D-State and Pickup Effects in Nucleon-Deuteron Scattering in the Impulse Approximation*

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The effects of the *D* state of the deuteron wave function and of the pickup scattering on elastic nucleon-deuteron scattering are investigated within the framework of the impulse approximation and with the neglect of the multiple scattering. A detailed examination is made of the various terms which arise from the identity of the incident nucleon with the target nucleons, and of these only the so-called pickup term is retained in the calculation. The latter term is treated only in Born approximation; the validity of this approximation is discussed. The computations which are carried out at 40 and 150 MeV indicate that the effects of the *D* state and of the pickup scattering are quite large for scattering in the backward hemisphere.

I. INTRODUCTION

QUITE recently, several works have appeared which deal with the consistent mathematical formulation of the integral equations for three-body scattering problems as well as some schemes for solving these equations in an approximate fashion.[1] It seems evident that further work along these lines will permit calculations of moderately high-energy[2–6] *N* (nucleon)-*d* (deuteron) scattering, in particular, which are superior to those existing at the present time.[6–16]

The standard[17,18] impulse-approximation (IA) calculations have yielded a fair description of *N*-*d* differential cross sections and polarizations for small scattering angles in the energy range from 40 to 150 MeV.[6–15] Those quantitative discrepancies which do exist in this range of angles and energies perhaps can be largely accounted for by a somewhat more refined evaluation of the single-scattering integrals as well as by the use of more accurate deuteron wave functions.

However, for large scattering angles and in the same energy range all existing calculations of elastic *N*-*d* scattering are at best only in qualitative agreement with the observed cross sections and polarizations and at worst in complete disagreement with experiment.[19] In view of this, it would seem that the need for more refined calculational techniques such as those alluded to above is now manifest.

Nevertheless, it cannot be said that the entire content of the simple IA approach to *N*-*d* scattering with the neglect of multiple scattering has been fully exploited. We have in mind here the fact that the combined contributions of the (so-called) off-the-energy-shell effects, the pickup scattering, and the *D* state of the deuteron have never been properly examined. The object of the present investigation is to estimate just how the predictions for the cross sections and polarizations, particularly at large scattering angles, are modified when these effects are considered simultaneously. It is quite conceivable that such estimates will be useful in constructing approximation procedures within the context of more sophisticated formalisms.

II. IMPULSE AND SINGLE-SCATTERING APPROXIMATIONS

If we employ the (IA) and neglect the multiple scattering the transition operator T for *N*-*d* scattering can be written as the sum[20]

$$T=T_d+T_p+T_e, \tag{2.1}$$

in which

$$T_d\equiv \sum_{\alpha} t(\alpha), \tag{2.2a}$$

$$T_p\equiv -\tfrac{1}{2}\sum_{\alpha} t(\alpha) \sum_{\beta\neq\alpha} P_{0\beta}, \tag{2.2b}$$

$$T_e\equiv -\tfrac{1}{2}\sum_{\alpha,\beta,\gamma} t(\alpha)(E-H_0+i\epsilon')^{-1}[P_{0\beta},V_{0\gamma}], \tag{2.2c}$$

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; S. Weinberg, Phys. Rev. **133**, B232 (1964); L. Rosenberg, *ibid.* **135**, B715 (1964); R. D. Amado, *ibid.* **132**, 485 (1963); C. Lovelace, *ibid.* **135**, B1225 (1964). The last-named work contains a rather extensive bibliography.

[2] Throughout this work we will be concerned only with elastic nucleon-deuteron scattering for incident nucleon energies high enough to be considered within the normal range of validity of the impulse approximation (IA) in its usual form (cf. Refs. 3–6).

[3] G. F. Chew and G. C. Wick, Phys. Rev. **85**, 636 (1952).

[4] J. Ashkin and G. C. Wick, Phys. Rev. **85**, 686 (1952).

[5] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

[6] K. L. Kowalski and D. Feldman, Phys. Rev. **130**, 276 (1963).

[7] M. Verde, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 142.

[8] *Proceedings of the Conference on Nuclear Forces and the Few-Nucleon Problem, London, 1959*, edited by T. C. Griffith and E. A. Power (Pergamon Press, Inc., London, 1960), Vol. 1.

[9] T. Fulton and P. Schwed, Phys. Rev. **115**, 973 (1959).

[10] L. Favella and M. Olivetti, Nuovo Cimento **11**, 679 (1959).

[11] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N. Y.) **8**, 551 (1959).

[12] J. Sawicki and S. Watanabe, Nucl. Phys. **10**, 151 (1959).

[13] Y. Sakamoto and T. Sasakawa, Progr. Theoret. Phys. (Kyoto) **21**, 879 (1954).

[14] H. Postma and R. Wilson, Phys. Rev. **121**, 1229 (1961).

[15] L. Castillejo and L. S. Singh, Nuovo Cimento **11**, 131 (1959). See also Ref. 8, p. 193.

[16] N. M. Queen, Nucl. Phys. **55**, 177 (1964).

[17] By "standard" we mean those calculations which neglect the multiple scattering. Queen (Ref. 16) has carried out an approximate partial summation of the Watson (Ref. 18) multiple-scattering expansion with some improvement over the single-scattering approximations.

[18] K. M. Watson, Phys. Rev. **89**, 575 (1953).

[19] H. E. Conzett, H. S. Goldberg, E. Shield, R. J. Slobodrian, and S. Yamabe, Phys. Letters, **11**, 68 (1964).

[20] K. L. Kowalski, Nuovo Cimento **30**, 266 (1963).

represent the contributions of the direct, the pickup, and the target-exchange[20,21] scattering processes, respectively. Here $t(\alpha)$ is the two-body transition operator which satisfies

$$t(\alpha)=V_s(\alpha)+\tfrac{1}{2}V_s(\alpha)(E_{0\alpha}-K_0-K_\alpha+i\epsilon')^{-1}t(\alpha)\,, \quad (2.3)$$

with $E_{0\alpha}$ a two-body energy which will be specified later. Also,

$$V_s(\alpha)=V_{0\alpha}(1-P_{0\alpha})$$

and

$$H_0=K_0+H_n\,,$$

where $V_{0\alpha}(=V_{\alpha 0})$ represents the interaction between the incident projectile (0) and the target nucleon α ($\alpha=1$, 2), H_n is the deuteron Hamiltonian, and K_0, K_α are kinetic-energy operators for the indicated particles. The exchange operator for all the variables of particles 0 and α is denoted by $P_{0\alpha}$. Finally, we remark that in the formulation of Eqs. (2.1) and (2.2) an isotopic spin convention has been employed so that all nucleons are regarded as identical.[20]

The physical transition probability amplitudes are obtained from the matrix elements of T with respect to the unperturbed states $|\phi\rangle$ which satisfy

$$H_0|\phi\rangle=E|\phi\rangle\,,$$

where

$$E=E_0+\epsilon\,,$$

E_0 is the energy of the incident particle, and ϵ is the energy of the target including its c.m. motion. The particular value of E which occurs in Eq. (2.2c) corresponds to the initial values of E_0 and ϵ. The states $|\phi\rangle$ are presumed to be completely antisymmetrized with respect to all the target-nucleon variables.

