it may be possible to establish the existence and then the spin of a weak boson. We predict that double lepton events will proceed via an $S-P$, rather than a $V-A$ interaction; but this may not be a spectacular effect because neither the nuclear recoil nor the final neutrino momentum is measured.[19] Angular distributions of the "elastic" events may provide the distinguishing analyzer; a report on this will appear shortly. If the experiments should favor a scalar intermediary boson, then it will be worthwhile to study the mathematical structure of the theory in a more thorough fashion.

[19] I am indebted to Professor R. E. Norton for the following remark: If $g_2=0$, then the events in which a μe lepton pair is produced via the decay of a real scalar boson into $\nu+e$ will be suppressed relative to $\mu\mu$ events (by the decay of the scalar boson into $\nu+\mu$) to the same extent, and for the same reason, as $\pi\to\nu+e$ is suppressed relative to $\pi\to\nu+\mu$. Until the existence of an intermediary meson is definitely established and a selection of events proceeding through a *real* intermediary can be made, this test cannot be applied.

PHYSICAL REVIEW VOLUME 136, NUMBER 4B 23 NOVEMBER 1964

Inelastic Effects of the N^* and ρ on Pion-Nucleon Scattering*†

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(Received 6 July 1964)

A formalism is developed for calculating the pion-nucleon scattering amplitude which conveniently includes the inelastic production of a (3,3) isobar, and is extended to include ρ-meson production. The exact unitarity relations are found to contain nondiagonal terms which, when simplified, are interpreted as the overlap between resonances. Calculations based on the N/D method are performed in which the unphysical discontinuities are evaluated from single-nucleon or pion-exchange diagrams. The numerical solution of the resulting integral equations is compared to simpler approximations and to experiment. The importance of both the isobar and the ρ is pointed out. For the $D_{3/2}$ channel, a resonance was found for $T=\frac{1}{2}$ but not for $T=\frac{3}{2}$. No resonance was found for either of the $F_{5/2}$ or $D_{5/2}$ channels.

I. INTRODUCTION

THE importance of production channels in explaining the higher resonances of pion-nucleon scattering has been pointed out by several authors,[1–3] who concentrated primarily on the inelastic channel $\pi+\pi+N$ with the two pions resonating to form a ρ meson. We wish to calculate the scattering amplitude for the process $\pi+N\to\pi+N$ using partial-wave dispersion techniques to include the effects of production channels in which either a (3,3) pion-nucleon isobar, the N^*, or a ρ meson is produced. The formalism here also includes the overlap of these resonances. Although the threshold for N^* production is slightly further from the observed resonances than the threshold for ρ production, the higher spin of the N^* allows its effects to reach further than might otherwise be expected.

The detailed calculations are not designed as a fit to the data but proceed from known masses and three known coupling constants, so that no adjustable parameters are available.

Unitarity of the S matrix $S_{jk}=\delta_{jk}+iT_{jk}$, gives $T_{jk}(s_+)-T_{jk}(s_-)=i\sum_n T_{jn}(s_+)T_{nk}(s_-)$ for s real and greater than threshold. In the sum, we keep those two- and three-body states which can contribute to pion-nucleon scattering, namely $\pi+N$ and $\pi+\pi+N$ states. The three-body state increases the complexity of the problem considerably. However, using the observed resonances of the two-body pion-nucleon and pion-pion systems, several authors[1–4] have suggested reducing the complexity of the production channel by considering only those three-body states where two of the particles emerge in a resonant state and considering this two-body resonant state as a distinct, but unstable, particle. This would lead to processes $\pi+N\to\pi+N^*$ or π^*+N.

Federbush *et al.*[4] have shown a self-consistent treatment of unstable particles by coupling them to pions and nucleons as though they were stable elementary particles, and only using the properties of their decay to calculate the appropriate coupling constants from experiment. Cook and Lee[5] have developed an extended N/D formalism which imposes unitarity on the coupled $\pi+N$ and $\pi+\pi+N$ channels, while at the same time it easily permits the inclusion of a $\pi\pi$ resonance in the

* This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract AT(30-1)-2098.

† A portion of this work was submitted to the Massachusetts Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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[1] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962).

[2] J. Ball, W. Frazer, and M. Nauenberg, Phys. Rev. **128**, 478 (1963).

[3] V. Teplitz, University of Maryland Technical Report 270, 1962 (unpublished).

[4] P. G. Federbush, M. Grisaru, and M. Tausner, Ann. Phys. (N. Y.) **18**, 23 (1962).

[5] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 297 (1962).

three-body state. Calculations, simulating the inelastic effects by a delta function of variable strength in the channels appropriate to the ρ meson, were able to fit several of the observed pion-nucleon resonance positions.

We wish, first, to extend the N/D formalism to permit formation of a pion-nucleon resonance (N^*) and a pion-pion resonance (ρ) in the three-particle state; and second, to carry through a more or less realistic calculation for the amplitudes, starting from diagrams where a single nucleon or pion is exchanged. To do this, a combination of the above formalisms is used. First the problem is formulated in such a way as to introduce readily a πN resonance into the $\pi+\pi+N$ state. Specifically, a partial-wave expansion is used in which the three-particle angular momentum state $\pi(\pi N)$ appears as a pion coupled to a πN system of definite spin but variable mass ω. For $\omega=$ the mass of the N^*, this is interpreted as a $\pi+N^*$ state. The exact meaning of this simplification is given.[2,6] The connection between this expansion and that of Lee and Cook, who first couple the two pions together to obtain a $(\pi\pi)N$ state, permits the inclusion of the ρ meson in the calculations. For values of the variables corresponding to the processes $\pi+N\rightarrow\pi+N^*$ or $\rho+N$, we use the semiphenomenological approach of Federbush[4] to calculate the absorptive part of the amplitudes, necessary for the N/D method, from Feynman diagrams. The inclusion of both the N^* and the ρ leads to a multichannel scattering problem which, for given parity and $J>\frac{1}{2}$, couples six channels: one πN state, two πN^* states, and three ρN states. The four masses and three coupling constants are taken from experiment.[7] We wish to point out the effect of the N^* alone and the ρ alone, and the effect of the coupled $N^*\rho$. One interesting question is whether the pion-nucleon system can be considered as self-sustaining without specific $\pi\pi$ forces, i.e., using only the N^*.

The symmetry of the pions forces the appearance of a nondiagonal term in the exact partial-wave dispersion relations which persists when the three-particle amplitudes are simplified. It is then interpreted as the overlap of two N^* resonances. A similar nondiagonal term corresponding to an $N^*\rho$ overlap appears. The form of these terms is given in terms of integrals over the amplitudes defining the resonances which are a measure of their width. The extension of the N/D method to handle these nondiagonal terms is given. The actual numerical calculations start with the coupled dispersion relations for the functions N and D, where the inputs to these equations, the unphysical discontinuities, are calculated from the Born term diagrams, Fig. 1. The functions D are then eliminated to leave a system of coupled Fredholm integral equations for N. One common N/D approach is to take N directly to be the Born-term amplitude. This is similar to the determinantal method developed by Baker, and we call this the determinantal approximation. It actually means the usual Born terms forced to satisfy unitarity. This is, in fact, just the first-order iteration solution of the integral equations for N. A numerical technique is developed to solve the actual integral equations, and the results are compared to this determinantal approximation and to experiment.

[6] R. C. Hwa, Phys. Rev. **130**, 2580 (1963).

[7] See Sec. IV.

II. AMPLITUDES AND SYMMETRIZED UNITARITY

To specify a two-particle state, we use the usual variables: $s=$ (total energy)^2 in the center-of-mass system, and $\Omega=(\theta,\phi)$, the polar angles of the nucleon's momentum in the center of mass.

It is not possible to choose variables for the three-particle state which are convenient for handling both the N^* and the ρ. We choose those appropriate to the N^*, since this will allow most of the new aspects of the coupled problem to be dealt with directly. Therefore, to specify a three-particle state $|pk_1k_2\rangle$, the variables are chosen to be: $s=[(\text{total energy})^2$ in the three-body center-of-mass system$]=(p+k_1+k_2)^2$; $\omega^2=[(\text{total energy})^2$ of the π_1N in its center-of-mass system$]=(p+k_1)^2$; $\Omega=(\theta,\phi)=$ the polar angles of the total momentum of the π_1N system, i.e., $p+k_1$, measured in the three-body center-of-mass system, and $\Xi=(\alpha,\beta)=$ the polar angles of the nucleon's momentum measured in the π_1N center-of-mass system. In other words, we describe the π_1N system in its center-of-mass system and then treat it as a single unit in describing the three-particle state. The appropriate angles are shown in Fig. 1.

