Physical Significance of Operators in Quantum Optics*

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The relation between normally ordered and unordered products of creation and annihilation operators is examined, and it is emphasized that the former correspond to counting correlations and the latter to counting moments. Both can be measured. It is shown that there exists a particularly simple relation between the generating functions for the two kinds of products. This relation can also be obtained by semiclassical considerations, which give more insight into its significance. The result provides further indication of the very close connection between the semiclassical and quantum-mechanical theories of optical coherence.

1. INTRODUCTION

THERE has recently been a good deal of discussion of the relation between the principal elements of the newly developing quantum theory of optical coherence and of the older semiclassical theories.[1–11] A close correspondence between the two has already been noted.

The quantum theory makes extensive use of configuration-space creation and annihilation operators $A_j^\dagger(x)$ and $A_j(x)$ defined by

$$A_j(x)=\sum_s \int d^3k \exp(-ikx)a_{k,s}\epsilon_{k,s,j} \tag{1}$$

and its Hermitian conjugate. Here $a_{k,s}$ is the annihilation operator for a photon of momentum $\hbar k$ and spin s, and $\boldsymbol{\epsilon}_{k,s}$ is the complex unit polarization vector. The principal elements of the theory, as it has developed so far, are the expectation values of normal-ordered products of creation and annihilation operators[2,3]

$$\begin{aligned}G^{(N,M)}{}_{j_1\cdots j_N,j_{N+1}\cdots j_{N+M}}(x_1\cdots x_N,x_{N+1}\cdots x_{N+M})\\ =\langle A_{j_1}{}^\dagger(x_1)\cdots A_{j_N}{}^\dagger(x_N)A_{j_{N+1}}(x_{N+1})\cdots A_{j_{N+M}}(x_{N+M})\rangle,\end{aligned} \tag{2}$$

which have recently been shown[4–6] to be equivalent to the corresponding correlation functions of the classical theory.[7,8] It was pointed out by Glauber[2] that the $G^{(n,n)}{}_{j_1\cdots j_n,j_1\cdots j_n}(x_1\cdots x_n,x_1\cdots x_n)$ are measures of the n-fold coincidence counting rate for photoelectric detectors sensitive to the $j_1,\cdots,j_n$ components of the radiation field at the space-time points $x_1,\cdots,x_n$. As

$$N=\int_{\delta V} A_j{}^\dagger(x)A_j(x)d^3x \tag{3}$$

is the number operator for photons in a volume δV at a given time, it follows that

$$\langle N\rangle=\int_{\delta V} G_{j,j}{}^{(1,1)}(x,x)d^3x. \tag{4}$$

By analogy with (4), it is at first tempting to look on

$$\int\cdots\int_{\delta V} G_{jj\cdots j}{}^{(n,n)}(x_1\cdots x_n,x_1\cdots x_n)d^3x_1\cdots d^3x_n$$

as the nth moment of the number operator N. That this cannot be the case was recently pointed out by Jordan,[12] who showed that $G_{jj,jj}{}^{(2,2)}(xx,xx)-[G_{j,j}{}^{(1,1)}(x,x)]^2$ is negative for certain states of the field. As a result, Jordan suggested that a theory of quantum optics ought not to be confined to normal-ordered operators.

In the following, we wish to emphasize that normal-ordered and alternating or unordered operators both correspond to "observables," but that they have a different physical significance. An N-point counting correlation, even for coincident points, represents a different measurement from an Nth moment of counts. The first calls for the use of coincidence detectors (or their equivalent), while the second does not. We shall show, moreover, that there exists an interesting relation between the ordered and unordered operators, whose meaning is not immediately obvious. Surprising though it might seem, this relation was already anticipated by semiclassical considerations of photoelectric counts,[10,13] which lend it an immediate physical interpretation.

2. MOMENTS AND CORRELATIONS OF COUNTS

As $G_{jj,jj}{}^{(2,2)}(x_1x_2,x_1x_2)$ is a measure of the correlation between the numbers counted at x_1 and x_2 with some

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[1] R. J. Glauber in *Quantum Electronics* (Columbia University Press, New York, 1964), Vol. III, p. 111.
[2] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[3] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[4] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[5] E. C. G. Sudarshan in *Proceedings of the Symposium on Optical Masers* (Polytechnic Institute of Brooklyn, Brooklyn, New York, 1963), p. 45.
[6] J. R. Klauder, J. McKenna, and D. G. Currie (to be published).
[7] E. Wolf in *Proceedings of the Symposium on Optical Masers* (Polytechnic Institute of Brooklyn, Brooklyn, New York, 1963), p. 29.
[8] E. Wolf, in *Quantum Electronics* (Columbia University Press, New York, 1964), Vol. III, p. 13.
[9] L. Mandel, Phys. Letters **7**, 117 (1963).
[10] L. Mandel, in *Progress in Optics* edited by E. Wolf (North-Holland Publishing Company, Amsterdam, 1963), Vol. II, p. 181.
[11] L. Mandel, E. C. G. Sudarshan, and E. Wolf, Proc. Phys. Soc. (London) **84**, 435 (1964).
[12] T. F. Jordan, Phys. Letters (to be published).
[13] L. Mandel, Proc. Phys. Soc .(London) **74**, 233 (1959).