III. KINEMATIC AND ISOSPIN STRUCTURE

In this section we will study the kinematic and isospin structure of the matrix elements $\langle\phi_f|T_A|\phi_i\rangle$, where $A=d$, p, e for the specific case of elastic nucleon-deuteron scattering.[22] Parts of this material are readily available in the literature[6,16] but are reproduced for the sake of continuity as well as to indicate clearly the particular features and limitations of the present calculation.

The initial (final) states $|\phi_i\rangle$ ($|\phi_f\rangle$) of the nucleon and the deuteron (in its ground state) will be written as

$$|\phi_{i,f}\rangle=|\mathbf{k}_{i,f},\mathbf{P}_{i,f},\zeta_{i,f}\rangle|\chi_{i,f}\rangle\,.$$

Here $\mathbf{k}_i$ ($\mathbf{k}_f$) and $\mathbf{P}_i$ ($\mathbf{P}_f$) are the initial (final) wave vectors of the projectile and the c.m. of the deuteron, respectively. The index ζ refers to a three-nucleon ordinary-spin state which is symmetric in the deuteron variables and $|\chi_i\rangle=|\chi_f\rangle$ represents the isospin state of the free nucleon and the isosinglet deuteron. We will adhere throughout this paper to the normalization conventions of Ref. 6.

It will prove convenient to express $V_{0\alpha}$ explicitly in terms of the isosinglet $V^{(0)}(\alpha)$ and the isotriplet $V^{(1)}(\alpha)$ parts of the nucleon-nucleon potential:

$$V_{0\alpha}=V^{(0)}(\alpha)P_0(\alpha)+V^{(1)}(\alpha)P_1(\alpha)\,.$$

The projection operators on the isosinglet ($\tau=0$) and the isotriplet ($\tau=1$) states of particles 0 and α are denoted by $P_\tau(\alpha)$.

Similarly, $t(\alpha)$ can be written in terms of the two-nucleon transition operators $t^{(\tau)}(\alpha)$ which correspond to scattering in each of the two isospin states:

$$t(\alpha)=t^{(0)}(\alpha)P_0(\alpha)+t^{(1)}(\alpha)P_1(\alpha)\,. \quad (3.1)$$

The operators $t^{(\tau)}(\alpha)$ satisfy Eq. (2.3) with $V_s(\alpha)$ replaced by $V_s^{(\tau)}(\alpha)$ which is defined by

$$V_s^{(\tau)}(\alpha)=V^{(\tau)}(\alpha)[1+(-1)^\tau\mathcal{P}_{0\alpha}]\,,$$

where $\mathcal{P}_{0\alpha}$ is the exchange operator in all the variables of particles 0 and α excepting isospin.

A. Direct Terms

The direct amplitude is given by[23]

$$\langle\phi_f|T_d|\phi_i\rangle=\tfrac{1}{2}\langle\mathbf{k}_f,\mathbf{P}_f,\zeta_f|3t^{(1)}(\alpha)+t^{(0)}(\alpha)|\mathbf{k}_i,\mathbf{P}_i,\zeta_i\rangle\,. \quad (3.2)$$

The matrix elements of $t^{(\tau)}(\alpha)$ are[6]

$$\langle\mathbf{k}_f,\mathbf{P}_f,\zeta_f|t^{(\tau)}|\mathbf{k}_i,\mathbf{P}_i,\zeta_i\rangle =\delta(\mathbf{K}_f-\mathbf{K}_i)\langle\zeta_f|t_d^{(\tau)}(\mathbf{k}_f',\mathbf{k}_i')|\zeta_i\rangle\,, \quad (3.3)$$

with

$$t_d^{(\tau)}(\mathbf{k}_f',\mathbf{k}_i')=\int d\mathbf{q}\phi^\dagger(\mathbf{q}-\boldsymbol{\kappa})\langle\tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{2}\boldsymbol{\kappa} -\tfrac{1}{2}\mathbf{q}|t_c^{(\tau)}|\tfrac{3}{4}\mathbf{k}_i'-\tfrac{1}{2}\mathbf{q}\rangle\phi(\mathbf{q})\,, \quad (3.4)$$

and

$$\mathbf{K}=\mathbf{k}+\mathbf{P}\,,$$
$$\mathbf{k}'=\tfrac{2}{3}(\mathbf{k}-\tfrac{1}{2}\mathbf{P})\,,$$
$$\boldsymbol{\kappa}=\tfrac{1}{2}(\mathbf{k}_f'-\mathbf{k}_i')\,.$$

The deuteron ground-state wave function in momentum space $\phi(\mathbf{q})$ and the matrix element of $t_c^{(\tau)}$ which appears in (3.4) are regarded as operators in ordinary spin space. The c.m. motion of the deuteron has been separated out of $\phi(\mathbf{q})$.

The operator $t_c^{(\tau)}(\alpha)$ is defined in terms of $t^{(\tau)}(\alpha)$ by

$$\langle\mathbf{k}_f,\mathbf{k}_f^\alpha|t^{(\tau)}(\alpha)|\mathbf{k}_i,\mathbf{k}_i^\alpha\rangle =\delta(\mathbf{k}_f+\mathbf{k}_f^\alpha-\mathbf{k}_i-\mathbf{k}_i^\alpha)\langle\mathbf{k}_f,\mathbf{k}_f^\alpha|t_c^{(\tau)}(\alpha)|\mathbf{k}_i,\mathbf{k}_i^\alpha\rangle \quad (3.5)$$

which is permissible provided the two-nucleon interaction is translationally invariant.[24] In Eq. (3.5) $\mathbf{k}^\alpha$ denotes the wave vector of the relevant target nucleon.

