Coupling of Magnetohydrodynamic Waves in Stratified Media

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The theory presented deals with propagation of magnetohydrodynamic waves in a plasma whose propagation characteristics vary along the path of propagation. Plane waves with wave normals in the direction of the gradient of the medium are assumed. Modified Alfvén waves and acoustic waves obey a set of coupled wave equations. Alfvén waves are decoupled by a restricting assumption on the variation of the static magnetic field. The main aim of the paper is to demonstrate the generation of fairly intense acoustic waves in propagation of modified Alfvén waves in a steep gradient of the propagation conditions. This effect is investigated at a boundary which is equivalent to a very steep gradient.

1. INTRODUCTION

It is assumed that waves are propagated in the $\pm z$ direction in a medium varying with z, i.e., in a one-dimensionally stratified medium. All wave functions that will appear are consequently of the form

$$u=A(z)\exp\left[-i\int k(z)dz\right] \tag{1}$$

or

$$w=B(z)\exp\left[i\int k(z)dz\right]. \tag{2}$$

Symbols u and w are used to characterize waves traveling in the two opposite directions $\pm z$.

In the case of *electromagnetic waves* in a plasma with presence of a static magnetic field, one is concerned with *two* modes of propagation (termed "ordinary" and "extraordinary") with different propagation constants (k_1,k_2). There are accordingly four wave functions u_1, w_1, u_2, w_2, referring to the two modes in ascent and in descent. In the present study of *magnetohydrodynamic waves*, *three* modes have to be distinguished: Alfvén waves, modified Alfvén waves, and acoustic (more strictly termed "ion-acoustic") waves. Varying nomenclature for these modes is in use in the literature.

All formulations will be kept one-dimensional. Refraction of waves does not appear in the one-dimensional considerations.

In a slowly varying medium, the individual wave functions (u_i,w_i) are (nearly) independent of each other. The amplitude of any of the waves (A_i or B_i), though varying slowly with z, is in this case not affected by the presence of other waves. This is the geometric-optical situation, in which Eqs. (1) and (2) represent the WKB approximations.

With a more rapid variation of the medium, one wave may gain intensity at the expense of some other wave or it may lose while the other gains. This is *coupling* between the waves, caused by the gradient of the parameters of the medium. The wave functions are determined by a set of coupled wave equations. Written as a single vector equation, this set is expected to have the form

$$\frac{\partial \mathbf{u}}{\partial z}+\frac{i\omega}{c}\mathsf{N}\cdot\mathbf{U}=\mathsf{M}\cdot\mathbf{U}. \tag{3}$$

The vector quantity $\mathbf{U}$ denotes the combination of all individual wave functions. With inclusion of four wave functions (corresponding to two propagation modes) there is

$$\mathbf{U}=\begin{pmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{pmatrix}. \tag{4}$$

Boldface roman type is used for vectors, sans-serif type for matrices. The matrix N, a diagonal matrix composed of the refractive indices $n_i=k_ic/\omega$, is defined as

$$\mathsf{N}=\begin{pmatrix} n_1 & 0 & 0 & 0 \\ 0 & -n_1 & 0 & 0 \\ 0 & 0 & n_2 & 0 \\ 0 & 0 & 0 & -n_2 \end{pmatrix}. \tag{5}$$

The matrix M is supposed to disappear in a homogeneous medium. Its elements consequently must be terms or sums of terms, that contain some derivative with respect to z as a factor.

The matrix M, generally speaking, determines the variation of amplitudes (A_i and B_i) of the waves that results from the inhomogeneity of the medium. The nondiagonal elements of M express coupling between the component equations of Eq. (3); they refer to variation of the amplitudes by coupling between waves. The diagonal elements of M correspond to amplitude variations independent of the presence of other waves. These amplitude variations may simply indicate that the amplitudes are functions of some parameters of the medium. As an example may be noted the case of electromagnetic waves in a nonhomogeneous dielectric, in which the amplitude of $\mathbf{E}$ is a function of ϵ. The entire term $\mathsf{M}\cdot\mathbf{U}$ in Eq. (3) will be called the "coupling term," despite the two types of amplitude variations included in it.

Coupling occurs between any two wave functions. Coupling between the two waves of one mode traveling in opposite directions (u_i and w_i) is nothing but partial reflection.

Coupling of electromagnetic waves under ionospheric conditions has been treated in the literature[1] to some

[1] K. G. Budden, *Radio Waves in the Ionosphere* (Cambridge University Press, Cambridge, England, 1961), in particular, pp. 394–408.

extent. (More references were given elsewhere.[2]) The aim now is a similar representation of coupling of magnetohydrodynamic waves. The formulations that will follow resemble those given by Budden[1] for electromagnetic waves. In order to obtain coupled wave equations of the form of the above Eq. (3), it will be necessary first to define the wave functions in an appropriate way. The elements of the coupling matrix **M** for these wave functions have to be derived. It will be seen that coupling leads to generation of acoustic waves in passage of modified Alfvén waves through an inhomogeneous medium, a process believed to be of interest in applications of the theory.

As mentioned above, it will be assumed that the wave normals are in (or opposite to) the direction of the gradient. The magnetic field may have arbitrary and varying direction, but with the restraint that it has to stay in one plane with the direction of the gradient. Attenuation of waves, known to result from collisions, friction, and Landau damping, will not be considered.

S. Weinberg[3] (in Sec. III and V of his paper) developed a method for the study of coupling between magnetohydrodynamic waves on the basis of more general formulations (not limited, for example, to one-dimensional stratification). Weinberg includes coupling between Alfvén waves and modified Alfvén waves, which now is eliminated by the specializing assumption on the magnetic field, but he omits in his more specific formulations acoustic waves, with which the present paper is particularly concerned. Fejer[4] indicated already how to use coupling theory in the case of magnetohydrodynamic waves in the atmosphere.

