PHYSICAL REVIEW VOLUME 136, NUMBER 4B 23 NOVEMBER 1964

Amplitude Bound in Ladder Graph Models. II*

GEORGE TIKTOPOULOS AND S. B. TREIMAN
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 2 July 1964)

A procedure described earlier for bounding ladder-graph scattering amplitudes is applied here to two models: exchange of pairs of scalar particles (bubbles), and exchange of vector particles. The forward-scattering absorptive amplitudes are bounded from above. Apart from logarithmic factors, the correct amplitudes are known to grow with energy like S^α, and our results determine upper bounds on the exponents α. For the case where the exchanged particles are massless, our upper bounds in fact coincide with the exact results for α as given in the literature.

I. INTRODUCTION

IN a recent paper,[1] we discussed a technique for setting bounds on the absorptive part of an amplitude for scattering of spinless particles in a ladder-graph model. The essential idea is this: The absorptive amplitude satisfies a Bethe-Salpeter integral equation of the Volterra type, in which the kernel and inhomogeneous term are both positive. The amplitude is thus bounded from above (or below) by the solution of a comparison equation in which the kernel and/or inhomogeneous term have been majorized (minorized). Bounds can then be obtained by a trial function procedure, as discussed in our earlier work.

In Ref. 1 we were concerned with finding an upper bound for the forward ($t=0$) scattering amplitude in a ladder model involving a trilinear scalar interaction. Of special interest was the asymptotic behavior in the limit of large scattering energy s. In this limit, the amplitude varies like $s^{\alpha(0)}$, where $\alpha(0)$ is the leading singularity in the angular momentum plane for the crossed channel reaction at $t=0$. Subsequently, Nakanishi[2] has succeeded in finding an exact solution to this problem for the special case where the rungs of the ladder correspond to massless particles. For this situation our upper bound on $\alpha(0)$ exactly coincides with the true result.

Two other ladder models for scattering of spinless particles have been much discussed in the literature.[3–6] One involves the exchange of pairs of scalar particles (bubble exchange) according to a basic quadrilinear scalar interaction. The other corresponds to exchange of a vector particle. Both models have been studied in detail for the case of forward scattering. In the limit of large energies, the amplitudes behave like s^α, apart from logarithmic factors which reflect the existence of fixed cuts in the crossed-channel angular momentum plane. Exact expressions for the exponent α are given in the literature[5,6] for the special situation where the exchanged particles have zero mass.

Our purpose in the present paper will be to test the amplitude-bounding techniques of Ref. 1 on the above models, namely, to determine by these methods upper bounds on the absorptive amplitudes, hence on the exponents α. We do not restrict ourselves to the case of vanishing masses for the exchanged particles. However, for the situation where these masses are zero, we can compare our bounds on α with the known expressions; and here we find that our bounds in fact coincide with the exact results for each of the two models under discussion. From these comparisons, it therefore appears that the bounding techniques of Ref. 1 are, in fact, quite useful in practice.

A special remark is in order here concerning the vector exchange model. The vector particle propagator is properly given by $(\mu^2-k^2)^{-1}(\delta_{\alpha\beta}-k_\alpha k_\beta/\mu^2)$. In the cited literature, however, it has been replaced by the first term $(\mu^2-k^2)^{-1}\delta_{\alpha\beta}$ taken alone. This is of course not physically legitimate, unless the vector particle is coupled to a conserved current and unless one considers simultaneously graphs other than those which just correspond to the simple ladder. Nevertheless, for the purposes of our present concern we shall adopt the vector exchange model in this form, the more so because it permits us to compare our bounding techniques with results available in the literature.

II. EXCHANGE OF BUBBLES

We consider forward scattering of spinless particles with momenta p and k, in the ladder approximation corresponding to Fig. 1, which involves a quadrilinear scalar interaction. The heavy lines describe particles of mass m; the exchanged particles (wavy lines) have mass μ.

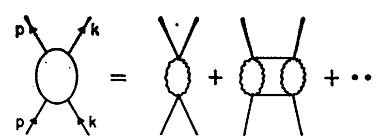

FIG. 1. The set of graphs under consideration.

* Work supported by the U. S. Air Force Office of Research, U. S. Air Research and Development Command.