[21] G. Takeda and K. M. Watson, Phys. Rev. **97**, 1336 (1955).

[22] We will neglect any possible electromagnetic effects. At the energies and angles with which we will be concerned this is certainly a valid approximation.

[23] Equation (3.2) is, of course, independent of the particle index α. When no confusion is likely to arise we will, for simplicity of notation, write $t^{(\tau)}$ instead of $t^{(\tau)}(\alpha)$ and similarly for related operators.

[24] We suppose, in addition, that the interactions are Galilean invariant.

In order to specify the energy $E_{0\alpha}$ in Eq. (2.3), let us suppose that both $\mathbf{k}_i$ and $\mathbf{k}_f$ refer to the laboratory system. Then it is consistent with the introduction of the (IA) to take $E_{0\alpha}$ as E_0, namely, the lab energy of the incident particle.[25,26] With this agreed upon, the integral equation in the space of relative coordinates satisfied by the operator $t_c^{(\tau)}(\alpha)$ appropriate to the matrix element in (3.4) is

$$t_c^{(\tau)}(\alpha)=V_s^{(\tau)}(\alpha)+\tfrac{1}{2}V_s^{(\tau)}(\alpha)\times(E_c-K_{0\alpha}+i\epsilon')^{-1}t_c^{(\tau)}(\alpha)\,, \quad (3.6)$$

where

$$E_c=E_i'-\tfrac{1}{2}(\hbar^2/2M)\mathbf{q}^2\,,$$
$$E_i'=(\hbar^2/2M)(\tfrac{3}{4}\mathbf{k}_i-\tfrac{1}{2}\mathbf{q})^2\,,$$

and $K_{0\alpha}$ is the kinetic-energy operator for the relative motion of particles 0 and α. The reduced two-nucleon mass is denoted by M. The extent of the condensation in the notation used in Eqs. (3.3) and (3.4) should now be evident.

Several methods for evaluating the integral (3.4) were discussed in Ref. 6. We will employ only one of these techniques, namely, we will set $\mathbf{q}=\frac{1}{2}\boldsymbol{\kappa}$ in the matrix element of $t_c^{(\tau)}$ which appears in (3.4) and, in addition, we take $E_c\approx E_i'$.[27] Then the momentum integration in (3.4) involves only the deuteron wave functions and, moreover, the only matrix elements of $t_c^{(\tau)}$ which appear are related to the two-nucleon transition *amplitudes*. This approximation was found[6] to yield results not appreciably different from those obtained with a more refined evaluation of the so-called off-the-energy-shell effects.

B. Pickup Terms

It is clear for the case at hand that

$$\langle\phi_f|T_p|\phi_i\rangle=-\tfrac{1}{2}\langle\mathbf{k}_f,\mathbf{P}_f,\zeta_f|t^{(0)}(\alpha)\mathcal{P}_{0\beta}\times|\mathbf{k}_i,\mathbf{P}_i,\zeta_i\rangle,\ (\alpha\neq\beta).$$

Thus, the simple pickup scattering involves only the isosinglet part of the nucleon-nucleon interaction as one might expect.

One finds that

$$\langle\mathbf{k}_f,\mathbf{P}_f,\zeta_f|t^{(0)}(\alpha)\mathcal{P}_{0\beta}|\mathbf{k}_i,\mathbf{P}_i,\zeta_i\rangle=\delta(\mathbf{K}_f-\mathbf{K}_i)\langle\zeta_f|t_p(\mathbf{k}_f',\mathbf{k}_i')|\zeta_i\rangle\,,$$

with

$$t_p(\mathbf{k}_f',\mathbf{k}_i')=\phi^\dagger(\tfrac{1}{2}\mathbf{k}_f'+\mathbf{k}_i')\int d\mathbf{q}\langle\mathbf{k}_f'+\tfrac{1}{2}\mathbf{k}_i'|t_c^{(0)}(\alpha)\times|\mathbf{q}\rangle\mathcal{S}_{0\beta}\phi(\mathbf{q}),\quad(\alpha\neq\beta),\quad(3.7)$$

where $\mathcal{S}_{0\beta}$ is the ordinary spin-exchange operator. If we adopt the previous definition of the two-body operator $t(\alpha)$ which satisfies (2.3) then one finds that the integral equation satisfied by $t_c^{(0)}(\alpha)$ appropriate to the matrix element in (3.7) is just Eq. (3.6) with E_c replaced by E_0, namely, the incident-particle energy in the lab system.

In contrast to the direct terms, the factorization of the integral (3.7) by somehow extracting some average value of the matrix element of $t_c^{(0)}(\alpha)$ is not possible owing to the divergence of the residual integral. However, the fact that the Green's function in the integral equation for $t_c^{(0)}(\alpha)$ is defined with respect to the energy E_0 rather than to an energy on the order of $\frac{1}{2}E_0$ [cf. Eq. (3.6)] suggests that when E_0 is large a reasonable first approximation to (3.7) can be achieved by employing the Born approximation for $t_c^{(0)}(\alpha)$, viz.,

$$t_c^{(0)}(\alpha)\approx V_s^{(0)}(\alpha)\,.\quad(3.8)$$

Then, if we use the wave equation for the deuteron, Eq. (3.7) becomes simply

$$t_p(\mathbf{k}_f',\mathbf{k}_i')=-2[|\epsilon_d|+(\hbar^2/2M)(\mathbf{k}_f'+\tfrac{1}{2}\mathbf{k}_i')^2]\times\phi^\dagger(\tfrac{1}{2}\mathbf{k}_f'+\mathbf{k}_i')\mathcal{S}_{0\beta}\phi(\mathbf{k}_f'+\tfrac{1}{2}\mathbf{k}_i')\,,\quad(3.9)$$

where $|\epsilon_d|$ is the deuteron binding energy. We have recovered in (3.9) the usual Born-approximation result for the pickup scattering.[7,28]

Naturally, it is of interest to inquire how much an improved approximation to $t_c^{(0)}(\alpha)$ will change the result (3.9). If E_0 is large enough, then in accord with the philosophy which led to (3.8), it seems reasonable to obtain a correction to (3.9) by considering $t_c^{(0)}(\alpha)$ in second Born approximation,

$$t_c^{(0)}(\alpha)\approx V_s^{(0)}(\alpha)+\tfrac{1}{2}V_s^{(0)}(\alpha)\times(E_0-K_{0\beta}+i\epsilon')^{-1}V_s^{(0)}(\alpha)\,,\quad(3.10)$$

rather than (3.8).