To define invariant scattering amplitudes, the appropriate S-matrix element is contracted on one of the pions and multiplied by $\pi_i(E_i)^{1/2}$. We define

$$\begin{aligned}
&M_{22}(s_+,\Omega,\Omega',\lambda\lambda')\\
&\quad=(p_0/m)^{1/2}\langle p\lambda|j(0)|p'\lambda',k\rangle_{\text{in}}(p_0'/m)^{1/2}(2k_0)^{1/2},\\
&M_{23}(s_+,\omega_+,\Omega,\Omega'\Xi',\lambda\lambda')\\
&\quad=(p_0/m)^{1/2}(2k_0)^{1/2}\,{}_{\text{out}}\langle p\lambda,k|j(0)|p'\lambda',k_1'\rangle_{\text{in}}\\
&\qquad\times(p_0/m)^{1/2}(2k_{10}')^{1/2},\\
&M_{32}(s_+,\omega_+',\Omega'\Xi',\Omega,\lambda'\lambda)\\
&\quad=(p_0'/m)^{1/2}(2k_{10}')^{1/2}\,{}_{\text{out}}\langle p'\lambda',k_1|j(0)|p\lambda,k\rangle_{\text{in}}\\
&\qquad\times(p_0/m)^{1/2}(2k_0)^{1/2},\\
&M_{33}(s_+,\omega_+,\omega_+',\Omega\Xi,\Omega'\Xi',\lambda\lambda')\\
&\quad=(2k_{10}2k_{20})^{1/2}(p_0/m)^{1/2}\,{}_{\text{out}}\langle p\lambda,k_1k_2|j(0)|p'\lambda',k_1'\rangle_{\text{in}}\\
&\qquad\times(2k_{10}')^{1/2}(p_0'/m)^{1/2},
\end{aligned}\tag{1}$$

where $j(0)$ is the pion current operator, $s=$ (total energy)^2, $\omega^2=(p+k_1)^2$, $\omega'^2=(p'+k_1')^2$, and λ, λ' are the nucleon helicity. M_{33} does not include those diagrams where one pion does not scatter.

Some of the variables are redundant. We later eliminate these by choosing the center-of-mass system so that the initial angles are zero, but it proves convenient to keep all variables here explicitly.

We wish to decompose the amplitudes into partial waves and obtain their unitarity relations. The details

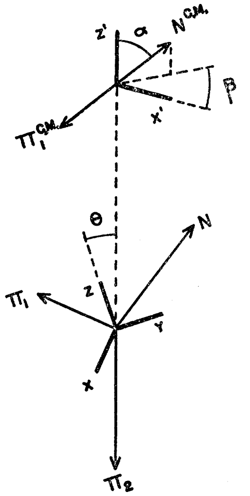

FIG. 1. Angles for three-particle state.

are in the Appendix. Two-particle states are familiar. For three-particles states, the above choice of variables leads to a state of total angular momentum J in which the two-body π_1N system has angular momentum j in its own center of mass. This two-body system acts like a particle of spin j and mass ω when coupled to the remaining pion to form the total angular momentum J.

One complication is the transformation of the helicity of the nucleon. More serious is the fact that the state must be symmetric between the two pions. The choice of variables was not symmetric, and this has repercussions by complicating the unitarity relations. If the two pions are coupled together first, the complications are only postponed until the effects of a πN resonance on the three-particle state are considered.

Contraction rules are used to express Blankenbecler's[8] generalized unitarity relations. To write a dispersion relation in s for the amplitudes, one needs, for example, $M_{32}(s_+,\omega_+)-M_{32}(s_-,\omega_+)$. Using the unitarity of S as $S^+S=1$ gives an expression for $M_{32}(s_+,\omega_+)-M_{32}(s_-,\omega_-)$, which is not the discontinuity in s if ω is also on a cut. Also, the three-particle scattering amplitude which enters is not M_{33}, but M_{33} plus the disconnected diagrams mentioned above.

After decomposition, these unitarity relations couple only amplitudes of the same total angular momentum and parity, and these indices will be suppressed.

If the pions are treated, for the moment, as distinguishable, the unitarity relations read, for a given J and parity,

$$\text{disc}_s M_{ij}(s)=\textstyle\sum_k M_{ik}(s_+,\xi_+)\rho_k(s,\xi)M_{kj}(s_-,\xi_-)\,, \quad (2)$$

where ξ refers to any intermediate auxiliary variables, and

$$\rho_2(s)=(m/16\pi^2)(P(s,m^2,\mu^2)/\sqrt{s})\,,$$

$$\rho_3(s,\omega)=(m/32\pi^2)(P(s,m^2,\omega^2)/\sqrt{s})(P(s,m^2,\mu^2)/2\omega)\,,$$

$$4sP^2(s,x,y)=s^2-2s(x+y)+(x-y)^2\,,$$

$$\sum_3=\sum_{j\lambda}\int_{(m+\mu)^2}^{(\sqrt{s}-\mu)^2}d\omega^2\,,$$

and $\text{disc}_s M(s)=[M(s_+)-M(s_-)]/2i$ with any auxiliary variables held fixed.

The symmetrized unitarity relations are exactly the same except that for the three-particle intermediate states

$$\sum_{j\lambda}\int_{(m+\mu)^2}^{(\sqrt{s}-\mu)^2}d\omega^2\,\rho_3(s,\omega)$$

is replaced by the nondiagonal term

$$\sum_{j\lambda j'\lambda'}\int\int_{(m+\mu)^2}^{(\sqrt{s}-\mu)^2}d\omega^2 d\omega'^2[\rho_{33}(s,\omega,\omega',j\lambda,j'\lambda')]\,. \quad (3)$$

The details of how this term arises and the exact expression for ρ_{33} are given in the Appendix. The extra term is essentially the overlap of the two-body states π_1N and π_2N, which must be present in the three-body state, since it is symmetric between the pions.

III. SIMPLIFICATION OF THREE-PARTICLE AMPLITUDES

The three-particle angular-momentum state was formed by coupling one pion to the system of mass ω and spin j, formed from the nucleon and second pion. A two-body resonance in this nucleon-pion system is expected to manifest itself in the three-body amplitudes as a peak in the mass variable ω at m_*. For $\omega=m_*$, the amplitudes are highly suggestive of describing the scattering to a state of a pion and an unstable particle of mass m_*, spin j; and this suggestion is exploited in the calculations. In the meantime, however, we only assume that the ω dependence of the three-particle amplitude is dominated by the strongest of the πN resonances, namely, the (3,3) resonance at 1238 MeV; and further, that this occurs only in the π_1N system.

Specifically, for a given J and parity, the amplitude $M_{23}(s,\omega,j,\lambda)$ as a function of ω is expected to have a strong maximum near $\omega=m_*=1238$ MeV for $j=\frac{3}{2}$, and to be negligible otherwise. If this dependence is divided out,

$$M_{23}(s,\omega,j,\lambda)=\delta_{j3/2}M_{22}(\omega)\tilde{M}_{23}(s,\omega,\lambda)\,, \quad (4)$$

then $\tilde{M}_{23}(s,\omega,\lambda)$ can be considered a smooth function of ω. Similarly,

$$M_{33}(s,\omega,\omega',j\lambda,j'\lambda) =\delta_{j3/2}\delta_{j'3/2}M_{22}(\omega)M_{22}(\omega')\tilde{M}_{33}(s,\omega,\omega',\lambda\lambda')\,. \quad (5)$$

The integrals over ω can now be simplified.[2] For instance,

[8] R. Blankenbecler, Phys. Rev. 122, 983 (1961).

the quadratic term

$$Q=\sum_{j\lambda}\int_{(m+\mu)^2}^{(\sqrt{s}-\mu)^2} d\omega^2\, M_{23}(s_+,\omega_+,j\lambda)\rho_3(s,\omega)M_{32}(s_-,\omega_-,j\lambda)$$
$$\approx\sum_{\lambda}\tilde{M}_{23}(s_+,m_{*+},\lambda)\left[\int d\omega^2\rho_3(s,\omega)\,|M_{22}(\omega)|^2\right]$$
$$\times\tilde{M}_{32}(s_-,m_{*-},\lambda)\Theta(s-(m_*+\mu)^2),\quad (6)$$

where any factors except $M_{22}(\omega)$ may be evaluated at $\omega=m_*$.