[1] G. Tiktopoulos and S. B. Treiman, Phys. Rev. 135, B711 (1964); denoted by I.
[2] N. Nakanishi, Phys. Rev. 135, B1430 (1964).
[3] R. F. Sawyer, Phys. Rev. 131, 1384 (1963); P. Suranyi, Phys. Letters 6, 59 (1963).
[4] A. Bastai, L. Bertocchi, S. Fubini, G. Furlan, and M. Tonin, Nuovo Cimento 30, 512 (1963); A. Bastai, L. Bertocchi, G. Furlan, and M. Tonin, Nuovo Cimento 30, 1532 (1963).
[5] M. Baker and I. J. Muzinich, Phys. Rev. 132, 2291 (1963).
[6] G. Cosenza, L. Sertorio, and M. Toller, Nuovo Cimento 31, 1086 (1964).

Following the notation of Fig. 2, we write down an integral equation for the off-mass-shell absorptive part of the forward scattering amplitude:

$$A(p,k)=\pi g^2\rho[(p+k)^2]+\frac{g^2}{(2\pi)^3}\int_{4\mu^2}^{(s^{1/2}-2\mu)^2}dy\,\rho(y)\times\int d^4p'\frac{\delta[(p-p')^2-y]}{(m^2-p'^2)^2},$$

where,

$$\rho(y)=\frac{1}{32\pi^2}(1-4\mu^2/y)^{1/2}.$$

In terms of the invariants $k^2=m^2$, p^2, p'^2, $s=(p+k)^2$, $s'=(p'+k')^2$, we can rewrite this as follows:

$$A(s,p^2)=\pi g^2\rho(s)+\frac{g^2}{16\pi^2}[(s-p^2-m^2)^2-4m^2p^2]^{-1/2}\times\int_{4\mu^2}^{(s^{1/2}-2\mu)^2}dy\,\rho(y)\int_{4\mu^2}^{(s^{1/2}-y^{1/2})^2}ds'\times\int dp'^2\frac{A(s',p'^2)}{(m^2-p'^2)^2}. \quad (1)$$

The upper and lower limits on the p'^2 integration are given by

$$f_\pm(y)=p^2+y-\frac{1}{2s}(s+p^2-m^2)(s-s'+y)\pm\frac{1}{2s}[(s+p^2-m^2)^2-4p^2s]^{1/2}\cdot[(s-s'+y)^2-4ys]^{1/2}.$$

Following the method of Ref. 1, we now majorize the kernel and the inhomogeneous term of the integral Eq. (1). For convenience we shall assume that $m\leqslant\mu$. First, notice that $p^2\leqslant 0$ implies $p'^2\leqslant 0$, so that temporarily we can restrict ourselves to negative values of p^2 and p'^2. But then we can make use of the inequalities

$$\rho(y)\leqslant 1/32\pi^2,$$
$$[(s-p^2-m^2)^2-4m^2p^2]^{-1/2}\leqslant(s-p^2-m^2)^{-1}.$$

With these we are led to the majorizing equation

$$A_1(s,p^2)=\frac{g^2}{32\pi}+\frac{1}{2}\left(\frac{g}{16\pi^2}\right)^2(s-p^2-m^2)^{-1}\times\int_{4\mu^2}^{(s^{1/2}-2\mu)^2}ds'\int_{f_-(4\mu^2)}^{f_+(4\mu^2)}dp'^2\frac{A_1(s',p'^2)}{(m^2-p'^2)^2}\times\int_{4\mu^2}^{y_0(s,p^2,p'^2)}dy, \quad (2)$$

where we have made an interchange in order of integration. After some tedious but straightforward algebra, one can verify that

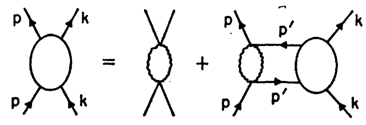

FIG. 2. Notation for the integral equation.

$$y_0\leqslant\frac{(s-p^2-s'+p'^2)[p^2(s'-m^2)-p'^2(s-m^2)]}{(s-p^2-m^2)(s'-p'^2-m^2)};$$

also

$$-s+s'+p^2\leqslant f_-(4\mu^2)\leqslant f_+(4\mu^2)\leqslant p^2\frac{s'-m^2}{s-m^2}.$$