In order to estimate the contribution of the second term in (3.10) to (3.7) we employed a simple, pure S-state Hulthén deuteron wave function and an appropriate exponential isosinglet potential.[29] All integrals involved can then be evaluated in closed form. For $E_0=40$ MeV the magnitude of the correction to (3.7) was about 70% of the magnitude of (3.7) with the predominant contribution arising from the imaginary part of the second Born term. At 150 MeV the magnitude of the corrections was about one-third of the magnitude of (3.7). In this case the real and imaginary parts of the corrections were roughly the same.

[25] The choice of the two-body energy in Eq. (2.3) is not uniquely determined in the course of invoking the (IA) at least within the context of the Watson (Ref. 18) form of multiple-scattering theory. The procedure followed here seems reasonable enough for the single-scattering terms. However, owing to the recoil of the target during the intermediate scatterings, the use of E_0 in (2.3) is not free from contradiction. Indeed, in general the proper two-body energy would appear to be the consequence of some sort of self-consistency requirement. (See, for example, Ref. 26.) These considerations merely reflect the well-known lack of a precise characterization of the (IA) in a multiple-scattering theory except under certain additional restrictions.

[26] K. M. Watson, Phys. Rev. **105**, 1388 (1957).

[27] In Ref. 6 this prescription was termed the linear approximation. Strictly speaking, the name does not apply when the D state of the deuteron is included.

[28] G. F. Chew and M. L. Goldberger, Phys. Rev. **77**, 470 (1950).

[29] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), p. 201.

It is well known, of course, that for energies below 100 MeV the Born series is quite unreliable particularly for the isosinglet part of the nucleon-nucleon potential.[30] The preceding results at 40 MeV are therefore not too surprising.[31] The very large differences between the observed and the calculated cross sections[16] obtained using (3.9) for backward scattering, where the pickup term is expected to dominate,[7,28] also tend to indicate that (3.9) is a poor approximation at this energy.

Despite the spurious nature of (3.9) at 40 MeV we will, nevertheless, employ it in our own calculations and thus our results at this energy and large-scattering angles are only of qualitative interest.[32] At 150 MeV the situation is somewhat improved, and in fact there is reason to suspect that the Born series for $t_c^{(0)}$ converges rather rapidly at this energy.[31] The apparent approximate validity of (3.9) at 150 MeV and the lack of it at 40 MeV will be seen to be reflected roughly in our calculations.

The integral (3.7) does not appear to possess any simplifying feature which would lend itself to the sort of analysis carried out in Ref. 6. In particular, any relation between the matrix element of $t_c^{(0)}$ in (3.7) to the nucleon-nucleon scattering amplitudes seems singularly remote. Possibly the most consistent way by which to evaluate (3.7) is by the use of a formalism developed previously specifically for the sort of highly off-shell two-body matrix elements which appear there.[33] One simplifying feature of such a calculation, in contrast to the direct terms,[6] is the circumstance that only those terms in the partial-wave expansion of $t_c^{(0)}$ with total angular momentum J such that $J\leq 3.0$ contribute to (3.7).

C. Target-Exchange Terms

We next consider, briefly, the matrix elements of T_e. Let us state at the outset, however, that we do not intend to calculate these matrix elements in any degree of approximation. Our primary motivation for including this material in this paper arises from the lack of explicit mention of these terms in most works on N-d scattering.

The states $|\phi_{i,f}\rangle$ are to be completely antisymmetrized with respect to the target nucleon variables. Therefore, the sum over intermediate states in Eq. (2.2c) will involve only such target states. With this in mind, one finds using the same techniques which were employed for the direct and pickup terms that

$$\begin{aligned}
\langle\phi_f|T_e|\phi_i\rangle &= \delta(\mathbf{K}_f-\mathbf{K}_i) \\
&\quad\times\sum_{\tau=0}^{1}\sum_{\mathbf{p}}\int d\mathbf{k}_n' t_e^{(\tau)}(\mathbf{k}_f',\mathbf{k}_n';\mathbf{p}) \\
&\quad\times[\phi_{\mathbf{p}}^{(\tau)}(\tfrac{1}{2}\mathbf{k}_n'+\mathbf{k}_i')]^\dagger \mathcal{S}_{0\beta}\phi(\mathbf{k}_n'+\tfrac{1}{2}\mathbf{k}_i'),\\
t_e^{(\tau)}(\mathbf{k}_f',\mathbf{k}_n';\mathbf{p}) &= \int d\mathbf{q}\phi^\dagger(\mathbf{q}-\boldsymbol{\kappa})\langle\tfrac{3}{4}\mathbf{k}_f'+\tfrac{1}{2}\boldsymbol{\kappa}_n-\tfrac{1}{2}\mathbf{q}\\
&\quad\times|t_e^{(\tau)}|\tfrac{3}{4}\mathbf{k}_n'-\tfrac{1}{2}\mathbf{q}\rangle\phi_{\mathbf{p}}^{(\tau)}(\mathbf{q}), \quad (3.11)\\
t_e^{(\tau)} &= \tfrac{1}{4}[3(-1)^\tau t_c^{(1)}+(1+2\tau)t_c^{(0)}],\\
\boldsymbol{\kappa}_n &= \tfrac{1}{2}(\mathbf{k}_f'-\mathbf{k}_n').
\end{aligned}$$

The sum over $\mathbf{p}$ in (3.11) refers to an integration over the intermediate continuum states and a discrete contribution from the deuteron ground state for $\tau=0$. The continuum two-nucleon wave functions in momentum space $\phi_{\mathbf{p}}^{(\tau)}(\mathbf{q})$ correspond to excited states of the target with (relative) energy $(\hbar^2/2M)\mathbf{p}^2$ and an isospin state τ.

The fact that the energy denominator in (2.2c) has disappeared upon evaluating the matrix element is a result peculiar to the deuteron target. A similar remark applies to the form of the pickup terms in Born approximation.

