Λ-Λ Potential from Analysis of ${}_{\Lambda\Lambda}Be^{10}$†

Y. C. TANG
School of Physics, University of Minnesota, Minneapolis, Minnesota
AND
R. C. HERNDON
Lawrence Radiation Laboratory, University of California, Livermore, California
(Received 28 December 1964)

The strength of the Λ-Λ interaction in the ${}^1S_0$ state is determined by analyzing the double hypernucleus ${}_{\Lambda\Lambda}Be^{10}$ treated as a four-body system of $2\alpha+2\Lambda$. The α-α potential chosen fits the ${}^1S_0$ phase shifts up to a c.m. energy of about 12 MeV. The α-Λ potential is determined from the binding energy of ${}_{\Lambda}He^5$, taking into proper account the size of the alpha particle and the range of the Λ-nucleon interaction. With these potentials, the binding energy of ${}_{\Lambda}Be^9$ is also given correctly. For the Λ-Λ part, a potential is used which has a hard core of radius 0.4 F and an attractive well of exponential shape. The intrinsic range is chosen as 1.5 F, corresponding to the mechanism of 2-pion exchange. Using a 12-parameter variational function, the Λ-Λ ${}^1S_0$ potential which yields the observed separation energy of the two Λ particles in ${}_{\Lambda\Lambda}Be^{10}$ is found to have a well-depth parameter of $0.732_{-0.034}{}^{+0.027}$, which is about 10% smaller than the value obtained by treating ${}_{\Lambda\Lambda}Be^{10}$ as a three-body system of $Be^8+2\Lambda$. The scattering length and effective range are equal to $-(1.04_{-0.22}{}^{+0.20})$ and $2.91_{-0.27}{}^{+0.46}$ F, respectively. Also, we found that the Be^8 core in ${}_{\Lambda\Lambda}Be^{10}$ is quite compressed, with the α-α separation about 12% smaller than the corresponding separation in ${}_{\Lambda}Be^9$.

I. INTRODUCTION

THE recent discovery of a double hypernucleus,[1] best interpreted as ${}_{\Lambda\Lambda}Be^{10}$, has made it possible to gain some information about the strength of the Λ-Λ interaction in the ${}^1S_0$ state. Using a three-body model of $Be^8+\Lambda+\Lambda$, it was found by the present authors that, with a hard-core Λ-Λ potential having an intrinsic range of 1.5 F suggested by the mechanism of 2-pion exchange, the well-depth parameter in the ${}^1S_0$ state has a value of 0.817 ± 0.029, which corresponds to a scattering length of $-(1.93_{-0.38}{}^{+0.51})$ F and an effective range of 2.21 ± 0.17 F.[2]

The major defect of the three-body model lies in the fact that the distortion of the Be^8 core is not taken into account. Since the Be^8 nucleus is not even bound, it is to be expected that the two Λ particles will cause an appreciable distortion of the core. Hence, for a more realistic analysis, one should consider the hypernucleus ${}_{\Lambda\Lambda}Be^{10}$ as a four-body system of $2\alpha+2\Lambda$. The detailed structure of the alpha particle does not need to be considered, since, due to its low compressibility, one can safely assume that the effect of the distortion of the alpha particle will be a rather unimportant one.

Using the four-body model of $2\alpha+2\Lambda$ for ${}_{\Lambda\Lambda}Be^{10}$, analyses have already been made by Deloff[3] and Nakamura.[4] In these analyses, purely attractive Λ-Λ potentials have been used. This is somewhat unrealistic, since the presence of a short-range repulsion in the nucleon-nucleon potential, usually represented by a hard core, suggests that a hard core of similar size may also be present in the Λ-Λ potential.[5] Moreover, the α-α potential used in their calculations is not sufficiently repulsive; it yields α-α ${}^1S_0$ phase shifts well above the experimental values. Furthermore, the variational wave functions they used do not seem to have enough flexibility to describe in a reliable way the α-Λ correlation and the relative motion between the alpha particles. Thus, in our opinion, their results may not correspond too well to the real situation and a more detailed analysis, free of all these uncertainties, is clearly desired.[6]