The situation is complicated by two new features, the symmetry of the pions and a $\pi\pi$ resonance. The influence of the ρ meson would show up as a peak in the invariant mass σ of the $\pi\pi$ system, $\sigma^2=(k_1+k_2)^2$. However, the dependence on σ, as well as the angular variables in the $\pi\pi$ system, is buried in the variables s, ω, Ξ which were chosen to describe the three-particle state. Clearly, however, the partial-wave expansion could have been carried out by coupling the two pions together first, using the $\pi\pi$ mass σ as a variable (simply interchange π_2 and N). The alternate partial-wave amplitudes $T_{23}(s,\sigma,l\nu)$ describe scattering to a three-particle state in which the $\pi\pi$ system of mass σ is in a state of angular momentum l in its center-of-mass system. The total angular momentum and parity are as before, and these indices are suppressed. The two alternatives are connected by

$$M_{23}(s,\omega,j\lambda)=\sum_{l\nu}\int d\sigma^2 c(s\omega\sigma,j\lambda,l\nu)T_{23}(s,\sigma,l\nu)\,,\quad (7)$$

with the recoupling coefficients given by Wick.[9]

We write the full amplitude[10] as $M_{23}=M_{23}{}^*+M_{23}{}^\rho$, where the separation is defined so that the first term contains the N^* directly as a peak in ω at $\omega=m_*$, and the second term contains the ρ through a peak in $T_{23}(s,\sigma)$ at $\sigma=$(mass of the ρ)$=m_\rho$. The sum in the quadratic terms of the unitarity relations is then extended. Writing $T_{23}(s,\sigma,l\nu)=\delta_{l1}T_{\pi\pi}(\sigma)\tilde{T}_{23}(s,\sigma,\nu)$, we have

$$Q=\sum_{\lambda}\tilde{M}_{23}(s_+,m_*,\lambda)\left[\int d\omega^2\,\rho_3(s,\omega)\,|M_{22}(\omega)|^2\right]$$
$$\times\tilde{M}_{32}(s_-,m_*,\lambda)+\sum_{\nu}\tilde{T}_{23}(s_+,m_\rho,\nu)$$
$$\times\left[\int d\sigma^2\rho_3(s,\sigma)\,|T_{\pi\pi}(\sigma)|^2\right]\tilde{T}_{32}(s_-,m_\rho,\nu)\quad (8)$$

plus overlap terms.

The first term is the one considered earlier; the second follows from

$$\sum_{j\lambda}\int d\omega^2 M_{23}{}^\rho(s_+,\omega_+,j\lambda)\rho_3(s,\omega)M_{32}{}^\rho(s_-,\omega_-,j\lambda)\,,$$

by the orthogonality of the recoupling coefficients, and it is the analog of the first term. The overlap terms are nondiagonal, containing cross terms like $\tilde{T}_{23}\rho_{33}\tilde{M}_{32}$, but they contribute only for a finite range of s. The expression for these is given in the Appendix.

For convenience, we define $f_{ij}=b_i\tilde{M}_{ij}b_j$, where i and j range over the states πN, πN^*, and ρN.

$$b_N{}^2=m/16\pi^2\,,$$
$$b_*{}^2=\frac{m}{32\pi^2}\int\frac{d\omega}{(2\pi)^3}|M_{22}(\omega)|^2P(m_*{}^2,m^2,\mu^2)\,,\quad (9)$$
$$b_\rho{}^2=\frac{m}{32\pi^2}\int\frac{d\sigma}{(2\pi)^3}|T_{\pi\pi}(\sigma)|^2P(m_\rho{}^2,\mu^2,\mu^2)\,.$$

Then the unitarity relations read:

$$\mathrm{disc}f_{ij}(s)=\sum_{kk'}f_{ik}(s_+,+)\rho_{kk'}(s)f_{k'j}(s_-,-)\,,\quad (10)$$

where the extra $\pm$ refer to any intermediate mass variables, and the diagonal parts of $\rho_{kk'}$ are given by

$$\rho_N=P(s,m^2,\mu^2)/\sqrt{s}\,,$$
$$\rho_*=P(s,m_*{}^2,\mu^2)/\sqrt{s}\,,$$
$$\rho_\rho=P(s,m^2,m_\rho{}^2)/\sqrt{s}\,.$$

These amplitudes contain kinematical singularities[11] which can be removed. For f_{NN} we let the "orbital angular momentum" l equal $J+\frac{1}{2}(-1)^\pi(-1)^{J-1/2}$, and we define

$$F_{NN}{}^l(s)=g_N{}^l(s)f_{NN}(s)g_N{}^l(s)\,,$$
$$g_N{}^l(s)=(\mu/E+m)^{1/2}((E+m)/P)^l.\quad (11)$$

$p=$relative momentum of πN state$=P(s,m^2,\mu^2)$, E $=$nucleon's energy$=(p^2+m^2)^{1/2}$.

$F_{NN}{}^l$ has the dynamical branch cuts described by Frazer and Fulco,[12] and is free of kinematical singularities, except possibly for the branch points of $s^{1/2}$. We prefer to keep the variable as s instead of $W=s^{1/2}$, which is the "basic" variable for states involving fermions, in order to minimize the problems of subtractions. The square-root branch points present no additional problems in the formalism or in the calculations.

In analogy with $F_{NN}{}^l$, we define

$$F_{ij}{}^{ll'}=g_i{}^l f_{ij}{}^{ll'}g_j{}^{l'}\,,\quad (12)$$

where i and j refer to the channel spin states of πN, πN^*, and ρN. The connection between channel spin and helicity amplitudes is given by Jacob and Wick[13] as

$$\langle JMls|JM\lambda_1\lambda_2\rangle=\left(\frac{2l+1}{2J+1}\right)^{1/2}C^{lsJ}_{0\lambda\lambda}C^{s_1\ s_2s}_{\lambda_1-\lambda_2\lambda}\,,$$

[9] G. Wick, Ann. Phys. (N. Y.) **18**, 65 (1962); referred to as (i).

[10] It has been suggested that this sum for the full production amplitude should be diminished in the overlap region. Y. C. Leung, University of Colorado (unpublished).

[11] M. Baker, Ann. Phys. (N. Y.) **4**, 271 (1958).

[12] W. Frazer and J. Fulco, Phys. Rev. **117**, 1603 (1960); **119**, 1429 (1960).

[13] M. Jacob and G. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

and the factors g are

$$g_*{}^l = (\mu/(E_*+m_*))^{1/2}((E_*+m_*)/P_*)^l\,,$$
$$g_\rho{}^l = (\mu/(E+m))^{1/2}((E+m)/P_\rho)^l\,,$$

P_* = relative momentum of πN^* state $= P(s,m_*{}^2,\mu^2)$,

P_ρ = relative momentum of ρN state $= P(s,m^2,m_\rho{}^2)$,

E_* = energy of N^* in the πN^* state $= (p_*{}^2+m_*{}^2)^{1/2}$,

E = energy of the nucleon in the ρN state $= (p^2+m^2)^{1/2}$,

m_* = mass of the N^*,

m_ρ = mass of the ρ.

It is hoped that this removes the kinematical singularities for F, except possibly for $s^{1/2}$. This is in fact true for the approximations used.

At this point, we note that for the three-particle state

$$\mathcal{P}|JM,j\lambda,\pi\rangle = (-1)^\pi|JM,j\lambda,\pi\rangle \tag{13}$$

and the parity operator, $\mathcal{P}$ does not relate $M_{23}(-\lambda)$ to $M_{23}(+\lambda)$ as might be expected. Hence, parity does not limit the value of l for the three-particle amplitudes. This allows the πN resonant system to take either parity. However, in the calculations, the N^* is assigned a definite parity, and this limits the values of l.

The unitarity relations for F have ρ replaced by $g^{-1}\rho g^{-1}$.

IV. INTERPRETATION OF THREE-PARTICLE AMPLITUDES AND CALCULATION

Federbush[4] has given the unitarity relations and Born-term amplitudes for the scattering of pion and nucleon into a pion and isobar or nucleon and ρ. We quote the relevant results: Let

$$\alpha_{jk}{}^J(s) = (s/P_iP_j)^{1/2}(S^J-1)_{jk}/2i\,, \tag{14}$$

where j, k refer to the states πN or πN^*, or ρN. Then

$$\mathrm{Im}\alpha_{jk}{}^J(s) = \sum_{k'} \alpha_{jk'}{}^J(s_+)\rho_{k'}(s)\alpha_{k'k}{}^J(s_-)\,, \quad \rho_k(s) = P_k(s)/\sqrt{s}\,. \tag{15}$$

We seek the connection between $M^J(s,m_*)$, or equivalently $f^J(s,m_*)$, and the $\alpha^J(s)$.