Physically, one may think of (3.11) as representing the distortion of the initial state of a free nucleon and the deuteron target due to identity effects.[34] This term then gives rise to the so-called "polarization" effects which are important in N-d scattering at low energies.[7]

There exist quite plausible physical reasons[20,21] to expect the target-exchange scattering to be negligible at energies for which the (IA) is expected to be valid, and these constitute our justification for not considering (3.11) in our calculations. Nonetheless, it is not at all obvious from Eq. (3.11) when this term is small compared to the direct and pickup matrix elements.[35,36] Clearly the difficulty in obtaining such an estimate arises from the sum over the continuum two-nucleon states. It would appear that some more quantitative investigation of the target-exchange scattering is necessary before including such supposed higher order effects as multiple scattering particularly at moderate energies and large-scattering angles.

IV. SPIN STRUCTURE

We will now address ourselves to the problem of evaluating the matrix elements of the operators (3.4) and (3.9) with respect to the three-nucleon spin states $|\zeta\rangle$. The target-exchange terms (3.11) will not be considered.

[30] R. Jost and A. Pais, Phys. Rev. 82, 840 (1951). The arguments of R. Aaron, R. D. Amado, and B. W. Lee, *ibid.* 121, 319 (1961), are also certainly relevant here.

[31] We recall, however, that owing to the peculiar structure of the pickup transition term, 40 MeV for the N-d problem corresponds to a two-body lab energy of 80 MeV. Similarly, 150 MeV for N-d scattering will correspond to 300 MeV for the two-body case.

[32] The same remark applies to the calculation of Ref. 16.

[33] K. L. Kowalski and D. Feldman, J. Math. Phys. 4, 507 (1963).

[34] Compare the interpretation given in Ref. 21.

[35] It should be pointed out that the rough estimates of Refs. 21 and 36 are not relevant to Eq. (3.11).

[36] J. Sawicki, Nuovo Cimento 15, 606 (1960).

The deuteron wave function $\psi(\mathbf{r})$ in coordinate space is related to $\phi(\mathbf{q})$ by

$$\phi(\mathbf{q})=\frac{1}{(2\pi)^{3/2}}\int d\mathbf{r}\exp(-i\mathbf{q}\cdot\mathbf{r})\psi(\mathbf{r}),$$

where $\mathbf{r}$ is the relative position vector. Here $\psi(\mathbf{r})$ is also an operator in the two-nucleon spin space and can be written in the form[37]

$$\psi(\mathbf{r})=[1/(4\pi)^{1/2}r][u(r)+\tfrac{1}{4}\sqrt{2}S_{12}w(r)],$$

with S_{12} denoting the ordinary tensor operator. The (scalar) functions $u(r)$ and $w(r)$ are the S- and D-state radial wave functions, respectively, which are normalized such that

$$\int_0^\infty dr[u^2(r)+w^2(r)]=1.$$

In our detailed calculations we will use the forms for $u(r)$ and $w(r)$ derived from the Yale potential.[38,39]

It will be convenient to write the states $|\zeta\rangle$ as

$$|\zeta\rangle=\sum_{\mu,s}C(\zeta;\mu,s)|\mu(1,2)\rangle|s(0)\rangle, \tag{4.1}$$

where $|\mu\rangle$ denotes a two-nucleon spin state and $|s\rangle$ is a single-nucleon spin state. We recall that $|\zeta\rangle$ is symmetric in the indices 1, 2. Then

$$\phi(\mathbf{q})|\zeta\rangle=\Sigma f_{\zeta,\nu}(\mu,\sigma,s')\phi_{\nu,\mu}(\mathbf{q})|\sigma(0,\alpha)\rangle|s'(\beta)\rangle,\ \alpha\neq\beta \tag{4.2}$$

with the sums on the right-hand side understood to be over repeated indices. The matrix elements of ϕ are defined by

$$\phi_{\nu,\mu}(\mathbf{q})\equiv\langle\nu(1,2)|\phi(\mathbf{q})|\mu(1,2)\rangle. \tag{4.3}$$

The (real) quantities C and f in Eqs. (4.1) and (4.2), respectively, can be expressed in terms of the Clebsch-Gordan coefficients in an elementary manner.[40,41]

One finds, using (4.2) and the "linear approximation" described in Sec. (III-A), that

$$\begin{aligned}\langle\zeta_f|t_d^{(\tau)}(\mathbf{k}_f',\mathbf{k}_i')|\zeta_i\rangle\\ =\Sigma f_{\zeta_f,\nu_f}(\mu_f,\sigma_f,s_f')f_{\zeta_i,\nu_i}(\mu_i,\sigma_i,s_i')F_\kappa(\nu_f,\mu_f|\nu_i,\mu_i)\\ \times\langle\mathbf{p}_f,\sigma_f|t_c^{(\tau)}|\mathbf{p}_i,\sigma_i\rangle\delta_{s_f',s_i'},\end{aligned} \tag{4.4}$$

where

$$F_\kappa(\nu_f,\mu_f|\nu_i,\mu_i)\equiv\int d\mathbf{q}\,\phi_{\nu_f,\mu_f}^*(\mathbf{q}-\boldsymbol{\kappa})\phi_{\nu_i,\mu_i}(\mathbf{q}), \tag{4.5}$$

and

$$\mathbf{p}_i=\tfrac{3}{4}\mathbf{k}_i-\tfrac{1}{4}\boldsymbol{\kappa},\quad \mathbf{p}_f=\tfrac{3}{4}\mathbf{k}_f+\tfrac{1}{4}\boldsymbol{\kappa}.$$

The functions F_κ are the generalizations of the usual deuteron form-factor integral when the D state is neglected.[6] We note that $|\mathbf{p}_i|=|\mathbf{p}_f|$.

For computational purposes the integrals (4.5) are more convenient when expressed in terms of the coordinate-space wave functions, viz.,

$$F_\kappa(\nu_f,\mu_f|\nu_i,\mu_i)=\int d\mathbf{r}\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})\psi_{\nu_f,\mu_f}^*(\mathbf{r})\psi_{\nu_i,\mu_i}(\mathbf{r}). \tag{4.6}$$

The matrix elements of $\psi(\mathbf{r})$ are defined by Eq. (4.3) with $\phi(\mathbf{q})$ replaced by $\psi(\mathbf{r})$.

The matrix elements of $\psi(\mathbf{r})$ have the general form[37]

$$\psi_{\nu,\mu}(\mathbf{r})=[1/(4\pi)^{1/2}r]\{u(r)\delta_{\nu,\mu}+A_{\nu,\mu}w(r)Y_2^m(\mathbf{r},\mathbf{p}_i)\}. \tag{4.7}$$

The numerical coefficients $A_{\nu,\mu}$ are real and $Y_l^m(\mathbf{r},\mathbf{p}_i)$ denotes the normalized spherical harmonic which is defined with respect to a polar axis in the direction of our axis of spin quantization $\mathbf{p}_i$. In (4.7) we have suppressed the dependence of m on ν and μ.