detectors, it should come as no surprise that, for states which are eigenstates of the number operators $a_{k,s}^\dagger a_{k,s}$, the correlation between the counting fluctuations is negative. For, when the total number of particles is determined, the more counts are registered by one detector, the fewer can be registered by the other. A contradiction only appears to arise when the space-time points x_1 and x_2 coincide. But the difficulty is apparent rather than real, for the result must vary smoothly as $x_1 \to x_2$. One would therefore expect the correlation $G^{(2,2)}-[G^{(1,1)}]^2$ between the counting fluctuations at two detectors to be negative for states which are eigenstates of the number operator. This simple example shows that there is a physical distinction between the measurement of correlations of counts and moments of counts. The latter can be derived from a simple counting histogram obtained with one detector, whereas the former can only be measured with coincidence detectors.

Let us explore the relation a little further. For simplicity we allow the two detectors registering photons within δV to (almost) coincide, and observe that the correlation C between the fluctuations registered is

$$C=\int_{\delta V}\int \langle A_j^\dagger(x)A_j^\dagger(x')A_j(x)A_j(x')\rangle d^3x d^3x' -\left[\int_{\delta V}\langle A_j^\dagger(x)A_j(x)\rangle d^3x\right]^2, \quad (5)$$

whereas the expectation value of the square of the number operator, according to (3), is

$$\langle N^2\rangle=\int_{\delta V}\int \langle A_j^\dagger(x)A_j(x)A_j^\dagger(x')A_j(x')\rangle d^3x d^3x'. \quad (6)$$

With the help of the equal time commutation rules

$$\begin{aligned}[][A_j(x),A_j(x')]=0=[A_j^\dagger(x),A^\dagger(x')],\\ [A_j(x),A_j^\dagger(x')]=\delta^3(x-x'),\end{aligned} \quad (7)$$

we readily find that

$$\begin{aligned}C&=\langle N^2\rangle-\langle N\rangle^2-\langle N\rangle\\ &=\langle(\Delta N)^2\rangle-\langle N\rangle,\end{aligned} \quad (8)$$

so that the correlation C is positive, negative, or zero according as the numbers fluctuate more than, less than, or as in a Poisson process. For an eigenstate of the N operator, $\langle(\Delta N)^2\rangle$ vanishes and C is negative. For the mixed state of a typical thermal radiation field, it is well known[14] that $\langle(\Delta N)^2\rangle$ exceeds $\langle N\rangle$, so that C is positive. On the other hand, it is easy to see directly from definition (5) that there may be states for which C will vanish. Among these states are the "classical" or "coherent" eigenstates[15] $|\{v_{k,s}\}\rangle$ of the $A_j(x)$ operator used by Glauber[1–3] and Sudarshan,[4,5] for which the complex classical fields[7,8]

$$V_j(x)=\sum_s\int d^3k\,\exp(-ikx)v_{k,s}\epsilon_{k,s,j} \quad (9)$$

with Fourier amplitudes $v_{k,s}$ are the corresponding eigenvalues. For these states $\langle(\Delta N)^2\rangle=\langle N\rangle$. The coherent states therefore mark a transition from positive to negative counting correlations.

3. HIGHER ORDER MOMENTS AND CORRELATIONS

It should be clear from the foregoing that the counting correlations are described by normal-ordered operators, whereas the alternating operators $A_j^\dagger(x)\times A_j(x)A_j^\dagger(x)A_j(x)\cdots$ describe the counting moments. We shall now examine the higher order products of operators and the relation between them.