Since $\alpha^J \propto S^J-1$, α^J contains any disconnected diagrams omitted from the f^J; $\alpha_{\pi N;\pi N^*}{}^J(s_+)$ and $f_{23}{}^J(s_+,\omega_+{}^*)$ both describe scattering to the state π, $(\pi N)^*$, i.e., π and resonating system of πN, so

$$\alpha_{\pi N;\pi N^*}{}^J(s_+) \propto f_{23}{}^J(s_+,m_{*+})\,. \tag{16}$$

However,

$$\alpha_{\pi N;\pi N^*}{}^J(s_-) \propto f_{23}{}^J(s_-,m_{*-})\,.$$

Therefore, the unitarity relations correspond to those for the simultaneous discontinuity in s and ω variables mentioned earlier. Comparison with the unitarity relations for f^J shows:

$$\begin{aligned} f_{22}{}^J(s) &= \alpha_{\pi N;\pi N}{}^J(s)\,,\\ f_{23}{}^J(s_+,m_{*+}) &= \alpha_{\pi N;\pi N^*}{}^J(s_+)\,,\\ f_{32}{}^J(s_+,m_{*+}) &= \alpha_{\pi N^*;\pi N}{}^J(s_+)\,,\\ f_{33}{}^J(s_+,m_{*+},m_{*+}) &= \alpha_{\pi N^*;\pi N^*}{}^J(s_+) - \text{disconnected parts.} \end{aligned} \tag{17}$$

The discontinuity desired is

$$\begin{aligned} f_{23}{}^J(s_+,m_{*+}) - f_{23}{}^J(s_-,m_{*+}) &= \alpha_{\pi N;\pi N^*}{}^J(s_+) - \alpha_{\pi N;\pi N^*}{}^J(s_-)\\ &\quad - [f_{23}{}^J(s_-,m_{*+}) - f_{23}{}^J(s_-,m_{*-})]\,. \end{aligned} \tag{18}$$

The term in brackets, the discontinuity in ω, can be obtained from the contraction rules as the discontinuity in s was.

The discontinuity in ω is given by cutting the diagram across the state described by ω:

$$\begin{aligned} \mathrm{disc}_\omega M_{23}(s,\omega,j\lambda) &= M_{23}(s,\omega_+,j\lambda)\rho_2(\omega)M_{22}(\omega_-)\\ &\quad + \sum_{j'\lambda'} \int_{(m+\mu)^2}^{(\omega-\mu)^2} d\omega'^2 M_{33}(\omega_+,s,\omega_+',j\lambda,j'\lambda')\\ &\quad \times \rho_3(\omega,\omega')M_{32}(\omega_-',\omega_-,j'\lambda')\,. \end{aligned} \tag{19}$$

Dividing out $M_{22}(\omega)$ from $M_{23}(s,\omega)$ removes the contribution from two-particle states,[2] and

$$f_{23}(s,\omega,\lambda) \propto M_{23}(s,\omega,\tfrac{3}{2},\lambda)/M_{22}(\omega)\,. \tag{20}$$

For the three-particle states, however,

$$\begin{aligned} \mathrm{disc}_\omega f_{23}(s,\omega,\lambda) \propto{}& M_{23}(s,\omega_+,j\lambda) \sum_{j'\lambda'} \int_{(m+\mu)^2}^{(\omega-\mu)^2} d\omega'^2 M_{23}(\omega_+,\omega_+',j'\lambda')\rho_3(\omega,\omega')M_{32}(\omega_-,\omega_-',j'\lambda')\\ &- M_{32}(\omega_+) \sum_{j'\lambda'} \int_{(m+\mu)^2}^{(\omega-\mu)^2} d\omega'^2 M_{33}(\omega_+,s,\omega_+',j\lambda,j'\lambda')\rho_3(\omega,\omega')M_{32}(\omega_-',\omega_-,j'\lambda')\,, \end{aligned} \tag{21}$$

and the terms no longer cancel. For $\omega = m_*$, however, the range of the ω' integration is from $m+\mu$ to $m_*-\mu$ and cannot include $\omega = m_*$, i.e., cannot include the resonant πN state, and neither term will contribute to $\mathrm{disc}_\omega f_{23}(s,\omega)$. Hence, we ignore the $\pm$ limits on ω and calculate the discontinuities appearing in the N/D formalism from the Born approximation. Similarly, discontinuities in σ from two-pion states are removed from $T_{23}(s,\sigma,\nu)$.

We use the interaction Lagrangians[4]

$$\begin{aligned} \mathcal{L}_{\pi NN} &= g\bar{\psi}\gamma_5\tau_i\psi\varphi^i\,,\\ \mathcal{L}_{\pi NN^*} &= (g_*/m)(\bar{\psi}\overleftrightarrow{\partial}_\mu\psi_\mu{}^i)\varphi^i + \text{H.c.},\\ \mathcal{L}_{\pi\pi\rho} &= (g_\rho/2)(\varphi^i\overleftrightarrow{\partial}_\mu\varphi^j)\rho_\mu{}^k\epsilon^{ijk}, \end{aligned} \tag{22}$$

where g is the pion-nucleon coupling constant and g_*, g_ρ are calculated from the widths of the N^* and ρ, respectively.

The detailed forms of the amplitudes for the single-particle exchange diagrams in Fig. 2 are given in Ref. 4. The α's are linear combinations of Legendre functions of the second kind $Q_l(A)$ where the argument A is a rational function of the center-of-mass energies and momenta. We let F^B be the corresponding amplitudes of Sec. III with the kinematical singularities removed.

The asymptotic limits of the α's are important when considering the subtractions necessary in the N/D formalism. These limits are different for the amplitudes F^B, but the limit of the product $\rho\alpha$ which enters will be essentially unchanged. For large s,

$$\begin{aligned}&\alpha_{\pi N;\pi N}{}^J\sim\text{constant},\\&\alpha_{\pi N;\pi N^*}{}^J\sim s^{1/2},\\&\alpha_{\pi N;\rho N}{}^J\begin{cases}\sim 0 & \text{for channel spin } \tfrac{1}{2}\\ \sim\text{constant} & \text{for channel spin } \tfrac{3}{2}.\end{cases}\end{aligned}\tag{23}$$

The singularities of $\alpha_{\pi N;\pi N}{}^J$ lie along the real axis below threshold. The singularities of $\alpha_{\pi N;\pi N^*}{}^J$ and $\alpha_{\pi N;\rho N}{}^J$ are similar and are complex. Let us concentrate on $\alpha_{\pi N;\pi N^*}{}^J$. $Q_l(A)$ has branch points at $A\pm 1$. The cut from $s^{1/2}$ is taken along the negative real axis, and the additional cuts from p_N and p_* are taken along the real intervals $((m-\mu)^2,(m+\mu)^2)$ and $((m_*-\mu),(m_*+\mu)^2)$, respectively, and the cuts for $Q_l(A)$ are taken along the line where $A(s)=t$, real and $-1\leq t\leq 1$. $\alpha(s)$ then has the cuts of p_N, p^*, $s^{1/2}$, and $Q_l(A)$. However, the kinematical cuts on the positive real axis are removed by the factors $(p_N)^l$, $(p_*)^l$ in $F_{23}{}^B$, and the cuts of $F_{23}{}^B$ are shown in Fig. 3.

The complex cut crosses the real axis at $s=(9.0\mu)^2$ for the N^* amplitude, and at $s=(10.8\mu)^2$ for the ρ terms.

The threshold behavior of the production amplitudes depends on which sheet of the Q_l is involved. For the highest threshold, $Q_l\sim 0$ on the physical sheet and $\alpha^{ll'}\sim(P_{\text{inelastic}})^{l'}$. If the complex cut is avoided, this behavior remains, and $\alpha^{ll'}\sim(p)^{l\text{ or }l'}$ as $p\sim 0$. The sheet reached by going through the complex cut has been considered by some authors, who found $\alpha^{ll'}\sim(p)^{-l}$. This is not relevant here, however, since we always use the physical sheet, and the functions are the most singular on the complex cut.