The angular integrations which are implied in (4.6) after one has employed (4.7) can be carried out in a straightforward manner with the aid of various expressions for the products of spherical harmonics.[41] Then one finds that the overlap integrals (4.6) can be expressed in terms of the five radial integrals

$$I_1(\kappa)\equiv\int_0^\infty dr\ j_0(\kappa r)u^2(r),$$

$$I_2(\kappa)\equiv\int_0^\infty dr\ j_2(\kappa r)u(r)w(r),$$

$$I_3(\kappa)\equiv\int_0^\infty dr\ j_0(\kappa r)w^2(r),$$

$$I_4(\kappa)\equiv\int_0^\infty dr\ j_2(\kappa r)w^2(r),$$

$$I_5(\kappa)\equiv\int_0^\infty dr\ j_4(\kappa r)w^2(r),$$

where j_l denotes the spherical Bessel function of order l. The numerical evaluation of these integrals will be discussed in the next section.

The matrix elements (4.4) can now be expressed entirely in terms of the integrals $I_i(\kappa)$, the spherical harmonics $Y_l^m(\boldsymbol{\kappa},\mathbf{p}_i)$, and the nucleon-nucleon scattering amplitudes. The explicit forms for these matrix elements are given elsewhere.[42,43]

[37] L. Hulthén and M. Sugawara, Ref. 7, p. 1.

[38] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

[39] H. Kottler and K. L. Kowalski, Nucl. Phys. **53**, 334 (1964).

[40] In our calculations we adopted the phase conventions of Ref. 41 for the Clebsch-Gordan coefficients and spherical harmonics.

[41] Appendix A of Ref. 29.

[42] H. Kottler, thesis, Case Institute of Technology (unpublished). Copies of the various unlisted matrix elements referred to in this paper will be supplied upon request.

[43] We might mention that our original motivation for our rather laborious approach to the evaluation of the spin matrix elements and their subsequent summation was to obtain expressions for the amplitudes which would be adaptable to a study of the off-the-energy-shell effects. (Cf. Ref. 6.)

By using procedures similar to those utilized in evaluating the direct terms, we find that in the approximation (3.9)

$$\langle\zeta_f|t_p(\mathbf{k}_f',\mathbf{k}_i')|\zeta_i\rangle = -2[|\epsilon_d|+(\hbar^2/2M)Q^2]\times\Sigma f_{\zeta_f,\nu_f}(\mu_f,\sigma,s_f') \times C(\zeta_i;\mu_i,s_i')\delta_{s_f',s_i'}\phi_{\nu_f,\mu_f}^*(\mathbf{Q}_f)\phi_{\sigma,\mu_i}(\mathbf{Q}_i)\,, \quad (4.8)$$

where

$$\mathbf{Q}_i=\mathbf{k}_f'+\tfrac{1}{2}\mathbf{k}_i',\quad \mathbf{Q}_f=\tfrac{1}{2}\mathbf{k}_f'+\mathbf{k}_i'.$$

We note that

$$|\mathbf{Q}_i|=|\mathbf{Q}_f|\equiv Q.$$

The evaluation of the deuteron wave function factors in (4.8) will entail the computation of the radial integrals

$$J_1(Q)=\int_0^\infty dr\, ru(r)j_0(Qr)\,,$$

$$J_2(Q)=\int_0^\infty dr\, rw(r)j_2(Qr)\,.$$

The details of this computation will be discussed in the next section. The explicit forms of the matrix elements (4.8) in terms of the integrals $J_i(Q)$ and the spherical harmonics $Y_l^m(\mathbf{Q}_i,\mathbf{p}_i)$, $Y_l^m(\mathbf{Q}_f,\mathbf{p}_i)$ are given elsewhere.[42]

V. NUMERICAL DETAILS

Our calculations were carried out at incident nucleon (lab) energies of 40 and 150 MeV for which there exists a reasonable body of experimental information.[14,19]

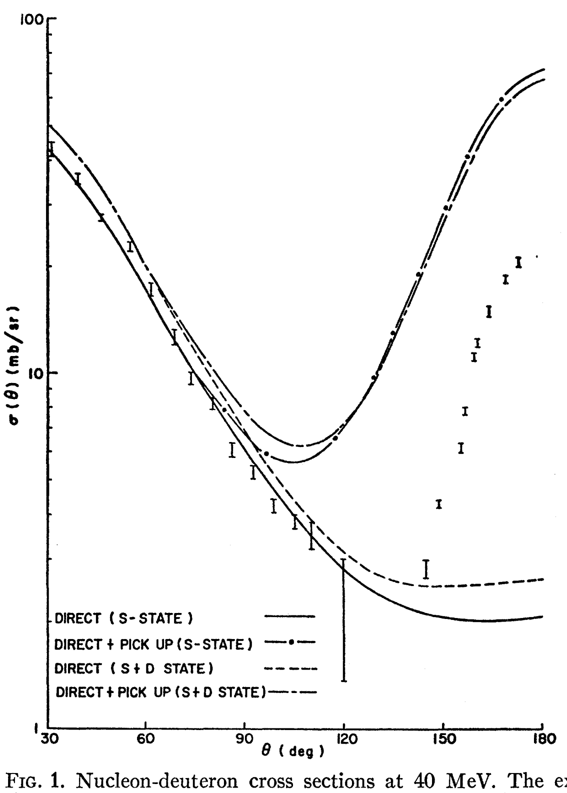

FIG. 1. Nucleon-deuteron cross sections at 40 MeV. The experimental points are taken from J. H. Williams and M. K. Brussel, Phys. Rev. **110**, 136 (1958).

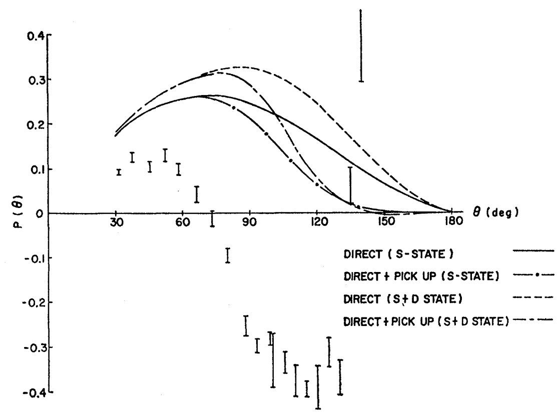

FIG. 2. Nucleon-deuteron polarizations at 40 MeV. The experimental points are taken from Ref. 19.