Effects of a stratification of the medium under conditions under which coupling between individual magnetohydrodynamic waves can be ignored have been investigated in a number of papers.[3–9] The reader is referred to these papers for effects not discussed here. (Some more references will be found in those given and in a paper on atmospheric phenomena that is in preparation.)

2. BASIC RELATIONSHIPS OF MAGNETOHYDRODYNAMICS

In the common simple formulation of magnetohydrodynamics the basic equations are Maxwell's equations, the hydrodynamic continuity condition for the plasma, and the two macroscopic equations connecting the motion of the plasma with the electromagnetic field,

$$\rho_0(\partial \mathbf{v}/\partial t)=\mathbf{J}\times\mathbf{B}_0-\nabla(a^2\rho)\,, \tag{6}$$

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}_0=0\,. \tag{7}$$

The notations introduced are:

- ρ_0 plasma density without oscillatory part,
- ρ oscillatory part of the plasma density,
- $\mathbf{v}$ gravity center velocity of a volume element of the plasma,
- $a^2\rho$ oscillatory part of the plasma pressure,
- a ionacoustic velocity (plasma sound velocity),
- $\mathbf{J}$ (oscillatory) current density,
- $\mathbf{E}$ electric field strength,
- $\mathbf{B}$ magnetic vector of the wave field,
- $\mathbf{B}_0$ magnetic vector of the static magnetic field.

The units are supposed to be in the mksA system.

All equations are linearized. The only spatial derivatives that appear are with respect to z (because of the limitation to the one-dimensional problem). The plasma density ρ_0, plasma sound velocity a, and direction and magnitude of the magnetic field $\mathbf{B}_0$ may now be functions of z. The time variation of all oscillatory quantities is supposed to follow $\exp(i\omega t)$. The symbol $\partial/\partial t$ will accordingly be replaced by $i\omega$.

In Eq. (6) it was assumed that the oscillatory preslure is $p=a^2\rho$. This is an approximation only. The resationship between p and ρ given in the literature[10] is more complicated. Proportionality between pressure variation and density variation exists in general not at a fixed location, but in a plasma parcel that travels with velocity $\mathbf{v}$. If however both the stationary quantities p_0, ρ_0 as functions of z and the oscillatory p and ρ in a plasma parcel follow the same adiabatic law, the proportionality assumed here is exact.

It will be assumed further on that the static magnetic field remains in one plane, the x,z plane. A variation of the magnetic field strength $\mathbf{B}_0$ with z requires under this assumption a curl $\mathbf{B}_0$ and a dc current density $\mathbf{J}_0$. If the plasma is the carrier of this current density, a force term $\mathbf{J}_0\times\mathbf{B}$ must be added in Eq. (6). The consequences of this term, which might be essential in discharge tubes, are not investigated now. The current density $\mathbf{J}_0$ may however result from processes foreign to the plasma, for example from corpuscular beams passing through the plasma. Only such dc currents are compatible with the present formulations.

It is necessary to select a set of variables for description of the wave field and the state of the plasma. The formulations will be simplified by using as these variables the quantities that appear under the $\partial/\partial z$ sign in the basic equations [Maxwell's equations, the continuity condition, and Eqs. (6) and (7)]. It proves advantageous to multiply these quantities by certain

[2] H. Poeverlein, in *Fortschritte der Hochfrequenztechnik* (Akademische Verlagsgesellschaft, Wiesbaden, 1959), Vol. 4, pp. 75–76.
[3] S. Weinberg, Phys. Rev. **126**, 1899 (1962).
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[6] J. H. Piddington, Geophys. J. Roy. Astronom. Soc. **2**, 173 (1959).
[7] W. E. Francis, M. I. Green, and A. J. Dessler, J. Geophys. Res. **64**, 1643 (1959).
[8] G. J. F. MacDonald, J. Geophys. Res. **66**, 3639 (1961).
[9] J. Bazer and J. Hurley, J. Geophys. Res. **68**, 147 (1963).
[10] C. O. Hines, Can. J. Phys. **38**, 1441 (1960).

constants and to combine them in two "vectors" consisting of four and two components respectively. These vectors are

$$\mathbf{F}=\begin{bmatrix} \rho_0 v_z \\ (1/c)a^2\rho \\ \epsilon_0 E_y \\ (\epsilon_0/\mu_0)^{1/2}B_x \end{bmatrix} \tag{8}$$

and

$$\mathbf{G}=\begin{bmatrix} \epsilon_0 E_x \\ (\epsilon_0/\mu_0)^{1/2}B_y \end{bmatrix} \tag{9}$$

(with the vacuum dielectric constant ϵ_0, the vacuum permeability μ_0, and the vacuum light velocity c). The vectors **F** and **G**, characterizing the electromagnetic field and the state of the plasma in the waves, may be called "field vectors."

Now it is desirable to have some differential equations for the field vectors. A possible way of deriving these differential equations will be sketched only.

The basic equations are a set of linear homogeneous equations for the components of **F** and **G**, their first-order derivatives, and the additional set of variables J_x, J_y, J_z, v_x, v_y, E_z, and B_z. The equations may be solved for the derivatives of the **F** and **G** components and the set of variables listed just now. All these quantities appear in the solution expressed in terms of the **F** and **G** components. In an earlier paper[11] two equations [Eqs. (25) and (27) of that paper] were given that are helpful in deriving the solutions for the **J**-components and $(\partial/\partial z)(a^2/\rho)$. (The equations were then written under the assumption of constant sound velocity a.)