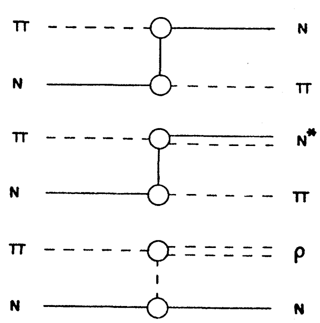

FIG. 2. Input Born diagrams.

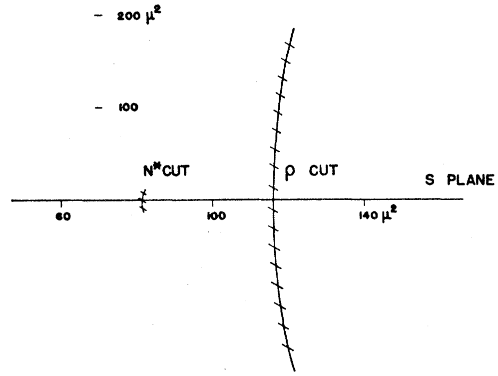

FIG. 3. Inelastic complex cuts.

The numerical values taken for the coupling constants and masses were $\mu=138$ MeV, $m=6.80\mu$, $m_*=8.96\mu$, $m_\rho=5.54\mu$; and $g^2/4\pi=15.0$, $g_*{}^2/4\pi=4.05$, and $g_\rho{}^2/4\pi=1.95$.

V. SOLUTION OF THE INTEGRAL EQUATION FOR N

The multichannel N/D method consists of expressing the scattering amplitude, in matrix notation, as $\mathbf{F}=\mathbf{N}\mathbf{D}^{-1}$ where

$$\mathbf{D}(s)=\mathbf{1}-\frac{(s-s_0)}{\pi}\int_{\text{Th}}^{\infty}ds'\frac{\boldsymbol{\varrho}(s')\mathbf{N}(s')}{(s'-s_0)(s'-s)},\tag{24}$$

where Th stands for threshold. In the usual formulation,[14] with a diagonal density-of-states matrix and without complex singularities, this solution satisfies unitarity for any $\mathbf{N}$ containing only unphysical singularities.

The nondiagonal density-of-states matrix presents no formal problem. If it is inserted into the definition of $\mathbf{D}$ as a matrix, it contributes only over a finite range of integration, and unitarity is still satisfied.

The effect on the definition of $\mathbf{N}$ and $\mathbf{D}$ of the complex singularities arising from single-pion exchange, appropriate to ρ-meson production, has been discussed in Refs. 1 and 2. The results for single-nucleon exchange, appropriate for N^* production, are similar.[15] They consist primarily of deforming the contour defining $\mathbf{D}$ around the complex singularity. Although of formal interest, these terms were dropped in the calculations as negligible. For instance,

$$\mathbf{D}_{N*}(s)=-\frac{(s-s_0)}{\pi}\int_{\text{Th}}^{\infty}ds'\frac{\varrho_N(s')\mathbf{N}_{N*}(s')}{(s'-s_0)(s'-s)}+\frac{(s-s_0)}{\pi}\int_{\text{complex contour}}ds'\frac{\varrho_N(s')\,\text{disc}\mathbf{N}_{N*}(s')}{(s'-s_0)(s'-s)}.\tag{25}$$

[14] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).
[15] F. T. Meiere, Massachusetts Institute of Technology, thesis (unpublished).

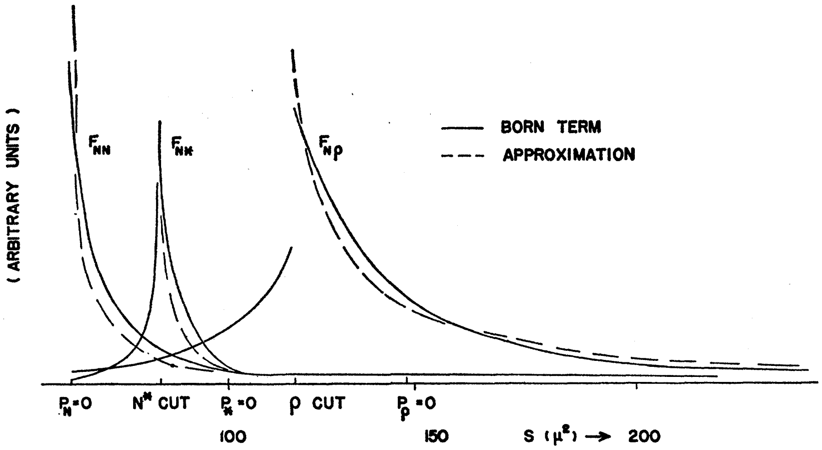

FIG. 4. Typical approximations used in the kernel of the integral equations.

The second term is expected to be small compared to the first for two reasons: the integration range is finite and short, and the integrand is the discontinuity of $\mathbf{N}$ which is smaller than $\mathbf{N}$ itself. We remark that the complex cut does not intersect the real cut arising from the overlap of resonances.

Hence, we write straightforwardly,

$$\mathbf{D}(s)=\mathbf{1}-\frac{(s-s_0)}{\pi}\int_{\mathrm{Th}}^{\infty}ds'\frac{\boldsymbol{\varrho}(s')\mathbf{N}(s')}{(s'-s_0)(s'-s)}, \tag{26}$$

$$\mathbf{N}(s)=\mathbf{F}(s)\mathbf{D}(s).$$

Letting C = the contour of unphysical discontinuities of $\mathbf{F}(s)$, and $\mathbf{A}(s)=\mathrm{disc}_s\mathbf{F}(s)$, we obtain:

$$\begin{aligned}\mathbf{N}(s)&=\mathbf{N}(s_0)+(s-s_0)\int_C ds'\frac{\mathbf{A}(s')\mathbf{D}(s')}{(s'-s_0)(s'-s)}\\&=\mathbf{N}(s_0)+(s-s_0)\int_C ds'\frac{\mathbf{A}(s')}{(s'-s_0)(s'-s)}\\&\quad-\frac{(s-s_0)}{\pi}\int_C ds'\frac{\mathbf{A}(s')}{(s'-s)}\int_{\mathrm{Th}}^{\infty}\frac{\boldsymbol{\varrho}(s'')\mathbf{N}(s'')}{(s''-s_0)(s''-s')}.\end{aligned} \tag{27}$$

Letting

$$\mathbf{G}(s)=\mathbf{N}(s_0)+(s-s_0)\int_C ds'\frac{\mathbf{A}(s')}{(s'-s_0)(s'-s)},$$

and using

$$\frac{-1}{(s'-s)(s''-s')}=\frac{1}{(s''-s)}\left[\frac{s''-s_0}{s'-s''}-\frac{s-s_0}{s'-s}\right]\frac{1}{(s'-s_0)},$$

we obtain

$$\mathbf{N}(s)=\mathbf{G}(s)+\frac{(s-s_0)}{\pi}\int_{\mathrm{Th}}^{\infty}ds''\frac{\mathbf{G}(s'')-\mathbf{G}(s)}{(s''-s_0)(s''-s)}\times\boldsymbol{\varrho}(s'')\mathbf{N}(s''). \tag{28}$$

The Born approximations to $\alpha^J(s)$ are finite at $s=0$, and we see no reason to think otherwise in general. However, $g_N{}^l$, $g_*{}^l$, and $g_\rho{}^l$ go to zero; $(g^l)^2\sim\sqrt{s}$ as s goes to 0. Therefore, we let s_0 go to 0 along the positive real axis and take $\mathbf{N}(s_0)=0$. Then $\mathbf{G}(s)=\mathbf{F}^B(s)$. This also insures that the first iteration of the N/D method gives the Born approximation back again. To avoid confusion, we keep the notation $\mathbf{G}$ where it appears in the kernel of the integral equation. The equations now read:

$$\mathbf{N}(s)=\mathbf{F}^B(s)+\frac{s}{\pi}\int_{\mathrm{Th}}^{\infty}ds'\frac{\mathbf{G}(s')-\mathbf{G}(s)}{s'(s'-s)}\boldsymbol{\varrho}(s')\mathbf{N}(s'). \tag{29}$$

No general closed-form solution to this is known. It is equivalent to an equation for $\boldsymbol{\varrho}^{1/2}(s)\mathbf{N}(s)/s$ with symmetric kernel, which is finite for $s=s'$, and Fredholm.