These two energies are of particular interest in that one (40 MeV) corresponds to a situation where the simple (IA) might begin to fail[3] while the other (150 MeV) is high enough to expect the (IA) to be excellent. We will now summarize the procedures followed in our evaluation of the direct and pickup amplitudes and the consequent cross sections and polarizations.

The two-nucleon amplitudes appearing in the direct terms were calculated using the Yale phase shifts.[44]

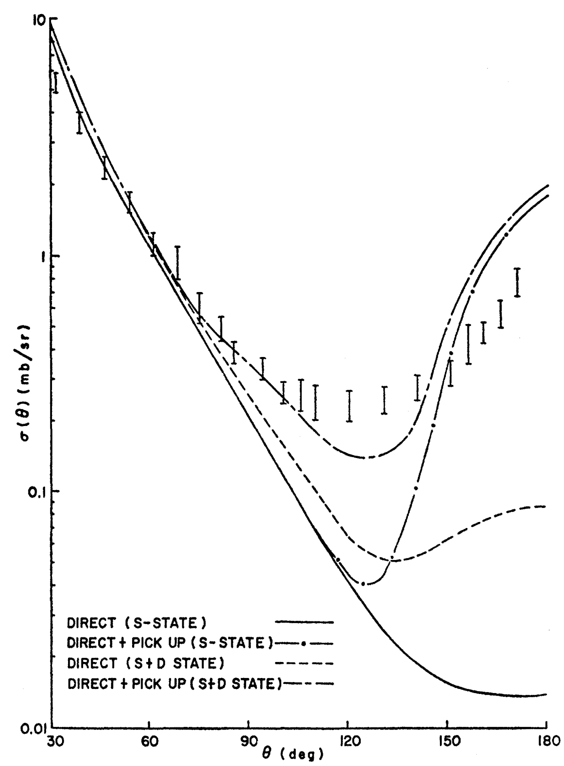

FIG. 3. Nucleon-deuteron cross sections at 150 MeV. The experimental points are taken from Ref. 14.

[44] G. Breit, M. H. Hull, Jr., K. E. Lassila, K. D. Pyatt, Jr., and H. M. Ruppel, Phys. Rev. **128**, 827 (1962). M. H. Hull, Jr., K. E. Lassila, H. M. Ruppel, F. A. Mc Donald, and G. Breit, *ibid.* **128**, 830 (1962).

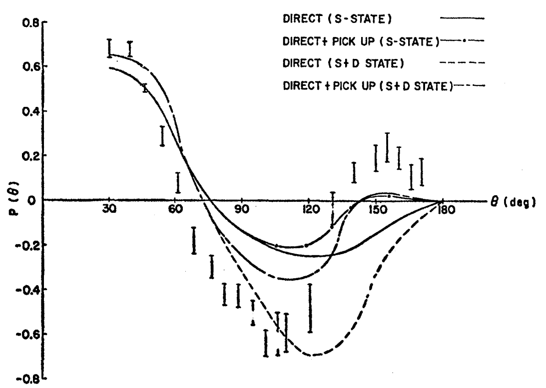

FIG. 4. Nucleon-deuteron polarizations at 150 MeV. The experimental points are taken from Ref. 14.

Since all of these amplitudes were on the energy shell the procedures followed were standard.[45]

All of the radial integrals I_i and J_i were evaluated using the specific deuteron wave function mentioned previously.[39] With this (analytic) form for $u(r)$, the integral J_1 could be integrated in closed form. The integral I_5 did not appear in our final expressions for the amplitudes and therefore was not computed. The remaining integrals were evaluated numerically with the upper limit of integration cut off at a distance of either six (I_2,I_3,I_4) or eight (J_2,I_1) pion Compton wavelengths. The maximum possible errors as a result of the cutoffs were estimated to be no more than a few percent. The largest possible errors were confined to the region of the smallest values of κ and Q considered.

The direct and pickup amplitudes were employed in the usual manner[46] to calculate for several different cases the cross section $\sigma(\theta)$ and the polarization $P(\theta)$, as a function of the three-body c.m. scattering angle θ. The results of these computations are presented in Figs. 1–4.

Four different cases were investigated. First of all, in order to provide a basis of comparison with the standard approximations we calculated $\sigma(\theta)$ and $P(\theta)$ using only the direct terms and the pure S-state deuteron wave function which in this case was normalized to unity for the sake of consistency. Next, the D state was properly incorporated into the direct terms but the pickup scattering was neglected. Then, the pickup scattering was included but the D state was neglected in both the direct and the pickup amplitudes. Finally, the D state was properly accounted for in both the direct and pickup amplitudes.

VI. RESULTS AND DISCUSSION

Let us consider first the various cross sections at 40 MeV which are plotted in Fig. 1. In the angular range $\theta < 75°$ the fit obtained to the data is somewhat of an improvement over most previous calculations carried out in the same degree of approximation.[6,12,15,16] This is most probably due to our use of a more accurate deuteron wave function. It is evident that in this angular range the pickup scattering is entirely negligible. On the other hand, the difference between the results obtained with and without the inclusion of the deuteron D state is small but not insignificant at least for these angles.

The cross sections calculated using only the direct terms are seen to agree rather well with the data up to about 120°. For $\theta > 75°$ this is probably fortuitous. As one expects, the agreement is poor for $\theta > 150°$. The effect of the D state on the cross section when the pickup scattering is neglected is about the same at all angles.

The pickup process dominates the scattering for $\theta > 120°$. In this region the cross sections computed using the combined direct and pickup amplitudes provide a very poor quantitative fit to the data. The contributions of the D state in these cases is insignificant.

An examination of the computed polarizations at 40 MeV (Fig. 2) shows virtually a complete failure of our calculations in any degree of approximation to provide even a qualitative fit over practically the entire angular range under consideration. This great disagreement is to be contrasted with our comparatively good results for $\sigma(\theta)$, particularly for $\theta < 75°$. Of course, the polarization is notoriously sensitive to effects which only slightly alter the cross section. In this regard we note the greater influence of the D state on $P(\theta)$ than on $\sigma(\theta)$.