We denote by K_n the nth moment of the counts in some volume δV at a given time

$$K_n=\int\cdots\int_{\delta V}\langle A_j^\dagger(x_1)A_j(x_1)\cdots A_j^\dagger(x_n)A_j(x_n)\rangle d^3x_1\cdots d^3x_n,\quad n=1,2,3,\text{ etc.}, \quad (10)$$

and consider the relation between K_n and the correlation

$$L_n=\int\cdots\int_{\delta V}\langle A_j^\dagger(x_1)\cdots A_j^\dagger(x_n)A_j(x_1)\cdots A_j(x_n)\rangle d^3x_1\cdots d^3x_n. \quad (11)$$

Clearly $K_1=L_1$, but in general $K_n\neq L_n$. With the help of the commutation rules (7), we may transform (10) to

$$K_n=\int\cdots\int_{\delta V}\langle A_j^\dagger(x_1)[A_j^\dagger(x_2)A_j(x_1)+\delta^3(x_1-x_2)]\cdots[A_j^\dagger(x_n)A_j(x_{n-1})+\delta^3(x_n-x_{n-1})]A_j(x_n)\rangle d^3x_1\cdots d^3x_n$$

[14] For a more general discussion of this question see Ref. 11.

[15] $|\{v_{k,s}\}\rangle$ is to be interpreted as $\prod_{k\,s}|v_{k,s}\rangle$. The states are labeled by the corresponding eigenvalues.

$$=\int_{\delta V}\langle A_j^\dagger(x)A_j(x)\rangle d^3x+\binom{n-1}{1}\iint_{\delta V}\langle A_j^\dagger(x_1)A_j^\dagger(x_2)A_j(x_1)A_j(x_2)\rangle d^3x_1d^3x_2$$
$$+\binom{n-1}{2}\iiint_{\delta V}\langle A_j^\dagger(x_1)A_j^\dagger(x_2)A_j(x_1)A_j^\dagger(x_3)A_j(x_2)A_j(x_3)\rangle d^3x_1d^3x_2d^3x_3+\cdots$$
$$+\binom{n-1}{n-1}\int\cdots\int_{\delta V}\langle A_j^\dagger(x_1)A_j^\dagger(x_2)A_j(x_1)\cdots A_j^\dagger(x_n)A_j(x_{n-1})A_j(x_n)\rangle d^3x_1\cdots d^3x_n.$$

By repeated application of the commutation rules in order to arrange the operators in normal order, together with definition (11), we arrive at the series

$$K_n=a_1^{(n)}L_1+a_2^{(n)}L_2+\cdots+a_n^{(n)}L_n, \tag{12}$$

where

$$a_r^{(n)}=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_{r-1}=1}^{i_{r-2}-1}\binom{n-1}{i_1}\binom{i_1-1}{i_2}\cdots\binom{i_{r-2}-1}{i_{r-1}};\quad r=2,\,3,\text{ etc., and } a_1^{(n)}=1. \tag{13}$$

This expresses the general relation between the moments K_n and the correlations L_n holding for any state of the radiation field. We note that $K_2=L_1+L_2$, but that in general the relation appears to be far from simple.

We shall see, however, that there exists an extremely simple relation between the generating functions

$$M_K(y)=1+\sum_{n=1}^{\infty}\frac{1}{n!}K_ny^n \tag{14}$$

and

$$M_L(y)=1+\sum_{n=1}^{\infty}\frac{1}{n!}L_ny^n. \tag{15}$$

With the help of the explicit form for $a_r^{(n)}$ given by (13), it may be shown that

$$\sum_{n=1}^{\infty}\frac{a_r^{(n)}y^n}{n!}=\frac{(e^y-1)^r}{r!}, \tag{16}$$

so that

$$M_K(y)=1+\sum_{r=1}^{\infty}\frac{(e^y-1)^r}{r!}L_r$$
$$=M_L(e^y-1). \tag{17}$$

As this result holds for any state of the field, we may also look on (17) as an equation connecting the operators themselves, rather than their expectation values.[16] Its significance is not at all obvious at this stage.

However, it is interesting to note that this mysteriously simple relation between M_K and M_L has already been derived from semiclassical considerations[13,10] which lead to it almost at once,[17] and indicate its significance. We outline the argument very briefly below.

[16] This relation can also be obtained directly, without explicit use of the expression for $a_r^{(n)}$. I am indebted to Dr. C. L. Mehta for a particularly simple proof. Operators of the same form have also been used by J. Schwinger, J. Math. Phys. 2, 407 (1961).

4. DISCUSSION OF THE SEMICLASSICAL TREATMENT

If $V_j(x)$ given by (9) is an eigenvalue of $A_j(x)$ corresponding to the state $|\{v_{k,s}\}\rangle$, then, by expressing the density operator ρ of the field in Sudarshan's universal "diagonal" representation for free boson fields[4,5] in the basis $|\{v_{k,s}\}\rangle$,[15] we find

$$\langle N\rangle=\mathrm{Tr}\int d^2\{v_{k,s}\}\int_{\delta V}d^3x\,\rho(\{v_{k,s}\})|\{v_{k,s}\}\rangle$$
$$\times\langle\{v_{k,s}\}|A_j^\dagger(x)A_j(x)$$
$$=\int d^2\{v_{k,s}\}\int_{\delta V}d^3x\,\rho(\{v_{k,s}\})V_j^*(x)V_j(x)$$
$$=\left\langle\!\left\langle\int_{\delta V}V_j^*(x)V_j(x)d^3x\right\rangle\!\right\rangle, \tag{18}$$