In our investigation with the four-body model, we use an α-α potential which fits the ${}^1S_0$ phase shifts up to a c.m. energy of about 12 MeV. The α-Λ potential is chosen to yield the binding energy B_Λ of the Λ particle in the hypernucleus ${}_{\Lambda}He^5$, with the alpha-particle size and the range of the Λ-nucleon interaction taken properly into account. For the Λ-Λ interaction, a hard-core potential of core radius 0.4 F and an intrinsic range 1.5 F, corresponding to the mechanism of 2-pion exchange, will be used. As for the variational wave function, we shall adopt a form originally proposed by Austern and Iano.[7] In this form, the trial function is written as a product of two-particle correlation functions, each depending individually on the interparticle distance. For each of these functions, the solution of an appropriate two-body Schrödinger equation is used up to a certain interparticle separation, which is then connected to a variational function for larger distances.

† Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] M. Danysz, K. Garbowska, J. Pniewski, T. Pniewski, J. Zokrzewski *et al.*, Phys. Rev. Letters **11**, 29 (1963).

[2] Y. C. Tang, R. C. Herndon, and E. W. Schmid, Phys. Letters **10**, 358 (1964). With the three-body model, similar analysis has also been performed by R. H. Dalitz and G. Rajasekaran, Nucl. Phys. **50**, 450 (1964). In their analysis, however, a purely attractive Λ-Λ potential has been assumed.

[3] A. Deloff, Phys. Letters **6**, 83 (1963).

[4] H. Nakamura, Phys. Letters **6**, 207 (1963).

[5] J. J. de Swart, Phys. Letters **5**, 58 (1963).

[6] We should also mention here that, in spite of the fact that these authors have used similar potentials and variational wave functions, they have obtained contradictory results. Whereas Deloff concludes that the Λ-Λ interaction is quite attractive, Nakamura concludes that the Λ-Λ interaction is very weak.

[7] N. Austern and P. Iano, Nucl. Phys. **18**, 672 (1960).

To convince ourselves that the wave function described above has enough flexibility to yield reliable results for ${}_{\Lambda\Lambda}Be^{10}$, we shall give in the Appendix[8] a demonstration of the usefulness of this type of trial wave function by considering the nuclear three- and four-body problems with purely attractive potentials. There we shall see that the upper and lower bound to the eigenvalue are quite close together, which indicates that not only is the upper bound very close to the eigenvalue, but also the trial function with the optimum parameters is a good approximation to the eigenfunction.

In the next section, we shall discuss the two-body potentials to be used in this calculation. It will be seen that, with the α-α and α-Λ potentials used here, the binding energy of the hypernucleus ${}_{\Lambda}Be^{9}$ is also given correctly. This is somewhat to be expected, of course, since the addition of a comparatively weakly bound Λ particle should not destroy the alpha-particle feature of the Be^8 core. In Sec. III, the double hypernucleus ${}_{\Lambda\Lambda}Be^{10}$ will be analyzed. Using the observed separation energy $B_{\Lambda\Lambda}$ equal to 17.5 ± 0.5 MeV of the two Λ particles, the strength of the Λ-Λ potential in the ${}^1S_0$ state can be determined. This latter strength will be compared with that obtained by us earlier with the three-body model. Also in this section, we shall give the value of $B_{\Lambda\Lambda}$ calculated for the double hypernucleus ${}_{\Lambda\Lambda}He^6$ with the Λ-Λ potential determined from ${}_{\Lambda\Lambda}Be^{10}$. Finally, in Sec. IV, we shall present a discussion of the results of this investigation.

II. TWO-BODY POTENTIALS AND ${}_{\Lambda}Be^{9}$

A. Two-Body Potentials

The α-Λ potential is chosen in the same way as in the calculation of Dalitz and Downs on ${}_{\Lambda}He^5$.[9] Using a Gaussian shape for the nucleon distribution in the alpha particle with rms radius of 1.44 F and a Λ-nucleon interaction of a Gaussian form with an intrinsic range corresponding to the mechanism of 2-pion exchange, we get

$$V_{\alpha\Lambda}(r)=-v_{\alpha\Lambda}\exp(-\lambda r^2), \quad (1)$$

with $\lambda=0.408$ F^{-2}. The depth $v_{\alpha\Lambda}$ can be deduced from the solution of the Schrödinger equation for the motion of the Λ particle relative to the alpha particle. With $B_\Lambda=3.1$ MeV,[10] numerical integration leads to $v_{\alpha\Lambda}=43.98$ MeV, which corresponds to a volume integral of 940 MeV-F^3.