The solutions obtained for the derivatives of **F** and **G** express these derivatives in terms of **F** and **G**. They are the two equations

$$\frac{\partial \mathbf{F}}{\partial z}=-\frac{i\omega}{c}\mathbf{P}\cdot\mathbf{F} \tag{10}$$

and

$$\frac{\partial \mathbf{G}}{\partial z}=-\frac{i\omega}{c}\mathbf{Q}\cdot\mathbf{G}, \tag{11}$$

with the two matrices

$$\mathbf{P}=\begin{bmatrix} 0 & \dfrac{c^2}{a^2} & 0 & 0 \\ \dfrac{B_0^2}{B_{0z}^2} & 0 & \dfrac{\rho_0}{\epsilon_0}\dfrac{B_{0x}}{B_{0z}^2} & 0 \\ 0 & 0 & 0 & -1 \\ -\dfrac{B_{0x}}{B_{0z}^2} & 0 & -\dfrac{B_{0z}^2+\rho_0/\epsilon_0}{B_{0z}^2} & 0 \end{bmatrix} \tag{12}$$

and

$$\mathbf{Q}=\begin{bmatrix} 0 & 1 \\ \dfrac{B_0^2+\rho_0/\epsilon_0}{B_{0z}^2} & 0 \end{bmatrix}. \tag{13}$$

Equations (10) and (11) are the desired differential equations for the field vectors or the "field equations." The two equations are independent of each other. This fact justifies the definition of *two* field vectors **F** and **G**, as made in Eqs. (8) and (9).

The formulas for the two matrices will not be needed later on, but it is essential that the elements of the matrices are functions of ρ_0, a, B_0, B_{0x}, and B_{0z} only and not of any derivative of these parameters, which characterize the medium (cf. similar formulations by Budden[1] and Fejer[4]). This is a consequence of choosing as components of **F** and **G** all the quantities whose spatial derivatives appear in the basic equations.

The independence of Eqs. (10) and (11) from one another results ultimately from the assumption that the magnetic field remains in the x,z plane. Authors who assumed a *constant* magnetic field observed this separation of equations (cf. MacDonald[8] and Grad[12]). Alfvén waves are deducible from Eq. (11), whereas Eq. (10) leads to the other two modes, modified Alfvén waves and acoustic waves. Thus, Alfvén waves are now not coupled with the other two modes. They are consequently not of interest in this study of coupling. Only the two coupled modes, obtained from Eq. (10), will be considered in the following.

3. MODIFIED ALFVÉN WAVES AND ACOUSTIC WAVES

Treatment of coupling requires a definition of the wave functions of the coupled waves. The coupled waves, as noted, are modified Alfvén waves and acoustic waves, both derived from Eq. (10) and described by the behavior of the field vector **F**. One of the components of **F** may be selected to be split up into the contributions of individual waves, which then are taken to be the wave functions.

A quantity of particular interest in atmospheric problems is the plasma flux density. Its vertical or longitudinal component $\rho_0 v_z$, i.e., the first component of **F**, is therefore chosen for the definition of wave functions. The wave functions represent the vertical components of the flux densities of the individual waves. Hence, there is

$$\rho_0 v_z = u_1+w_1+u_2+w_2. \tag{14}$$

The four wave functions are those of modified Alfvén waves traveling in the $+z$ and $-z$ directions (u_1 and w_1) and of acoustic waves (u_2 and w_2). All wave func-

[11] H. Poeverlein, Zeit. Angew. Phys. **15**, 441 (1963).

[12] H. Grad, in *The Magnetodynamics of Conducting Fluids*, edited by D. Bershader (Stanford University Press, Stanford, California, 1959), p. 37.

tions in combination may be represented by one vector symbol **U** as shown in Eq. (4).

Two auxiliary quantities that will be used frequently are the ratio between vacuum light velocity c and plasma sound velocity a,

$$\lambda = c/a, \tag{15}$$

and the ratio between c and Alfvén velocity,

$$\mu = (\rho_0/\epsilon_0 B_0^2)^{1/2}. \tag{16}$$

It is assumed that

$$\lambda \gg \mu \gg 1. \tag{17}$$

That is to say, the Alfvén velocity is well below c and the sound velocity is well below the Alfvén velocity.

In a *homogeneous medium*, each of the four types of waves (corresponding to u_1, w_1, u_2, w_2) is characterized by its refractive index and the ratios between the various quantities oscillating in the wave. These characteristic data may be taken from the numerous literature on waves in plasmas.[9,12–18] The refractive indices under the present assumptions [including, in particular, inequalities (17)] are

$$n_1 = \mu \tag{18}$$

and

$$n_2 = \lambda/|\cos\alpha|. \tag{19}$$

The first refractive index relates to modified Alfvén waves, the second one to acoustic waves, the sound waves in the ion gas, which are affected by electromagnetic forces. The angle α between $\mathbf{B}_0$ and the z-direction is supposed to be not close to 90 deg.

Some of the oscillating quantities have been combined in the vector **F**. The ratios of these quantities for the individual waves, as given in the literature for a homogeneous medium, are represented by

$$\mathbf{F} = \mathbf{T}\cdot\mathbf{U} \tag{20}$$

with the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \dfrac{\mu}{\lambda^2} & -\dfrac{\mu}{\lambda^2} & \dfrac{1}{\lambda|\cos\alpha|} & -\dfrac{1}{\lambda|\cos\alpha|} \\ -\dfrac{1}{\mu^2 B_0 \sin\alpha} & -\dfrac{1}{\mu^2 B_0 \sin\alpha} & \dfrac{\sin\alpha}{\lambda^2 B_0} & \dfrac{\sin\alpha}{\lambda^2 B_0} \\ \dfrac{1}{\mu B_0 \sin\alpha} & -\dfrac{1}{\mu B_0 \sin\alpha} & -\dfrac{\sin\alpha}{\lambda B_0|\cos\alpha|} & \dfrac{\sin\alpha}{\lambda B_0|\cos\alpha|} \end{bmatrix}. \tag{21}$$

In order to obtain from Eqs. (20) and (21) the known ratios of oscillatory quantities for a single wave, one lets all components of **U** but one disappear.