where the double angular brackets are to be interpreted as an ensemble average, in the sense that $\rho(\{v_{k,s}\})$ is a generalized weighting or "probability" function.[5] This relation suggests that, if the field is to be described classically, one might look on

$$U=\int_{\delta V}V_j^*(x)V_j(x)d^3x \tag{19}$$

as the expectation value of the number of counts for a single typical member of the ensemble of classical

[17] It should be noted that in Refs. 13 and 10, the moments refer to counts registered by a detector in a given time interval, whereas (3) refers to numbers in a given volume. However, for plane light waves striking a detector these numbers obviously correspond directly.

fields. This, in turn, leads one to expect that the probability $p(r)$ of counting r photons in δV will be given by a Poisson distribution with parameter U, which is then to be averaged over the ensemble of U. Thus,

$$p(r)=\left\langle\left\langle\frac{U^r}{r!}\exp(-U)\right\rangle\right\rangle. \qquad (20)$$

Further details of the argument leading to (20) are given in Refs. 10 and 11.

It will now be seen that the nth moment of r, i.e., of the counts, corresponds to the K_n defined quantum mechanically by (10), whereas the nth moment of U, i.e., of the classical integrated intensity, corresponds to the quantum correlation L_n given by (11). The moment-generating function for r is given by

$$M_r(y)=\sum_{r=0}^{\infty}\exp(ry)p(r);$$

and from (20), and with the help of the well-known properties of the Poisson distribution, we arrive at

$$M_r(y)=\langle\langle\exp[U(e^y-1)]\rangle\rangle = M_U(e^y-1), \qquad (21)$$

by definition of the moment-generating function for U. This relation is the semiclassical equivalent of the quantum-mechanical equation (17).

The result illustrates once again that normal-ordered operators correspond to correlations of the complex field in the semiclassical treatment. As the relations (17) and (21) hold for any state of the field, we see that the semiclassical theory may sometimes be just as accurate as the quantized field theory, while providing some valuable intuitive insight into the physics of the problem.

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Gravitational Radiation and the Motion of Two Point Masses

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The expansion of the field equations of general relativity in powers of the gravitational coupling constant yields conservation laws of energy, momentum, and angular momentum. From these, the loss of energy and angular momentum of a system due to the radiation of gravitational waves is found. Two techniques, radiation reaction and flux across a large sphere, are used in these calculations and are shown to be in agreement over a time average. In the nonrelativistic limit, the energy and angular momentum radiation and angular distributions are expressed in terms of time derivatives of the quadrupole tensor Q_{ij}. These results are then applied to a bound system of two point masses moving in elliptical orbits. The secular decays of the semimajor axis and eccentricity are found as functions of time, and are integrated to specify the decay by gravitational radiation of such systems as functions of their initial conditions.

I. INTRODUCTION

THE existence of gravitational radiation was predicted by Einstein[1,2] shortly after he formulated his general theory of relativity. Systems of moving masses should emit gravitational waves in analogy with the emission of electromagnetic waves by a system of moving charges. Early attempts to calculate the energy in these waves were based on the use of a pseudostress-energy tensor for the evaluation of the energy flux. One disadvantage of this method is that one can always choose a coordinate system in which the energy flux vanishes.[3] This led to much scepticism about the reality of gravitational radiation. Another disadvantage of the calculation is that it is valid only for systems which are not gravitationally bound. Thus, the important case of gravitational radiation from binary stars remained unsolved at that time.

Later, Eddington found the radiation from a system by calculating the radiation reaction of the system on itself.[4] However, like Einstein's method, this is not valid for gravitationally bound systems. For situations in which the radiation is constant, the two methods agree; for situations in which the radiation is time-dependent, the answers differ. One can show that over a time average of the motion the two answers are in agreement. Analogous results occur in the theory of electromagnetic radiation.

For systems in which the velocities of the masses are small compared to the velocity of light, the calculation of Einstein has been extended to include gravitationally

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[1] A. Einstein, Sb. Preuss. Akad. Wiss. 688 (1916).
[2] A. Einstein, Sb. Preuss. Akad. Wiss. 154 (1918).
[3] For a detailed discussion of the status of the theories of gravitational radiation and their objections, the reader is referred to the review article by F. A. E. Pirani, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), Chap. 6.
[4] A. S. Eddington, Proc. Roy. Soc. (London) **A102**, 268 (1922).