The integral equations for $\mathbf{N}$, Eq. (29), serve as the starting point for our approximation scheme. The unphysical discontinuities $\mathbf{A}$ will be approximated, allowing the functions $\mathbf{G}$ to be calculated, which in turn gives both the inhomogeneous term and the kernel appearing in the integral equation for $\mathbf{N}$. Once these are solved, it is a straightforward procedure to calculate $\mathbf{D}$, and hence the physical amplitudes F. One could iterate the integral equations: to first order $N=F^B$, and since $\mathbf{N}\mathbf{D}^{-1}=N\,\mathrm{Adj}(\mathbf{D})/\det(\mathbf{D})$, to first order in the numerator $\mathbf{F}=\mathbf{F}^B/\det\mathbf{D}$. In this simple form, the Born terms are just enhanced by the factor $1/\det\mathbf{D}$ and enable one to see the effect on $\mathbf{F}$ of changes in the input, such as increasing the coupling. We also remark that this approximation is symmetric and has no complex cut in the elastic channel, properties possessed by the full solution but not necessarily by all approximations, such as truncated iterations for $\mathbf{N}$.

As for solving the equations for $\mathbf{N}$ directly, we keep the terms $\mathbf{F}^B$ for $\mathbf{N}$ without modification and calculate the corrections given by the second term in (29). For the functions appearing in the kernel, the form $G(s)=(as+b)/(s-c)$ fits very well for $s\geq s_{\mathrm{cut}}$, which is where the complex cut crosses the real axis; see Fig. 4. Using this form in the second term of (29) makes the

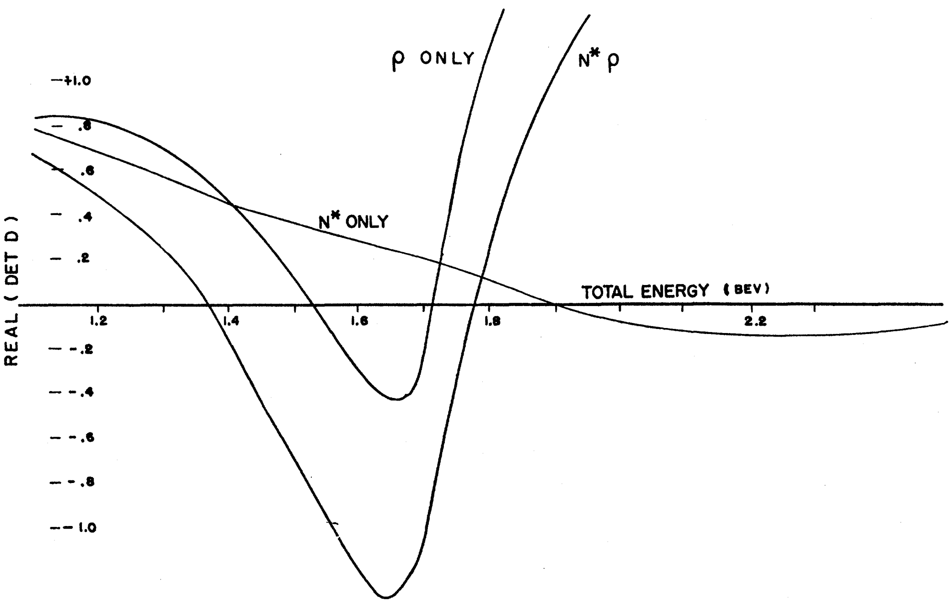

FIG. 5. Re(det$D$) using only Born term amplitudes. Resonance energies are determined by the first zero in Re(det$D$) in this lowest order determinantal solution.

kernel separable, for $s>s_{\text{cut}}$,

$$\frac{s}{\pi}(G(s')-G(s))/s'(s'-s)=\varphi(s)\psi(s'),$$
$$\varphi(s)=(s/\pi)(1/(s-c)),\quad \psi(s')=-(ac+b)/s'(s'-c). \tag{30}$$

For $s<s_{\text{cut}}$ the kernel does not necessarily separate. In this integral, s' is always greater than threshold, so the form holds for $G_{jk}(s')$; but for $(m+\mu)^2\leq s\leq s_{\text{cut}}$, $G_{23}(s)$ definitely does not have the same form.

To calculate D, only s greater than s_{cut} is needed for N_{32}, N_{33}, so the troublesome term is in N_{2j} for $(m+\mu)^2\leq s\leq s_{\text{cut}}$, namely:

$$\frac{s}{\pi}\int_{\text{Th}_3}^{\infty} ds'\frac{\mathbf{G}_{23}(s')-\mathbf{G}_{23}(s)}{s'(s'-s)}\varrho_3(s')\mathbf{N}_{3j}(s').$$

We drop this term but keep the term

$$\frac{s}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\frac{\mathbf{G}_{22}(s')-\mathbf{G}_{22}(s)}{s'(s'-s)}\varrho_2(s')\mathbf{N}_{2j}(s'),$$

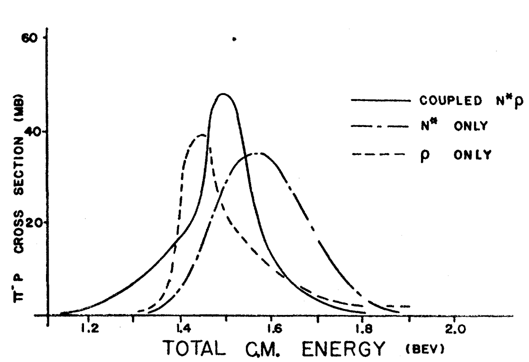

FIG. 6. Cross section given by the $D_{3/2}$, $T=\frac{1}{2}$ resonance.

noting first that this error is introduced only into the relatively small range from $(m+\mu)^2$ to s_{cut}; and second, that s is not included in the range of integration for the term dropped, but is included in the range of integration for the term kept. The effect is to make $\varphi_{23}=0$ and $\psi_{23}=0$ for $(m+\mu)^2\leq s\leq s_{\text{cut}}$. Equation (29) now reads:

$$N_{ij}(s)=F_{ij}{}^B(s)+\sum_k \varphi_{ik}(s)I_{ik}{}^j,$$
$$I_{ik}{}^j=\int_{\text{Th}}^{\infty} ds'\psi_{ik}(s')\rho_k(s')N_{kj}(s') =C_{ik}{}^j+\sum_{i'k'} H_{ik;i'k'}I_{i'k'}{}^j, \tag{31}$$
$$C_{ik}{}^j=\int_{\text{Th}_k}^{\infty} ds'\psi_{ik}(s')\rho_k(s')F_{kj}{}^B(s'),$$
$$H_{ik,i'k'}=\delta_{i'k'}\int_{\text{Th}_k}^{\infty} ds'\psi_{ik}(s')\rho_k(s')\varphi_{kk'}(s').$$

Considering the index ik as a single index in a direct product space, matrix multiplication gives I as $I^j=(1-H)^{-1}C^j$, and Eq. (31) gives N. The constants defining φ and ψ were determined by least squares, and the integrations were done numerically.

It should be pointed out where the approximate form $F^{\text{pole}}=(as+b)/(s-c)$ was used for F^{Born}. In all determinantal calculations for coupled N^*, N^* alone, or ρ alone, F^{Born} was used exactly as calculated. When solving the integral equations for the coupled $N^*\rho$ or the ρ alone, F^{Born} was not approximated in the first term of the expression

$$N=F^B+\varphi I,$$

but the approximation was used in the second term. For simplicity, when solving the integral equations with the N^* alone, the approximation $F^{\text{Born}}=F^{\text{pole}}$ was used throughout.[15]

Using the functions F^{pole} corresponds, in a certain sense, to a generalization of the single-pole approximation of Lee and Cook for the ρ.[5]

VI. GENERAL STRUCTURE OF THE SOLUTIONS

Several general statements can be made about all partial waves considered. As remarked earlier, the lowest order iteration gives $F_{NN}=F^{\text{Born}}/\det(D)$, and any resonances are given by $\text{Re}(\det D)=0$, Fig. 5. The inelastic force introduced was proportional to $g_{\text{inel}}=gg_*$ or gg_ρ. The expression for this case,

$$\text{Re}(\det D)=1-d_{22}-d_{23}d_{32}+\rho_2N_{23}\rho_3N_{32}\,,$$

$$d(s)=\frac{s}{\pi}\text{P.V.}\int_{\text{Th}}^{\infty}ds'\frac{\rho(s')N(s')}{s'(s'-s)}\,, \tag{32}$$

shows that g_{inel} enters only quadratically, and hence the resonance is independent of the sign of the inelastic force. Increasing the magnitude of g_{inel} increases the effect of the product $d_{23}d_{32}$ and allows the resonance to move down in energy.