Transverse and longitudinal magnetic fields $\mathbf{B}_0$ have to be excluded. In case of a transverse field (corresponding to $\alpha = 90$ deg), the refractive index formula Eq. (19) becomes invalid. In case of a longitudinal field the present choice of wave functions appears inappropriate.

4. COUPLED WAVE EQUATIONS

The various parameters of the medium are now supposed to be functions of z. The field vector **F** will be used as before to describe the electromagnetic field and the state of the plasma in the waves. The four wave functions, which were introduced with reference to the well-distinguishable four waves in a homogeneous medium, are now considered in an *inhomogeneous medium.* It is assumed that the individual waves exhibit at any location the same characteristics as in a homogeneous medium of the local conditions. In other words, Eq. (20) and the matrix **T** of Eq. (21) are left unchanged. The two equations with a given field vector **F** determine now how $\rho_0 v_z$ and the field vector at any location are split up into contributions of individual waves.

It may be shown now that the wave functions (combined in the vector **U**) obey a set of coupled wave equations such as Eq. (3). An expression for the coupling matrix will be deduced.

The field vector **F** is subject to Eq. (10). The matrix **P** in Eq. (10) is supposed to be known. Replacement of **F** in Eq. (10) by $\mathbf{T}\cdot\mathbf{U}$ and subsequent rearrangement of terms yields

$$\mathbf{T}\cdot\frac{\partial\mathbf{U}}{\partial z} + \frac{i\omega}{c}\mathbf{P}\cdot\mathbf{T}\cdot\mathbf{U} = -\frac{\partial\mathbf{T}}{\partial z}\cdot\mathbf{U}. \tag{22}$$

Multiplication of this equation by $\mathbf{T}^{-1}$ from the left leads to the wave equation

$$\frac{\partial\mathbf{U}}{\partial z} + \frac{i\omega}{c}\mathbf{T}^{-1}\cdot\mathbf{P}\cdot\mathbf{T}\cdot\mathbf{U} = -\mathbf{T}^{-1}\cdot\frac{\partial\mathbf{T}}{\partial z}\cdot\mathbf{U}. \tag{23}$$

[13] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1962), 2nd ed., pp. 61–67.
[14] J. Bazer and O. Fleischman, Phys. Fluids 2, 366 (1959).
[15] R. Lüst, Fortschr. Physik 7, 503 (1959).
[16] J. F. Denisse and J. L. Delcroix, *Théorie des Ondes dans les Plasmas* (Dunod et Cie., Paris, 1961), pp. 117–132.
[17] T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), pp. 32–34, 41, 42.
[18] W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas* (MIT Press, Cambridge, Massachusetts, 1963), pp. 47, 58, 78–85.

Equation (23) has the form of Eq. (3). In Eq. (3) the term on the right-hand side was supposed to disappear in a homogeneous medium. The right-hand term of Eq. (23) apparently does so too. Equation (3) thus is seen to be valid with

$$\mathbf{N}=\mathbf{T}^{-1}\cdot\mathbf{P}\cdot\mathbf{T} \tag{24}$$

and

$$\mathbf{M}=-\mathbf{T}^{-1}\cdot\frac{\partial\mathbf{T}}{\partial z}. \tag{25}$$

The matrix $\mathbf{N}$ was defined by Eq. (5) as a diagonal matrix whose elements are $\pm n_1$ and $\pm n_2$. Equation (24) is necessarily fulfilled if the two refractive indices [Eqs. (18) and (19)] and the matrix $\mathbf{T}$ [Eq. (21)] were given correctly. The equation pertains to the problem of finding the wave solutions in a homogeneous medium, an eigenvalue problem with $\pm n_i$ as eigenvalues.

The matrix $\mathbf{M}$ is the coupling matrix, which appears now expressed in terms of the matrix $\mathbf{T}$ and its derivative. Of interest should be mainly the nondiagonal elements of $\mathbf{M}$. It was noted in Sec. 1 that only these elements relate to coupling between waves.

Not all oscillatory quantities of physical significance appear as components of the field vector $\mathbf{F}$. The transverse components of the plasma flux density, for example, have to be derived from Eq. (7). With insertion of the wave functions they are found to be, regardless of homogeneity or inhomogeneity of the medium,

$$\left.\begin{aligned}\rho_0 v_x&=-(u_1+w_1)\frac{\cos\alpha}{\sin\alpha}+(u_2+w_2)\frac{\sin\alpha}{\cos\alpha},\\ \rho_0 v_y&=0.\end{aligned}\right\} \tag{26}$$

The longitudinal component of the flux density was simply the sum of all wave functions. Thus it becomes evident from Eqs. (26) that the plasma flux of modified Alfvén waves (corresponding to subscript 1) is normal to $\mathbf{B}_0$, while that of acoustic waves is parallel to $\mathbf{B}_0$. Distinction of the two modes of propagation consequently means also separation of the two plasma fluxes, normal and parallel to $\mathbf{B}_0$.

It should be noticed that Eqs. (26) are approximate only. Equation (7) requires a $\mathbf{v}$-component normal to $\mathbf{B}_0$ also in acoustic waves because of presence of an $\mathbf{E}$. This $\mathbf{v}$-component, however, is found to be small of the order μ^2/λ^2 compared to the other $\mathbf{v}$-component.

5. GENERATION OF ACOUSTIC WAVES BY MODIFIED ALFVÉN WAVES

It is assumed in this section that the primary wave passing through the one-dimensionally varying medium is a modified Alfvén wave (u_1). The coefficients of u_1 in the coupling term of Eq. (3) will be investigated. The corresponding part of the coupling term leads to generation of acoustic waves (u_2,w_2), to reflection of modified Alfvén waves (w_1), and to amplitude variation of the primary wave (u_1).