This leads to a further majorized equation

$$A_2(s,p^2)=\frac{g^2}{32\pi}+\frac{1}{2}\left(\frac{g}{16\pi^2}\right)^2(s-p^2-m^2)^{-1}\times\int_{4\mu^2}^{(s^{1/2}-2\mu)^2}ds'\int_{-s+s'+p^2}^{p^2(s'-m^2)/(s-m^2)}dp'^2A_2(s',p'^2)\times\frac{(s-p^2-s'+p'^2)[p^2(s'-m^2)-p'^2(s-m^2)]}{(m^2-p'^2)^2(s-p^2-m^2)(s'-p'^2-m^2)}.$$

Finally, we introduce the new variables

$$u=s-p^2-m^2,$$
$$r=-p^2/(s-p^2-m^2),$$

perform some further majorizations, and obtain

$$A_3(u,r)=\frac{g^2}{32\pi}+\frac{1}{2}\left(\frac{g}{16\pi^2}\right)^2\frac{1}{u}\int_{4\mu^2-m^2}^{u}du'(u-u')u'\times\int_r^1dr'(r'-r)\frac{A_3(u',r')}{(u'r'+m^2)^2}. \quad (3)$$

Since we are dealing with integral equations of the Volterra kind, whose iterative solutions always converge, our majorizations of the kernel and inhomogeneous term imply that

$$A_3(u,r)\geqslant A_2(s,p^2)\geqslant A_1(s,p^2)\geqslant A(s,p^2),$$

for all $p^2<0$. To find an explicit bound for A_3 (and therefore A), we now employ the "trial function" method of Ref. 1. It is based on the following remark. If the integral equation satisfied by A_3 is, symbolically,

$$A_3(\xi)-\int_a^b K(\xi,\xi')A_3(\xi')d\xi'=\varphi(\xi),$$

then any (trial) function $\psi(\xi)$ which for *all* ξ in (a,b) satisfies

$$\psi(\xi)-\int_a^b K(\xi,\xi')\psi(\xi')d\xi'\geqslant\varphi(\xi)$$

is an upper bound for A_3, i.e., $A_3(\xi)\leqslant\psi(\xi)$ for all ξ in (a,b). This follows immediately from the convergence of the iterative series and the positivity of K and φ.

For the present situation we take a trial function of the form

$$\psi(u,r)=cu^{\alpha}/(ur+m)^{\beta},$$

where c,α,β are real parameters to be determined from the inequality

$$cu^{\alpha+1}(ur+m^2)^{-\beta}-\frac{c}{2}\left(\frac{g}{16\pi^2}\right)^2\int_{4\mu^2-m^2}^{u}du'(u-u')u'^{\alpha+1}\times\int_r^1 dr'(r'-r)(ur'+m^2)^{-\beta-2}-\frac{g^2}{32\pi}u\geqslant 0. \quad (4)$$

This must hold for $u\geqslant 4\mu^2-m^2$ and for $0\leqslant r\leqslant 1$. Setting $u=4\mu^2-m^2$, we obtain the condition

$$c(4\mu^2-m^2)^{\alpha}(4\mu^2)^{-\beta}\geqslant g^2/32\pi.$$

For the rest, it will be sufficient in order to satisfy Eq. (4) if we require (i) that the first derivative of the left-hand side of Eq. (4) with respect to u be positive at $u=4\mu^2-m^2$; and (ii) that the second derivative be positive for all $u>4\mu^2-m^2$. The requirement (i) leads to

$$c[(\alpha+1)/(4\mu^2-m^2)-\beta/4\mu^2](4\mu^2-m^2)^{\alpha+1}(4\mu^2)^{-\beta}\geqslant g^2/32\pi. \quad (5)$$

The requirement (ii) gives

$$\frac{\partial^2}{\partial u^2}\frac{u^{\alpha+1}}{(ur+m^2)^{\beta}}-\frac{1}{2}\left(\frac{g}{16\pi^2}\right)^2u^{\alpha+1}\times\int_r^1 dr(r'-r)(ur'+m^2)^{-\beta-2}\geqslant 0.$$