The principal inadequacy of the polarization predictions is the complete failure to account for the very large negative peak in the observed $P(\theta)$. Although data is available only at a few energies[14,19,47] it seems to be a characteristic feature of N-d scattering even at relatively low energies[47] for $P(\theta)$ to be quite negatively peaked in the neighborhood of $\theta = 100°$. It also appears to be characteristic of the standard (IA) calculations to underestimate this peak.[6,11,14,15]

It should be pointed out that of all the various calculations[6,11,15] of $P(\theta)$ at 40 MeV the prediction of Ref. 11 seems to be in best qualitative accord with experiment in the angular range $30° \leqslant \theta \leqslant 75°$. This calculation appears to be unique in predicting the change of sign in $P(\theta)$ at about 70°.[48,49] Of course, for $\theta > 80°$ the magnitude of $P(\theta)$ is grossly underestimated. It is interesting to point out that the approximation procedures used in Ref. 11 were considerably less refined than those employed here. This inverse correlation between the degree of refinement in calculation

[45] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[46] L. Wolfenstein, Ann. Rev. Nucl. Sci. **6**, 43 (1956).

[47] H. E. Conzett, G. Igo, and W. J. Knox, Phys. Rev. Letters **12**, 222 (1964).

[48] This calculation is also unique in that the Gammel-Thaler (Ref. 49) N-N phase shifts are employed. It is unclear how sensitive our 40-MeV polarization calculations are with respect to any inaccuracies in the assumed N-N phase shifts (Ref. 42).

[49] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291, 1337 (1957).

procedures and the agreement with the 40-MeV polarization data has been observed before.[19]

Next let us examine our results at 150 MeV. Again, we consider first the cross sections (Fig. 3). The overestimation of $\sigma(\theta)$ at $\theta=30°$ is particularly interesting. At this angle and energy it is customarily expected that the simple (IA) should yield an excellent representation of the scattering amplitude. Yet several calculations carried out with varying degrees of refinement[6,11,14,15] and with the use of a variety of N-N data and deuteron wave functions or form factors all seem to yield a somewhat larger cross section than experiment at this particular angle even when the neighboring data are reasonably well approximated.[14] Further experimental data in the region of $\theta=30°$ for both $\sigma(\theta)$ and $P(\theta)$ would be most informative in identifying any theoretical shortcomings.

Excepting the point $\theta=30°$, a fair fit for $\sigma(\theta)$ is obtained with all of the different approximations for $\theta<75°$. For larger values of θ the pickup scattering and the D state both play an important role. Considering the approximations employed, we obtain a rather remarkable fit to $\sigma(\theta)$ over the entire angular range when both the pickup scattering and the D state are included. We note that the cross sections computed without the pickup scattering are lowered relative to the corresponding curves obtained in Ref. 6. Again we attribute this to the use of an improved deuteron wave function over that employed in Ref. 6.

One phenomenon associated with the D state should be pointed out. It is clear from Fig. 3 that with the D state included the pickup amplitude becomes important at considerably smaller angles than is the case without the D state. This effect is present in all our calculations although it is somewhat subdued at 40 MeV.

Finally, we consider the polarizations at 150 MeV (Fig. 4). The results for $\theta<75°$ are about the same as those obtained in previous calculations[6,11,14,15] except for the somewhat better agreement at 30°. In this angular range the pickup amplitude contributes negligibly; however, the effect of the D state is quite noticeable.

Certainly the most striking feature of Fig. 4 is the behavior of $P(\theta)$ in the neighborhood of $\theta=120°$. The effect of the D state on $P(\theta)$ in this angular range when the pickup scattering is neglected is enormous. It should be remarked that the calculation of $P(\theta)$ with the D state included is extremely sensitive to apparently small changes in the matrix elements (4.4). Our original calculations were carried out neglecting all but the dominant D-state terms in the matrix elements (4.4). In this case, we obtained curves for $P(\theta)$ more in accord with the S-state curves. The present computation evaluated all terms in (4.4).

Since we have treated the pickup term only in Born approximation this amplitude by itself will not give rise to a polarization of the outgoing nucleon. Therefore, in our approximation, the pickup process influences $P(\theta)$ only through its interference with the direct terms.[50] With this in mind it is very interesting to note the large "damping" of the $P(\theta)$ curves, particularly in the D-state case, for $\theta>100°$. This appears to be due primarily to the fact that $\sigma(\theta)$ is rapidly increasing for large θ when the pickup term is included as compared to $\sigma(\theta)$ calculated using only the direct terms. It is not entirely clear why the polarizations found at large θ using both the direct and pickup terms appear to be independent of whether the D state is included or not.[51]

VII. CONCLUSION

Our calculations indicate that the D state of the deuteron and the pickup process contribute to elastic N-d scattering to a greater degree and at smaller angles than was previously suspected. The approximations which were introduced (impulse approximation, neglect of multiple scattering, etc.) in the course of this computation were all physically plausible and we expect that they form a reasonable first-order representation of the "actual" situation. Therefore, there exists some justification for supposing that the qualitative aspects of our results will be valid even when the scattering amplitude is evaluated by more refined techniques.[52]

Note added in proof. After this manuscript was submitted for publication, we became aware of a calculation of the nucleon-deuteron cross section at 155 MeV by Benoist-Gueutal and Gomez-Gimeno [Phys. Letters **13**, 68 (1964)]. These authors took into account the deuteron D state as well as the pickup term. Consequently, their results are in close accord with the direct+pickup ($S+D$ state) curve presented in Fig. 3 of the present paper.

ACKNOWLEDGMENTS

One of us (K. L. K.) would like to thank Dr. Murray Peshkin for the hospitality extended to him at the Argonne National Laboratory where part of this work was carried out. We both extend our thanks to R. Stieglitz for his considerable aid with the programing. Thanks are also due to C. Lennon for his advice.

[50] It is a matter of taste whether or not one calls a change in the magnitude of $P(\theta)$ arising from a change in $\sigma(\theta)$ an interference effect.

[51] Our semiquantitative reproduction of the backward positive peak in $P(\theta)$ offers support to the conjecture of Postma and Wilson (Ref. 14) as to the origin of this peak.

[52] The identification of part of the scattering amplitude as the "pickup" term may or may not be useful or meaningful depending upon the formalism and approximation procedures employed.