Neglect of all wave functions but u_1 in the coupling term represents a first approximation, which might be used without correction in case of weak coupling or might be taken as a first step of an iterative procedure. In each further step the set of wave functions obtained in the preceding step would have to be inserted in the coupling term in order to derive corrected wave functions. A similar iterative procedure was suggested by Weinberg.[3]

The elements of the coupling matrix that appear as coefficients of u_1 are

$$M_{i1}=-\sum_{k=1}^{4}(\mathbf{T}^{-1})_{ik}\frac{\partial\mathbf{T}_{k1}}{\partial z}, \tag{27}$$

with $i=1, 2, 3, 4$. For the matrix inverse to $\mathbf{T}$ one finds, disregarding terms of higher order in μ/λ,

$$\mathbf{T}^{-1}=\frac{1}{2}\begin{bmatrix}\frac{\mu^2}{\lambda^2}(\sin\alpha)^2 & \mu(\sin\alpha)^2 & -\mu^2B_0\sin\alpha & \mu B_0\sin\alpha\\ \frac{\mu^2}{\lambda^2}(\sin\alpha)^2 & -\mu(\sin\alpha)^2 & -\mu^2B_0\sin\alpha & -\mu B_0\sin\alpha\\ 1 & \lambda|\cos\alpha| & \mu^2B_0\sin\alpha & -\frac{\mu^2}{\lambda}B_0\sin\alpha|\cos\alpha|\\ 1 & -\lambda|\cos\alpha| & \mu^2B_0\sin\alpha & \frac{\mu^2}{\lambda}B_0\sin\alpha|\cos\alpha|\end{bmatrix}. \tag{28}$$

The desired elements of the coupling matrix Eq. (27) become, after introduction of $\mathbf{T}$ and $\mathbf{T}^{-1}$ and with use of approximations as will be outlined,

$$\left.\begin{aligned}M_{11}&=-\frac{1}{2}\frac{1}{\mu}\frac{\partial\mu}{\partial z}+\frac{1}{\mu^2B_0\sin\alpha}\frac{\partial}{\partial z}(\mu^2B_0\sin\alpha), & M_{21}&=\frac{1}{2}\frac{1}{\mu}\frac{\partial\mu}{\partial z},\\ M_{31}&=-\frac{1}{2}\frac{1}{\mu^2B_0\sin\alpha}\frac{\partial}{\partial z}(\mu^2B_0\sin\alpha), & M_{41}&=-\frac{1}{2}\frac{1}{\mu^2B_0\sin\alpha}\frac{\partial}{\partial z}(\mu^2B_0\sin\alpha).\end{aligned}\right\} \tag{29}$$

The assumptions made in the derivation of Eqs. (29) are that μ/λ is small and that the relative orders of magnitudes of terms are not disturbed by taking derivatives. This requires a sufficiently normal behavior of all functions involved and a sufficiently slow variation of λ, which itself depends on temperature. The necessary condition for λ is: $(1/\lambda)(\partial\lambda/\partial z)$ must not become large (of the order λ/μ) in comparison with M_{31} and M_{41}. In the upper atmosphere this is believed to be fulfilled.

The elements of the coupling matrix shown in Eqs. (29) are composed of two terms, which express the relative rate of variation of two quantities, μ and $\mu^2 B_0 \sin\alpha$. The first quantity μ is identical to the refractive index n_1. Its variation determines according to Eqs. (29) the interchange between u_1 and w_1 or "reflection" of the wave u_1. Variation of the other quantity, which is transformed by means of Eq. (16) as

$$\mu^2 B_0 \sin\alpha = (\rho_0/\epsilon_0 B_0) \sin\alpha\,, \qquad (30)$$

leads to interchange between u_1 on one side and u_2 and w_2 on the other. This is coupling with the result of generation of acoustic waves traveling in the two opposite directions, u_2 and w_2.

There is no generation of acoustic waves when the expression of Eq. (30) stays constant, even though some of the individual parameters may vary. This is true, no matter how rapid the variation of individual parameters is. A gradient of λ, which was neglected in Eqs. (29), may however show up in case of disappearing Eq. (30).

All waves originating from reflection or coupling become negligible in a very slowly varying medium, but the diagonal matrix element M_{11} is of significance also in this case, as was explained in Sec. 1. Combining the two terms in M_{11}, with subsequent insertion of μ from Eq. (16), leads to the following expression for M_{11}:

$$M_{11} = \frac{B_0^{1/2}}{\rho_0^{3/4} \sin\alpha} \frac{\partial}{\partial z}\left(\frac{\rho_0^{3/4} \sin\alpha}{B_0^{1/2}}\right). \qquad (31)$$

The wave equation, Eq. (3), with Eq. (31) shows that the amplitude of u_1 in a slowly varying medium varies as

$$A_1 \propto \rho_0^{3/4} \sin\alpha / B_0^{1/2}. \qquad (32)$$

An amplitude variation of acoustic waves (eventually those generated by coupling) would become apparent only from a consideration of u_2 and w_2 in the coupling term.

6. COUPLING IN A STEEP GRADIENT

In a steep gradient of the propagation conditions, coupling is strong and a fairly simple treatment of coupling is possible. The gradient is considered steep when the variation is remarkable over a distance much shorter than a wavelength (strictly speaking, than the shorter of the two occurring wavelengths). The steep gradient may be the transition between two homogeneous (or almost homogeneous) media. The field shows clearly in this case the nature of various superimposed waves outside the steep gradient. The amplitudes of the individual waves vary in passing through the steep gradient. This is in the present notion the effect of coupling.

The gradient may be arbitrarily steep. It becomes in the limiting case of infinite steepness a discontinuous transition or boundary between media. The gradual variation of the waves in passing through the arbitrarily steep gradient is described by the above formulations. The formulations are, however, not needed when the behavior of the waves within the gradient is not of interest and when the gradient is sufficiently steep. A boundary may then be assumed in its place. The variation of the waves in crossing the boundary is easily derivable from the known continuity conditions for field and plasma quantities. This method will be used in Sec. 7. First the wave equation, Eq. (3), will be simplified for steep gradient and it will be shown how the simplified form leads over to the condition of continuity at a boundary.