The integration is elementary and we find the following sufficient condition on the parameters α and β:

$$(\alpha-\beta)(\alpha-\beta+1)\beta(\beta+1)\geqslant\tfrac{1}{2}(g/16\pi^2)^2.$$

The smallest value of α which is compatible with our inequalities is

$$\alpha_0=-1+(1+g/4\sqrt{2}\pi^2)^{1/2}, \quad (6)$$

with $\beta_0=\alpha_0/2$. The smallest value for the parameter c, correspondingly, is

$$c_0=\frac{g^2}{32\pi}\frac{(4\mu^2)^{\alpha_0/2}}{(4\mu^2-m^2)^{\alpha_0}}.$$

For $p^2\leqslant 0$ we have thus obtained the bound

$$A(s,p^2)\leqslant\frac{g^2}{32\pi}\left(\frac{4\mu^2}{m^2-p^2}\right)^{\alpha_0/2}\left(\frac{s-p^2-m^2}{4\mu^2-m^2}\right)^{\alpha_0}\equiv\bar{A}(s,p^2).$$

In order, finally, to obtain an upper bound for the absorptive amplitude $A(s,m^2)$ on the mass shell, we substitute $\bar{A}(s',p'^2)$ for $A(s',p'^2)$ under the integral in Eq. (1), since there p'^2 runs over negative values only. We then find

$$A(s,m^2)\leqslant\frac{g^2}{32\pi}\left(1-\frac{4\mu^2}{s}\right)^{1/2}+\frac{1}{2}\left(\frac{g}{16\pi^2}\right)^2[s(s-4m^2)]^{-1/2}\times\int_{4\mu^2}^{(s^{1/2}-2\mu)^2}ds'\int_{f-(4\mu^2)}^{f+(4\mu^2)}dp'^2\bar{A}(s',p'^2)\times\int_{4\mu^2}^{y_0}dy\left(1-\frac{4\mu^2}{y}\right)^{1/2}.$$

For large values of s, the right side grows like s^{α_0}. Without affecting the exponent here, we can simplify the above expression by making some further majorizations. Using the inequalities

$$y_0\leqslant(m^2-p'^2)\frac{(s-p^2)-(s'-p'^2)}{s-p'^2+m^2}$$

and

$$(4\mu^2s'/s-s'-4\mu^2)<f_{\pm}(4\mu^2)+m^2<s-s'-4\mu^2,$$

we obtain the bound

$$A(s,m^2)\leqslant\frac{g^2}{32\pi}\left(1-\frac{4\mu^2}{s}\right)^{1/2}+\frac{1}{2}\left(\frac{g}{16\pi^2}\right)^2[s(s-4m^2)]^{-1/2}I(s),$$

where

$$I(s)=\frac{g^2}{32\pi}\frac{(4\mu^2)^{\alpha_0/2}}{(4\mu^2-m^2)^{\alpha_0}}\int_{4\mu^2-m^2}^{s-4\mu^2-2m^2}du'u'^{\alpha_0-1}(u-u')\times\int_{u'4\mu^2/(s-4\mu^2)}^{u'}dx(x+m^2)^{-(\alpha_0/2)-1}.$$

It follows that

$$A(s,m^2)\leqslant B(s)s^{\alpha_0},$$

where $B(s)$ is a *bounded* function of $s[\lim_{s\to\infty}B(s)\neq 0]$ and α_0 is given by Eq. (6). In fact, apart from logarithmic factors, s^{α_0} is the asymptotic behavior of the exact solution for $\mu=0$ obtained by Baker and Muzinich[5] and by Cosenza *et al.*[6] For small values of the coupling constant g, our result for α_0 also reproduces the (mass-independent) exact weak-coupling limit obtained by Sawyer.[3]

III. EXCHANGE OF VECTOR PARTICLES

We consider next the ladder approximation for scattering of scalar particles of mass m, with vector mesons (mass μ) as the exchanged particles. The propagator for the vector particle is taken to be $(\mu^2-p^2)^{-1}\delta_{\alpha\beta}$.