If $\rho N(s)$ remained finite as $s\to\infty$, $d(s)$ would dominate $\rho N(s)$, and the term $\rho_2N_{23}\rho_3N_{32}$ would have little effect on $\text{Re}(\det D)$ for large s. However, the effect of the spin-$\frac{3}{2}$ N^* on the inelastic term is to make ρ_2N_{23} and $\rho_3N_{32}\sim s^{1/2}$ for large s. Since $d(s)$ goes at most like $\ln s$, the ρN terms force $\text{Re}(\det D)>0$ for large s. Hence, as s is increased from threshold, where $\text{Re}(\det\mathbf{D})>0$, the term $d_{23}d_{32}$ can drive $\text{Re}(\det\mathbf{D})$ down through zero and negative, producing a resonance, but it must go through zero again to reach its asymptotic positive value. The amplitude at this second zero is necessarily cut down and in the actual calculations did not produce a peak in the cross sections. It is of course possible that neither zero appears. On the other hand, if g_{inel} were so large as to force $\text{Re}(\det\mathbf{D})<0$ at threshold (which is not the case here), then at least one zero is forced to appear.

At best this simple solution can give only the position of the resonance and not its detailed properties, but since it is easier to calculate than the solution of the integral equations, it is interesting to check its reliability against a better solution.

As to the full solution of the coupled problem, no such sweeping statements can be made, although some remarks are pertinent. First, unlike before, the relative sign of the coupling constants g can influence the resonance. Second, the full solution is $F=N\,\text{Adj}(\mathbf{D})/\det(\mathbf{D})$, and since $\text{Adj}(\mathbf{D})$ has an imaginary part, the position of the resonance is no longer given exactly by the point where $\text{Re}(\det\mathbf{D})=0$, i.e., now phase (F) $=$phase $N(\text{Adj}\mathbf{D})-$phase $(\det\mathbf{D})$. Third, the asymptotic behavior of the input Born terms in one channel can influence the numerator function N for all other channels through the coupling of the integral equations. This can be significant for the coupled problem, since the N^* production amplitudes contain asymptotically at least one higher power of $s^{1/2}$ than do the elastic or ρ-production amplitudes.

VII. RESULTS AND CONCLUSIONS

Six partial waves were calculated, namely, $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states for the $D_{3/2}$, $F_{5/2}$, and $D_{5/2}$ angular-momentum states. In all cases, all available channels were kept, meaning for a given J, one πN state, two πN^* states, and three ρN states. Hence, the matrices $\mathbf{N}$ and $\mathbf{D}$ are of dimension 6×6 for the coupled $NN^*\rho$ case. The main contributions to the inelastic cross sections were from the lower angular-momentum states available, but contributions to $\det(D)$, and hence to the resonance positions, were important for both values of l.

The $D_{3/2}$, $T=\frac{1}{2}$ partial wave was the only one to exhibit a resonance (Fig. 6). This occurred at 1490 MeV with a width of 110 MeV, and compares favorably with the observed position of 1520 MeV and width of 100 MeV.[16] The curve contains a large inelastic contribution. For comparison, the lowest order determinantal solution predicted a resonance at 1370 MeV.

The effect of "turning off" either the N^* or the ρ does not eliminate the resonance. A $D_{3/2}$, $T=\frac{1}{2}$ resonance was predicted when only the N^* was used, coming at 1560 MeV with a width of 250 MeV, and also when only the ρ meson was used, coming at 1430 MeV with a width of 125 MeV. It is interesting to note that the width of 250 MeV in the case of the N^* alone is reduced considerably by the addition of the closed channel $\rho+N$. Physically, this would be expected if the N^* were the dominant mechanism for producing the resonance, since it could spend part of its time in the $\rho+N$ virtual state and lengthen its lifetime.[17] However, the $\rho+N$ state also plays an important role in producing the resonance, and hence this separation is perhaps artificial.

There was no evidence for any other of the observed πN resonances. The $T=\frac{1}{2}$ shoulder and the $F_{5/2}$, $T=\frac{1}{2}$ resonance[16] were particularly noticeable in their absence. It should, however, be mentioned that the solutions are very sensitive to the inelastic scattering, and hence, resonances are easy enough to obtain by varying the parameters of the theory, in particular the coupling constants g_* or g_ρ. This we have specifically tried to avoid. Also, states with $J=\frac{1}{2}$ were not calculated. Here the asymptotic behavior of the inelastic Born terms causes some of the integrals to diverge and forces the inclusion of arbitrary subtraction constants.

The lowest order determinantal method, in which the Born terms alone are used to calculate D, agreed in all cases with the more accurate solution in whether or not a resonance appears (Figs. 5 and 7). The positions were shifted by solving the integral equations.

[16] D. P. Lichtenberg, SLAC Report No. 13, 1963 (unpublished).
[17] I. P. Gyuk and S. F. Tuan, Purdue University (unpublished).

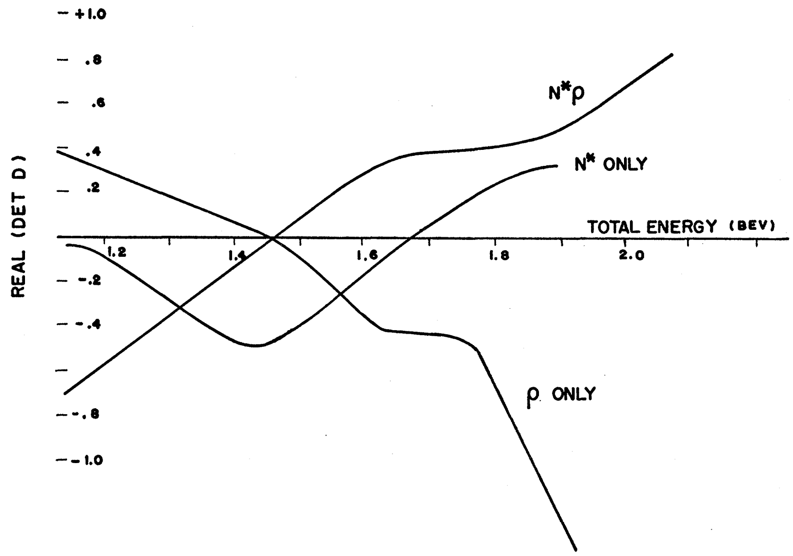

FIG. 7. Re(det$D$) after numerical solution of the integral equations for the numerator function. Resonance energies are no longer determined by the zeros in Re(det$D$) for the full solution.

The possible values of the available angular momentum for a given J are important. Because the terms (ρN) go to zero like p^{2l+1} near threshold, high angular-momentum shifts the effects of the principal-value integrals defining the elements of D to higher energies, where their value is reduced. Hence, it is expected that for a given J, the parity state which allows the lower angular momentum in the inelastic channel has the best chance of exhibiting a resonance.

The approximation of keeping only the two- and three-particle states coupled to the elastic channel is probably good. Four-body states with three-body resonances would seem to be the only possible significant correction. The variable mass dependence of the three-body amplitudes was taken to be the resonance behavior of the appropriate two-body system. This approximation is probably reasonably good also, if all the important resonances are kept.

Crossing symmetry is destroyed by the approximations used on the unphysical cut. This is reflected in the appearance of a "ghost" pole between zero energy and threshold in the full solution for the amplitude, and emphasizes the fact that analytic continuation below threshold (toward the crossed channels) is not justified. The problem of ghost poles is not infrequent in this type of calculation.[3,5]

For the elastic force, the Born term is surely bad for this strongly coupled problem. The results are more sensitive to the inelastic processes, where the Born term is not as bad as it first looks. The same type of approximation is used to relate the parameters of the "unstable particles" to experiment, and hence much of our ignorance about the three-particle amplitudes is lumped into the width or coupling constant of the resonance, which is found from experiment. A next interesting step would be the inclusion of the Born terms for the elastic N^* scattering,[17] i.e., the Peierls mechanism.

There were no free parameters in the theory. The width and position of the (3,3) isobar and the ρ meson determine the inelastic force, while the well-known πN coupling constant determines the direct elastic force. The isotopic-spin dependence is determined by the particular diagram considered. The technique used to solve the integral equations for N, given the unphysical discontinuities, is suitable for only a restricted class of functions, but is probably more accurate than the approximation used for the discontinuities.