The term $(i\omega/c)\mathbf{N}\cdot\mathbf{U}$ in Eq. (3), referring to the oscillatory variation of $\mathbf{U}$ within a wavelength, may be ignored in a steep gradient as negligible compared to the coupling term, which contains the spatial derivatives of the parameters characterizing the medium. Equation (3), with this neglect and with insertion of $\mathbf{M}$ from Eq. (25), becomes

$$\partial\mathbf{U}/\partial z = -\mathbf{T}^{-1}\cdot(\partial\mathbf{T}/\partial z)\cdot\mathbf{U}. \qquad (33)$$

Multiplication by $\mathbf{T}$ from the left and merging the two terms yield

$$\frac{\partial}{\partial z}(\mathbf{T}\cdot\mathbf{U}) = 0. \qquad (34)$$

It should be noticed that $\mathbf{T}\cdot\mathbf{U}$ according to Eq. (20) is the field vector. Eq. (34) thus is seen to postulate constancy of the field vector $\mathbf{F}$ in the steep gradient. The components of the field vector, which are to stay constant, are (with omission of constant factors) $\rho_0 v_z$, $a^2\rho$, E_y, and B_x. They are known to be the quantities of which continuity is required at a *boundary*. The present postulate of constancy apparently becomes the continuity condition in the limiting case. This justifies neglecting a narrow transitional region between homogeneous media and replacing it by a boundary as long as the interior of the transitional region is not to be investigated.

7. TWO ADJACENT MEDIA

In order to arrive at a better idea of the coupling phenomenon, one might want to study a relatively simple case more thoroughly. Two homogeneous media separated by a narrow transitional region turned out to be a fairly simple case. It was seen in the last section that the transitional region can be disregarded and that the postulate of continuity of $\mathbf{F}$ is applicable to the virtual boundary between the media.

Because the boundary is now a substitute for the transitional region, a discontinuity or divergence of the static magnetic field at the boundary is permissible. The ignored transitional region has to provide for the continuity of magnetic force lines. In general this requires in the transitional region a variation of $\mathbf{B}_0$ also in a transverse direction.

The behavior of magnetohydrodynamic waves at a boundary was investigated by various authors.[19–22] (More references may be found in those cited.) Fejer[22] was concerned with moving plasmas; Simon[20] and Williams[21] dealt with coupling between modified Alfvén waves and acoustic waves. The present section offers a more quantitative study under specializing assumptions (as, for example, that of normal incidence of waves) and allows for a discontinuity of the static magnetic field, which is meaningful in consideration of a narrow transitional region between media. The only restrictive assumption imposed on the magnetic field is the one introduced earlier, that is, $\mathbf{B}_0$ stays in the x,z plane (with the z direction being normal to the boundary).

Superscripts at the various symbols may denote the two media. It is assumed that in medium 1 a modified Alfvén wave, corresponding to a wave function $u_1^{(1)}$, is propagated and impinges on the boundary (or transitional region). Modified Alfvén waves and acoustic waves are found to depart from the boundary toward the two sides. Reflected waves of both types appear in medium 1. In medium 2 the waves continue traveling in the direction of incidence. The continuity condition for $\mathbf{F}$ at the boundary determines the amplitudes of all waves departing from the boundary.

Equations (20) and (21) with parameters λ and μ of different orders of magnitude [as indicated by inequality (17)] suggest neglecting the contributions of modified Alfvén waves to $a^2\rho/c$ (the second component of $\mathbf{F}$) and of acoustic waves to E_y and B_x (essentially the third and fourth components of $\mathbf{F}$), provided that none of these two modes is strongly predominant. This simplifies greatly the computation of wave functions from the continuity condition.

The longitudinal (z) component of the plasma flux density of modified Alfvén waves in medium 1 $u_1^{(1)}+w_1^{(1)}$ is assumed to be given. It is left open how this flux density is subdivided into the two parts corresponding to incident and reflected waves. The ratio between the two parts can be determined afterwards. The electric field strength in medium 1 is derived from Eqs. (20) and (21) as being

$$E_y^{(1)}=-\frac{1}{\epsilon_0}\frac{1}{\mu^{(1)2}B_0^{(1)}\sin\alpha^{(1)}}(u_1^{(1)}+w_1^{(1)}). \quad (35)$$

The first question arising is: *Under what condition do no acoustic waves originate at the boundary?* Continuity of $\rho_0 v_z$ without the presence of acoustic waves requires that at the boundary

$$u_1^{(1)}+w_1^{(1)}=u_1^{(2)}. \quad (36)$$

Continuity of E_y leads [according to Eqs. (20) and (21)] to

$$\frac{u_1^{(1)}+w_1^{(1)}}{\mu^{(1)2}B_0^{(1)}\sin\alpha^{(1)}}=\frac{u_1^{(2)}}{\mu^{(2)2}B_0^{(2)}\sin\alpha^{(2)}}. \quad (37)$$

This equation becomes by means of insertion of μ from Eq. (16)

$$(B_0^{(1)}/\rho_0^{(1)}\sin\alpha^{(1)})(u_1^{(1)}+w_1^{(1)}) = (B_0^{(2)}/\rho_0^{(2)}\sin\alpha^{(2)})u_1^{(2)}. \quad (38)$$

The condition for the two media following from Eqs. (36) and (38) is

$$(\rho_0^{(1)}/B_0^{(1)})\sin\alpha^{(1)}=(\rho_0^{(2)}/B_0^{(2)})\sin\alpha^{(2)}. \quad (39)$$

Equation (39) shows that $(\rho_0/B_0)\sin\alpha$ must remain constant in transition from one medium to the other when no acoustic waves are to occur. For a continuously varying medium the same condition was obtained [in connection with Eqs. (29) and (30)]. In the case of nonvarying α, Eq. (39) simply requires proportionality between ρ_0 and B_0.

