In order to supply a ground for the peratization procedure, some authors[2,3] have considered repulsive potentials singular at the origin in nonrelativistic quantum theory. These potentials satisfy the following two conditions:

(i) $\int_0^b r|V(r)|dr$ is divergent for any fixed value of $b>0$.

(ii) $\int_c^\infty r^2|V(r)|dr$ exists for any $c>0$.

In particular the class of potentials considered by Khuri and Pais[2] and Tiktopoulos and Treiman[3] are of the type

$$V(r)=+g/r^m, \tag{1}$$

where $g>0$ and $m>3$. Since the general solution of the radial Schrödinger equation given in the form

$$\{(d^2/dr^2)+k^2-[l(l+1)/r^2]-V(r)\}\psi(k,l,r)=0 \tag{2}$$

can be obtained [for $k=0$ for the class of potentials (1)] in a closed form, this may be compared to the one obtained by the peratization procedure. It has in fact been shown[2,3] that the peratization procedure works and gives the correct answer for the class of potentials (1).

In order to see whether the peratization procedure can be extended to other classes of potentials, we consider below the applicability of the scheme for a physically more realistic potential (which may have some field-theoretic origin) given by

$$V(r)=+g[(\ln r)^2/r^4]. \tag{3}$$

This potential satisfies the required conditions (i) and (ii). We show that the scattering amplitude for this potential exists at zero energy. We then obtain the Born series for this potential by introducing a cutoff parameter a, where in the end $a\to 0$. In each order of the coupling constant, we obtain a leading power of logarithm in the cutoff parameter a. It is possible then to sum the series of the leading powers of logarithm of the cutoff parameter. The resulting sum, we show, does not tend to a finite limit when the cutoff parameter $a\to 0$. Thus we have an example of a physically realistic potential for which we know that the amplitude exists, and for which we show that the peratization procedure does not seem to apply, in the sense that the sum of the leading singularities does not go to a finite limit when the cutoff parameter $a\to 0$. The peratization procedure therefore must be viewed and applied with extreme caution. Below we give the details of the calculation.

The integral equation for the regular solution corresponding to $l=0$ and $k=0$ for the potential (3) is

$$\psi(r)=r-gr\int_r^\infty \frac{(\ln r')^2}{r'^4}\psi(r')dr'-g\int_0^r r'\frac{(\ln r')^2}{r'^4}\psi(r')dr' \tag{4}$$

while the s-wave amplitude is given by

$$\frac{\tan\delta}{k}=-g\int_0^\infty r\frac{(\ln r)^2}{r^4}\psi(r)dr. \tag{5}$$

From the differential Eq. (2) with potential (3), it is easy to see that the solution for $k=0$, $l=0$ is asymptotically given by

$$\psi(r)\to r \quad \text{as} \quad r\to\infty\,, \tag{6}$$

* Work supported by the U. S. Atomic Energy Commission and the U. S. Air Force.
† On leave of absence from the Instituto de Fisica Teorica, Sao Paulo, Brazil.
‡ Present address: Max-Planck Institut für Physik und Astrophysik, München.

[1] G. Feinberg and A. Pais, Phys. Rev. 131, 2724 (1963).
[2] N. N. Khuri and A. Pais, Rev. Mod. Phys. 36, 590 (1964).
[3] George Tiktopoulos and S. B. Treiman, Phys. Rev. 134, B844 (1964).

while near the origin, using the WKB method,[4] we obtain

$$\psi(r) \underset{r\to 0}{\sim} \frac{r}{(\ln r)^{1/2}} \exp\left(-g^{1/2}\int_{r_0}^{r} \frac{\ln r'}{r'^2} dr'\right). \tag{7}$$

Therefore the potential problem which we are considering admits a regular solution having the right boundary conditions, namely, $\psi(r)\to r$ as $r\to\infty$ and $\psi(r)\to 0$ as $r\to 0$, so that according to (5), we have a finite s-wave scattering amplitude.