Considering the fact that there are no free parameters at all in the theory, the agreement with experiment is reasonable. There is no reason to believe *a priori* that any resonances should appear at all, or that they would occur in the correct channels. It certainly indicates that the mechanism, i.e., coupling of the elastic channel to three-body inelastic channels through unitarity, plays an important role in the explanation of the higher resonances, and in particular that the πN^* state has an important effect even though the threshold is not close to the resonance.

ACKNOWLEDGMENTS

The author wishes to thank Dr. Paul Federbush for suggesting this problem, and Dr. E. L. Lomon, Dr. S. F. Tuan, Dr. M. Sugawara, and Dr. H. Goldberg for enlightening discussion.

APPENDIX

Wick[9] has shown how to construct angular-momentum states for three particles. As applied here, we pick one of the pions, π_1, and form the state of total angular momentum J in which the nucleon has helicity λ, while the $\pi_1 N$ system has angular momentum j and energy ω, all measured in the $\pi_1 N$ two-body center-of-mass system. This state, denoted $|sJM,\omega j\gamma,\lambda\rangle_{(\pi_1 N)\pi_2}$ is not symmetric between the pions. If this fact is ignored and the three-particle decomposition carried out using this state, the partial-wave unitarity relations are the usual ones given in Sec. II. One way to treat the identity of pions properly is to consider the three-particle state in a direct-product space as $\sqrt{2}|P\lambda,k_1k_2\rangle=|P\lambda,k_1\rangle_{\pi_1N}|k_2\rangle_{\pi_2}+|P\lambda,k_1\rangle_{\pi_2N}|k_2\rangle_{\pi_1}$. The first state can be decomposed using the states $|(\pi_1N)\pi_2\rangle$, and the second using $|(\pi_2N)\pi_1\rangle$ in which the variables are interchanged. Letting 1 refer to the original choice of variables and 2 to those with the pions interchanged, the production amplitude has two terms arising from the two decompositions, $\sqrt{2}M_{23}=M_{23}{}^{(1)}+M_{23}{}^{(2)}$, and the quadratic unitarity terms are extended,

$$Q=\sum_3 M_{23}\rho_3M_{32}=\sum_{i,j=1}^{2}\sum_3 M_{23}{}^{(i)}\rho_3M_{32}{}^{(i)}=\sum_{ij}Q^{ij}.$$

Since the three-particle sum is invariant under the interchange of variables 1 into 2, $Q^{22}=Q^{11}$ and $Q^{12}=Q^{21}$. Q^{11} is the usual term, while schematically

$$\begin{aligned}Q^{21}&=\textstyle\sum_3 M_{23}{}^{(2)}\rho_3M_{32}{}^{(1)}\\&=\textstyle\sum_3\langle(\pi_2N)\pi_1|(\pi_1N)\pi_2\rangle M_{23}{}^{(1)}\rho_3M_{32}{}^{(1)},\end{aligned}$$

where the recoupling coefficients defined by

$$\begin{aligned}|sJM,\omega_2j_2\gamma_2,\lambda_2\rangle_{(\pi_2N)\pi_1}&\\=\sum_{j\gamma\lambda}\int d\omega^2c^J(s\omega_2\omega,j_2\gamma_2\lambda_2,j\gamma\lambda)&\\\times|sJM,\omega j\gamma,\lambda\rangle_{(\pi_1N)\pi_2}\,,&\quad\text{(A1)}\end{aligned}$$

are given in (i). They are, apart from normalization factors, given by

$$\begin{aligned}&\Delta_{\gamma_1\gamma_2}{}^{\lambda_1\lambda_2}(s,\omega_1\omega_2,Jj_1j_2)\\&\quad=d_{\gamma_1\gamma_2}{}^J(\theta_{\pi\pi})d_{\gamma_1\lambda_1}{}^{j_1}(\alpha_1)d_{\gamma_2\lambda_2}{}^{j_2}(\alpha_2)d_{\lambda_1\lambda_2}{}^{1/2}(\theta_{\pi\pi}{}^N)\,,\quad\text{(A2)}\end{aligned}$$

where $\theta_{\pi\pi}$ and $\theta_{\pi\pi}{}^N$ are the angle between the two pions measured in the $\pi\pi N$ and N rest system respectively, and the angles α are the pion polar angles as defined in Sec. II. All arguments are functions of s, ω_1, ω_2.

The terms Q^{12} and Q^{21} give nondiagonal contributions to the exact symmetrized unitarity relations, e.g., for a given J and parity,

$$\begin{aligned}\mathrm{disc}M_{22}(s)&=M_{22}(s_+)\rho_2(s)M_{22}(s_-)\\&\quad+\sum_{j\gamma j'\gamma'}\int d\omega^2d\omega'^2M_{23}(s_+,\omega_+,j\gamma)\\&\qquad\times\rho_{\gamma\gamma'}(s\omega\omega',jj')M_{32}(s_-,\omega_-',j'\gamma')\,,\end{aligned}$$

where

$$\begin{aligned}\rho_{\gamma\gamma'}(s\omega\omega',jj')&=\rho_3(s,\omega)[\delta(\omega^2-\omega'^2)\delta_{jj'}\delta_{\gamma\gamma'}\\&\quad+N_{jj'}\Delta_{\gamma\gamma'}\theta(\omega_u-\omega')\theta(\omega'-\omega_l)]\,,\\\Delta_{\gamma\gamma'}&=\Delta_{\gamma\gamma'}+(-1)^\pi(-1)^{J+1/2}\Delta_{\gamma-\gamma'}\,,\\N_{jj'}&=(2j+1)^{1/2}(2j'+1)^{1/2}P(s,\omega^2,\mu^2)P(\omega^2,m^2,\mu^2)\,.\end{aligned}\quad\text{(A3)}$$

The three-particle integral over ω and ω' can be simplified as before, making use of the resonance at $\omega=m_*$, but now the density of states for the πN^* state is modified. With the definitions from Sec. III,

$$M_{23}(s,\omega,j\gamma)=\delta_{j^{3/2}}b_*{}^{-1}f_{23}(s,\omega,\gamma)\,,$$

$$\begin{aligned}\mathrm{disc}f_{22}(s)&=f_{22}(s_+)(P(s,m^2,\mu^2)/\sqrt{s})f_{22}(s_-)\\&\quad+\textstyle\sum_{\gamma\gamma'}f_{23}(s_+,m_*,\gamma)\rho_{\gamma\gamma'}f_{32}(s_-,m_*,\gamma')\,,\end{aligned}$$

where

$$\begin{aligned}\rho_{\gamma\gamma'}&=(P(s,m_*{}^2,\mu^2)/\sqrt{s})\delta_{\gamma\gamma'}\\&\quad+(\Gamma/s)(m_*{}^2/P(m_*{}^2,m^2,\mu^2))\Delta_{\gamma\gamma'}(s,m_*,m_*,J,\tfrac{3}{2}\tfrac{3}{2})\\&\qquad\times\theta(s_u-s)\theta(s-s_l)\,,\quad\text{(A4)}\end{aligned}$$

where the limits for s are such that $\omega_l(s_l,m_*)\leq m_*\leq\omega_u(s_u,m_*)$, i.e., that N^* resonance is possible in the π_1N and π_2N systems simultaneously, and

$$\Gamma=\frac{|\int d\omega M_{22}(\omega)|^2}{\int d\omega|M_{22}(\omega)|^2}.$$

Γ is a measure of the width of the two-body resonance, and its value depends slightly on the shape of the resonance. The value of Γ is the full width at half-maximum times a factor close to unity for a Breit-Wigner form, a normal curve, or a square wave, and we take Γ=full width.

The above analysis calculated essentially the expansion of $(\pi_2N)\pi_1$ in terms of $(\pi_1N)\pi_2$ states to find the effect of a πN resonance in the π_2N system. This was necessary, since the mass of the π_2N system did not appear directly as an independent variable. The analysis can be repeated for the $(\pi_1\pi_2)N$ state to find the effect of a resonance in the $\pi\pi$ system. The result is obtained from Eq. (A2) by the interchange of π_1 and N. This, for instance, sends $(\pi_2N)\pi_1$ into $(\pi_2\pi_1)N$ and ω_2 into σ= (invariant mass of the $\pi\pi$ system), but sends $(\pi_1N)\pi_2$ and ω_1 into themselves. The corresponding overlap terms are obtained from Eq. (A4) by this interchange. These overlap terms can be thought of in terms of diagrams, although these are not used in the derivation.[4,18–20]

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