It may be noted without proof that Eq. (39) is in agreement with the idea of motion of magnetic force lines together with the plasma, an idea commonly adopted under conditions of (nearly) perfect conduction. It has to be kept in mind that the plasma flux of modified Alfvén waves is normal to the force lines, thus carrying force lines with it, while the flux of acoustic waves would be parallel to the force lines.

In the general case, in which Eq. (39) is not fulfilled, acoustic waves are expected to arise. This case will be discussed now. For reason of simplification, the sound velocity a and the angle α are taken to be constant. Equation (38), following from the continuity of E_y, remains valid and determines $u_1^{(2)}$ for given $u_1^{(1)}+w_1^{(1)}$. The continuity condition for $\rho_0 v_z$, which now has to include modified Alfvén waves and acoustic waves, reads

$$u_1^{(1)}+w_1^{(1)}+w_2^{(1)}=u_1^{(2)}+u_2^{(2)}. \quad (40)$$

From continuity of $a^2\rho$ it follows that

$$-w_2^{(1)}=u_2^{(1)}. \quad (41)$$

Equation (40) with elimination of $w_2^{(1)}$ becomes

$$u_1^{(1)}+w_1^{(1)}=u_1^{(2)}+2u_2^{(2)}. \quad (42)$$

The wave function $u_2^{(2)}$ can be computed from this equation. The ratio of $w_1^{(1)}$ to $u_1^{(1)}$, which is not investigated now, is determined by the continuity condition for B_x.

The preceding deductions [Eqs. (38) and (40)–(42)]

[19] V. C. A. Ferraro, Astrophys. J. **119**, 393 (1954).
[20] R. Simon, Astrophys. J. **128**, 392 (1958).
[21] W. E. Williams, Astrophys. J. **131**, 438 (1960).
[22] J. A. Fejer, Phys. Fluids **6**, 508 (1963).

can be summarized as follows: The primary plasma flux in the first medium (corresponding to $u_1^{(1)}+w_1^{(1)}$) is connected with an electric field E_y. This electric field extends into the second medium and causes there a plasma flux corresponding to $u_1^{(2)}$. If ρ_0/B_0 varies in transition to the second medium, the two plasma fluxes would violate the continuity condition for $\rho_0 v_z$. Acoustic waves departing from the boundary are necessary to reestablish continuity of the longitudinal flux density.

With $\rho_0^{(2)}/B_0^{(2)}>\rho_0^{(1)}/B_0^{(1)}$, the flux density $u_2^{(2)}$ is negative for positive $u_1^{(1)}+w_1^{(1)}$ and the flux density $w_2^{(1)}$ is positive. This is to say, the modified Alfvén waves in the first medium, by means of their electric field, set too much plasma in motion in the second medium; $u_1^{(2)}$ is too large for continuity of the plasma flux densities of modified Alfvén waves. The acoustic waves which arise, exhibiting a negative $u_2^{(2)}$ and a positive $w_2^{(1)}$, serve to keep the flux densities in balance.

If the two adjacent media differ a great deal, the modified Alfvén waves do not have to be intense to produce acoustic waves with a considerable density oscillation. It may be assumed, for example, that

$$\rho_0^{(2)}\gg\rho_0^{(1)} \tag{43}$$

and

$$B_0^{(2)}=B_0^{(1)}. \tag{44}$$

The longitudinal component of the plasma velocity initially assumed in the first medium may be v_{z0}. This velocity is ascribed to the combination of the two wave functions $u_1^{(1)}$ and $w_1^{(1)}$. There is, consequently,

$$u_1^{(1)}+w_1^{(1)}=\rho_0^{(1)}v_{z0} \tag{45}$$

and, according to Eq. (38),

$$u_1^{(2)}=\rho_0^{(2)}v_{z0}. \tag{46}$$

On the other hand, Eq. (38) with inequality (43) yields

$$u_1^{(1)}+w_1^{(1)}\ll u_1^{(2)}. \tag{47}$$

Equation (42) with neglect of $u_1^{(1)}+w_1^{(1)}$ leads to

$$u_2^{(2)}=-\tfrac{1}{2}u_1^{(2)} \tag{48}$$

or

$$u_2^{(2)}=-\tfrac{1}{2}\rho_0^{(2)}v_{z0}. \tag{49}$$

The acoustic wave in the second medium thus is seen to be connected with a longitudinal plasma velocity component that amounts to $-v_{z0}/2$. The corresponding density variation is derived from Eq. (21) as

$$\rho^{(2)}=\frac{c}{a^2}\frac{1}{\lambda|\cos\alpha|}u_2^{(2)} = -\frac{1}{2|\cos\alpha|}\frac{v_{z0}}{a}\rho_0^{(2)}. \tag{50}$$

The relative density variation $\rho^{(2)}/\rho_0^{(2)}$ according to Eq. (50) is comparable to v_{z0}/a. When the ionacoustic velocity a is small, as inequality (17) postulates, a small flux velocity v_{z0} will be sufficient to cause a large relative density variation. From Eq. (41) it is seen that in medium 1 the longitudinal plasma velocity component of the acoustic wave is even greater than in medium 2.