Let us now try to obtain the solution $\psi(r)$ and hence $(\tan\delta)/k$ by using the peratization technique. For this purpose we replace the actual potential (3) by a cutoff potential

$$V_a(r)=\theta(r-a)V(r), \tag{8}$$

so that the Born series in g in (4) exists. We get (for $r>a$)

$$\begin{aligned}
\psi_0(r)&=r,\\
\psi_1(r)&=gr\left\{\left[\frac{(\ln r)^2}{2r^2}+\frac{3}{2}\frac{\ln r}{r^2}+\frac{7}{4r^2}\right]-\frac{1}{r}\left[\frac{(\ln a)^2}{a}+\frac{2\ln a}{a}+\frac{2}{a}\right]\right\},\\
\psi_2(r)&=g^2r\left\{\left[\frac{(\ln r)^4}{24r^4}+\frac{2}{9}\frac{(\ln r)^3}{r^4}+\cdots\right]-\left[\left(\frac{(\ln a)^2}{a}+\frac{2\ln a}{a}+\frac{2}{a}\right)\left(\frac{1}{6}\frac{(\ln r)^2}{r^3}+\cdots\right)\right]+\frac{1}{r}\left[\frac{1}{3}\frac{(\ln a)^4}{a^3}+\cdots\right]\right\},
\end{aligned} \tag{9}$$

etc. The corresponding amplitudes are given by

$$\begin{aligned}
(\tan\delta)/k|_0&=-g\left[\frac{(\ln a)^2}{a}+\frac{2\ln a}{a}+\frac{2}{a}\right],\\
(\tan\delta)/k|_1&=g^2\left[\frac{1}{3}\frac{(\ln a)^4}{a^3}+\frac{7}{9}\frac{(\ln a)^3}{a^3}+\frac{17}{18}\frac{(\ln a)^2}{a^3}+\frac{17}{27}\frac{\ln a}{a^3}+\frac{17}{81}\frac{1}{a^3}\right],\\
(\tan\delta)/k|_2&=-g^3\left[\frac{2}{15}\frac{(\ln a)^6}{a^5}+\cdots\right],
\end{aligned} \tag{10}$$

etc.

If we consider the sum of the leading singularities in (10) for $a\to 0$, we obtain

$$\begin{aligned}
(\tan\delta)/k&=-g^{1/2}\ln a\left[g^{1/2}\frac{\ln a}{a}-\tfrac{1}{3}g^{3/2}\frac{(\ln a)^3}{a^3}+\frac{2}{15}\frac{(\ln a)^5}{a^5}\cdots\right]\\
&=-g^{1/2}(\ln a)\tanh[g^{1/2}(\ln a/a)].
\end{aligned} \tag{11}$$

For $a\to 0$,

$$(\tan\delta)/k\sim -g^{1/2}(\ln a), \tag{12}$$

which is not defined. Hence in the peratization approach in the sense of summing only the leading singularities in each order of the perturbation series, the scattering amplitude is not even defined, whereas we know from the consideration of the Schrödinger equation that the scattering amplitude does in fact exist for potential (3). Essentially the same conclusion is also deduced if we sum the leading singularities as $a\to 0$ in the perturbation series of the wave function as given in Eq. (9) near $r=0$. We then obtain

$$\psi(r)\sim r\left[\cosh\left(g^{1/2}\frac{\ln r}{r}\right)-\frac{\ln a}{\ln r}\sinh\left(g^{1/2}\frac{\ln r}{r}\right)\tanh\left(g^{1/2}\frac{\ln a}{a}\right)\right],$$

which does not exist when $a\to 0$. We may then conclude that the wave function does not exist near $r=0$ in the peratization approach used in the sense discussed above, whereas we know from the solution of the Schrödinger equation that it does exist near $r=0$ for the potential (3) and is given in (7).

Our approach has some similarity to the one used by Landau *et al.*,[5] who concluded, by summing the leading logarithmic singularities in each order of perturbation series for the self-energy part in the renormalized photon propagator in quantum electrodynamics, that one is led to a nonsensical answer. In our approach we have an example where the exact amplitude at zero energy is defined and where the approximation of summing the leading singularities in the cutoff parameter in each order of the perturbation series gives a nonsensical answer when the cutoff parameter $a\to 0$. While we have not been able to sum up the next leading singularities as $a\to 0$, it is unlikely that they can cancel the infinity which we obtain from the summing of leading singularities. In any case we have demonstrated the nonapplicability of summing principal logarithms as $a\to 0$ in each order of perturbation series.

ACKNOWLEDGMENTS

We are deeply indebted to Professor N. N. Khuri for very helpful discussions. We also want to thank Professor S. Okubo for helpful comments.

[4] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955).

[5] N. N. Bogoliubuev and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), p. 528.