The integral equation for the absorptive (off-mass-shell) forward amplitude is

$$A(s,p^2)=\pi g^2(\mu^2-2m^2-2p^2)\delta(s-\mu^2)$$
$$+\frac{g^2}{8\pi}[(s-p^2-m^2)^2-4m^2p^2]^{-1/2}$$
$$\times\int_{\mu^2}^{(s^{1/2}-\mu)^2}ds'\int dp'^2\frac{\mu^2/2-p^2-p'^2}{(m^2-p'^2)^2}A(s',p'^2).\quad(7)$$

As mentioned in the Introduction, the vector meson propagator used here is not the correct one, unless the vector meson couples to a conserved current and unless one includes also compensating graphs. The artificiality of the present model can be seen in that the (generalized) unitarity condition does not imply termwise positivity of the iteration series solution of Eq. (7). Nevertheless, the present model has been much discussed in the literature because of its (relative) mathematical simplicity. We can, at least, guarantee positivity of the absorptive amplitude by insisting that $\mu\geqslant\sqrt{2}m$. In fact, for convenience in what follows, we shall restrict ourselves to the special case $\mu=\sqrt{2}m$.

The quantity

$$\psi(s,p^2)=A(s,p^2)+2\pi g^2p^2\delta(s-\mu^2)$$

satisfies the equation

$$\psi(s,p^2)=A^{(4)}(s,p^2)+\frac{g^2}{8\pi^2}[(s-p^2-m^2)^2-4m^2p^2]^{-1/2}$$
$$\times\int_{4\mu^2}^{(s^{1/2}-\mu)^2}ds'\int dp'^2\frac{m^2-p^2-p'^2}{(m^2-p'^2)^2}\psi(s',p'^2),\quad(8)$$

where $A^{(4)}(s,p^2)$ is the absorptive part of the fourth-order ladder graph. Following exactly the same steps as for the trilinear interaction model of Ref. 1, we arrive, in terms of the variables

$$u=s-p^2-m^2,$$
$$x=-p^2,$$

at the following inequality for the "trial function" $\bar{\psi}(u,x)$:

$$\bar{\psi}(u,x)-\frac{g^2}{8\pi^2}\frac{1}{u}\int_{4\mu^2-m^2}^{u}du'\int_{x(u'/u)}^{u'}dx'\frac{m^2+x+x'}{(x'+m^2)^2}\bar{\psi}(u',x')$$
$$\geqslant A^{(4)}(u,x)\geqslant\frac{g^4}{4\pi}\left(u+x\ln\frac{u}{m^2}\right).\quad(9)$$

This corresponds to Eq. (10) of Ref. 1, with $x=ru$. The trial function $\bar{\psi}$ is an upper bound to ψ if the above inequality is met.

A convenient form of trial function is $(\epsilon>0)$

$$\bar{\psi}(u,x)=cu^\alpha(x+m^2)^2[\theta(x-\epsilon)x^{-\beta-2}+\theta(\epsilon-x)\epsilon^{-\beta-2}],$$

a form for which the integrations of Eq. (9) are elementary.

It is a matter of straightforward manipulation to show that the parameters α, β, c, ϵ can be chosen to satisfy Eq. (9) for all $x\geqslant 0$ and $u\geqslant 4\mu^2-m^2$. The smallest possible value for α is

$$\alpha_0=-1+(1+g^2/\pi^2)^{1/2},\quad(10)$$

the corresponding value for β being $\beta_0=\alpha_0/2$. Explicit expressions for C_0 and ϵ_0 (which go along with $\alpha=\alpha_0$) could be written down but they are not of great interest here. We are mainly interested in the behavior of the absorptive amplitude for large values of s. Our bound on the mass-shell absorptive amplitude can then be written

$$A(s,m^2)\leqslant B(s)s^{\alpha_0},$$

where α_0 is given by Eq. (10) and $B(s)$ is a *bounded* function $[\lim_{s\to\infty}B(s)\neq 0]$ which we do not bother to display. Thus, we find an upper bound α_0 on the exponent of a possible power law behavior of the absorptive amplitude at large energies. It is noteworthy that our bound α_0 in fact coincides with the exponent corresponding to the exact solution for the special case $\mu=0$, as obtained by Cosenza *et al.*[6]