A numerical example may illustrate the present result. It is assumed that the Alfvén velocity in the first medium is 500 km/sec, corresponding approximately to atmospheric conditions in altitudes above $F2$ maximum,[7] and that the ionacoustic velocity is 1 km/sec in both media, furthermore that in the first medium a modified Alfvén wave is incident whose plasma velocity is 200 m/sec normal to the force lines. (Values given for oscillatory quantities may represent amplitudes.) The modified Alfvén wave is "weak"; the plasma velocity in the wave is only 1/2500 of the wave velocity. According to Eq. (21) the magnetic field of the wave amounts to $B_0/2500$ if the wave is a progressive wave. The vertical component of the plasma velocity is $v_{z0}=(200\text{ m/sec})\sin\alpha$. Incidence of the modified Alfvén wave on the boundary at which ρ_0 is supposed to increase considerably causes generation of acoustic waves, which proceed into the two media. The vertical plasma velocity component of the acoustic wave in the second medium obtained from Eq. (49) is $-v_{z0}/2=-(100\text{ m/sec})\sin\alpha$. This value indicates a fairly intense acoustic wave; the ratio between plasma velocity (now taken parallel to the force lines) and wave velocity is $(1/10)|\tan\alpha|$ and Eq. (50) shows that this is also the ratio $|\rho/\rho_0|$.

So far, the lateral extension of the wave fields has been of no concern. A lateral confinement that is not too narrow in terms of wavelengths is representative of a beam of waves or a ray. The wave field is concentrated around the ray, whose direction may deviate from the direction of wave normals (z direction). A theorem of geometric optics states that the ray direction is normal to the surface representing in polar coordinates the refractive index versus direction of the wave normal.[1,9,14,23] The refractive index of modified Alfvén waves does not depend on the direction. Their ray direction consequently coincides with the direction of the wave normal. The ray direction of acoustic waves, on the other hand, is under the present assumptions always parallel to the magnetic force lines. When modified Alfvén waves and acoustic waves depart from the same location on a boundary between media, their different ray directions will in general lead to a separation on the subsequent path of propagation.

8. CONCLUSION

Passage of modified Alfvén waves through a region of varying ρ_0/B_0 was seen to cause generation of acoustic waves ("ion-acoustic" waves, which are propagated in the plasma). This phenomenon appeared as a result of coupling between waves and was derived (in Sec. 5)

[23] H. Poeverlein, Phys. Rev. 128, 956 (1962).

from the coupling theory developed in Sec. 2 to 4. The case chosen as prototype (in Sec. 7), a narrow transitional region between media, was treated without recourse to the formulas of general coupling theory by means of continuity considerations, which strictly apply to a boundary between media. The coupling theory is usable for quite general stratifications—for slow and rapid transitions and for nonmonotonic variations of the medium—but some limitations are imposed by the specializations made from the beginning.

The various formulations seem to indicate two different ways of dealing with an oscillatory field (including the oscillatory quantities of the plasma). There is the obvious way of direct use of the basic equations, to which field and plasma variables are subject. The alternative way, suggested by the coupling theory, requires splitting of the oscillatory quantities into contributions of individual waves, whose behavior is described by the coupling theory. The practicability of this splitting into individual waves is quite evident in a homogeneous or almost homogeneous medium, because there the characteristics of individual waves are easily recognized.

In an arbitrary stratification, particularly in a steep gradient of parameters, the field (and the plasma quantities) will not show periodicities or interferences corresponding to a superposition of wave fields. A physical meaning of the individual waves derives however in such conditions from the behavior of the plasma flux in the waves. Under the assumptions made, each of the three waves is characterized by a particular direction of plasma flux. The flux was found to be parallel to $\mathbf{B}_0$ in acoustic waves, normal to $\mathbf{B}_0$ and in the x,z plane in modified Alfvén waves, and it is in the y direction in Alfvén waves.

Although the typical nature of waves does not become apparent in a rapidly varying medium, the characteristics of the individual waves will show up when the waves leave the region of rapid variation (i.e., the "steep gradient"); different ray directions of different waves, for example, may be observable there (cf. the last paragraph of Sec. 7). This too justifies usage of the wave concept in general cases with the implication of coupling in an inhomogeneous medium.

Only coupling of modified Alfvén waves into acoustic waves was investigated in the more specific part of the paper (Secs. 5 and 7). One might expect also reversed coupling, coupling from acoustic waves into modified Alfvén waves (described by the coupling coefficients M_{13}, M_{23}, M_{14}, and M_{24}). Occurrence of two-way coupling would complicate a case in which both waves show comparable amplitudes. When modified Alfvén waves and acoustic waves have comparable plasma fluxes, it is easily seen, however, that the energy flux is much greater in the modified Alfvén wave. This may prevent a noticeable degree of reversed coupling under a wide variety of conditions. The energy flux in any one of the waves is energy density times wave velocity; the energy density is twice the kinetic energy density. The energy fluxes with comparable plasma velocities thus are roughly in the ratio of the wave velocities. Despite its small energy flux the acoustic wave (possibly produced by coupling) may, however, appear intense with respect to the amplitude of some oscillatory quantity. Large density variation in the acoustic wave was found in the preceding section.

The coupling theory of this paper might permit conclusions on processes in the upper atmosphere, although some of the simplifications introduced may not appear quite realistic in this application. A phenomenon to be recalled is the first phase of a geomagnetic storm. It is commonly explained by an increase of the pressure of the arriving solar plasma wind with subsequent magnetohydrodynamic processes in the terrestrial magnetosphere.[24,25] These processes, though not periodic, have largely the character of modified Alfvén waves. The present theory suggests that "acoustic" processes originate as a secondary effect in altitudes in which the plasma density decreases rapidly with height. Acoustic processes or waves are modified by the gravitational force, but should still entail considerable plasma density variations. Varying vertical extension of the magnetosphere above an individual location on earth, caused by varying position with respect to the sun during the course of a day, might also lead to processes of the character of modified Alfvén waves and secondarily to acoustic processes modified by gravity. Atmospheric phenomena will be discussed under the present aspects in a paper to be published shortly.

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[25] A. J. Dessler and E. N. Parker, J. Geophys. Rev. **64**, 2239 (1959).