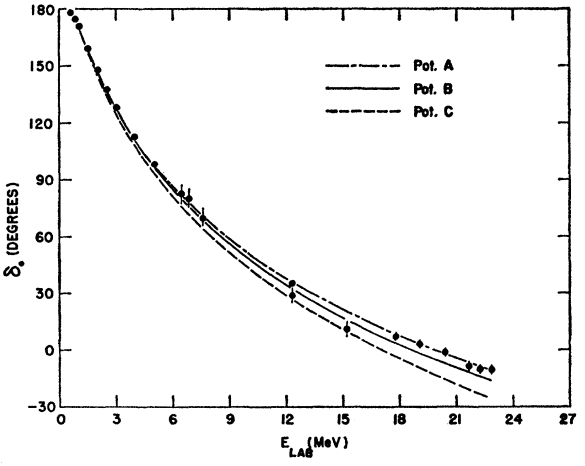

FIG. 1. α-α ${}^1S_0$ phase shifts as a function of incident energy in the laboratory system.

For the α-α potential, we use a type which has been used in the investigations of Suh[11] and Bodmer and Ali[12] on the hypernucleus ${}_{\Lambda}Be^9$. It has the form

$$V_{\alpha\alpha}(r)=v_R\exp(-\mu_R{}^2r^2)-v_A\exp(-\mu_A{}^2r^2)+V_c(r), \quad (2)$$

with $V_c(r)$ denoting the Coulomb interaction. Using a rms radius of 1.44 F for the alpha particle, this latter interaction can be approximately written as[13]

$$V_c(r)=(4e^2/r)\Phi(\nu r), \quad (3)$$

where $\Phi(x)$ is the error function, defined as

$$\Phi(x)=\frac{2}{\pi^{1/2}}\int_0^x\exp(-t^2)dt$$

and ν is equal to 0.602 F^{-1}.

To gain confidence in the validity of the $2\alpha+2\Lambda$ model for ${}_{\Lambda\Lambda}Be^{10}$, we would like to make sure that the $2\alpha+\Lambda$ model works for ${}_{\Lambda}Be^9$ at least. This means that, with the α-Λ potential given by Eq. (1) and the α-α potential yielding the α-α ${}^1S_0$ phase shifts in the low-energy region, we should get the binding energy of ${}_{\Lambda}Be^9$ correctly. For this purpose, we shall calculate with three different α-α potentials to see how sensitive the binding energy of ${}_{\Lambda}Be^9$ is to the variation in the α-α interaction. The parameters of these three potentials

TABLE I. Parameters of the α-α potential.

Potential type	μ_A (F^{-1})	v_A (MeV)	μ_R (F^{-1})	v_R (MeV)
A	0.475	160	0.6	300
B	0.475	160	0.635	400
C	0.475	160	0.7	750

[8] This Appendix is essentially an extension to an earlier investigation [Y. C. Tang, R. C. Herndon, and E. W. Schmid, Phys. Rev. 134, B743 (1964)] where the nuclear three-body problem with purely attractive two-body potentials has been considered. For a similar calculation involving hard-core potentials in nuclear three- and four-body problems, see Y. C. Tang, E. W. Schmid, and R. C. Herndon, Nucl. Phys. (to be published).

[9] R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958).

[10] R. Levi-Setti, in Proceedings of the International Conference on Hyperfragments, St. Cergue, Switzerland, 1963 (unpublished).

[11] K. S. Suh, Phys. Rev. **111**, 941 (1958).

[12] A. R. Bodmer and S. Ali, Nucl. Phys. **56**, 657 (1964).

[13] E. Van der Spuy, Nucl. Phys. **11**, 615 (1959).

TABLE II. Results for $_{\Lambda}Be^{9}$.[a]

α-α Potential	α_f (F^{-1})	β_f (F^{-1})	e_f (MeV)	d_f (F)	α_g (F^{-1})	β_g (F^{-1})	e_g (MeV)	d_g (F)	E_U (MeV)	B_Λ (MeV)	$\langle r_{12}^2\rangle^{1/2}$ (F)
A	0.43	2.5	−3.0	2.9	0.295	5.0	−6.0	1.5	−6.71±0.08	6.81±0.08	3.51
B	0.43	2.4	−3.2	3.0	0.280	5.0	−5.5	1.5	−6.43±0.08	6.53±0.08	3.65
C	0.46	2.0	−3.6	3.3	0.258	5.0	−4.2	1.5	−5.88±0.08	5.98±0.08	3.88

[a] The statistical accuracy in this table is achieved with 45 000 estimates in the Monte Carlo calculation.

are listed in Table I and they will be called potentials *A*, *B*, and *C*, respectively.[14]

The behavior of the α-α $^1S_0$ phase shifts calculated with these potentials is shown in Fig. 1, where a comparison with experimental data[15] is also made. It is seen that all three potentials fit quite well the experimental phases in the c.m. energy range of zero to about 12 MeV, with potential *B* giving the best over-all agreement.

The Λ-Λ potential used in this investigation has the same spatial dependence as that used in our previous study of $_{\Lambda\Lambda}Be^{10}$ with the three-body model,[2] i.e.,

$$\begin{aligned} V_{\Lambda\Lambda}(r) &= \infty\,, && (r<r_c)\\ &= -v_{\Lambda\Lambda}\exp[-\kappa(r-r_c)]\,, && (r>r_c) \end{aligned} \tag{4}$$

with r_c equal to 0.4 F and κ equal to 5.059 F^{-1}. This potential has an intrinsic range of 1.5 F, which corresponds to the mechanism of 2-pion exchange. The depth $v_{\Lambda\Lambda}$ is a variable parameter in our calculation; it will be varied to yield the observed separation energy $B_{\Lambda\Lambda}$ of the two Λ particles in the double hypernucleus $_{\Lambda\Lambda}Be^{10}$.

B. Binding Energy of $_{\Lambda}Be^{9}$

Using the potentials described above, the binding energy of $_{\Lambda}Be^{9}$ in the 2α+Λ model will be computed with a variational method. The variational wave function adopted is of the form

$$\Psi = F(r_{12})G(r_{13})G(r_{23})\chi_s\,, \tag{5}$$

where 1 and 2 denote the two alpha particles and 3 denotes the Λ particle. The function χ_s represents the appropriate spin function for the ground state of $_{\Lambda}Be^{9}$. For the function $F(r)$, we use a form originally proposed by Austern and Iano,[7] i.e.,

$$\begin{aligned} F(r) &= u_f(r)/r\,, && (r<d_f)\\ &= A_f r^{-1/2}[\exp(-\alpha_f r) && \\ &\quad +B_f\exp(-\beta_f r)]\,, && (r>d_f)\,, \end{aligned} \tag{6}$$

where $u_f(r)$ is a solution of the equation

$$-(\hbar^2/2\mu_f)(d^2/dr^2)u_f(r)+[V_{\alpha\alpha}(r)-e_f]u_f(r)=0\,, \tag{7}$$

with μ_f being the reduced mass of the two alpha particles. The constants A_f and B_f in Eq. (6) are adjusted such that the function $F(r)$ and its first derivative are continuous at the separation distance d_f. There are a total number of four variational parameters in this function, namely, α_f, β_f, e_f, and d_f. The function $G(r)$ is defined in an analogous manner, except that μ_f is replaced by μ_g, the reduced mass of the α and the Λ particle, and the potential function in Eq. (7) is replaced by the potential $V_{\alpha\Lambda}(r)$. The variational parameters in this latter function are α_g, β_g, e_g, and d_g.

The various expectation values are computed by a Monte Carlo method which has been described previously.[16] The results are shown in Table II,[17] where E_U denotes the upper bound to the eigenvalue, $\langle r_{12}^2\rangle^{1/2}$ denotes the rms distance of separation between the two alpha particles, and B_Λ denotes the binding energy of the Λ particle in $_{\Lambda}Be^{9}$. To obtain the values of B_Λ, a resonant energy of about 0.1 MeV for the ground state of Be^8 has been used.[18]

As demonstrated by the three- and four-body cases in the Appendix, we expect the upper bound obtained by this type of trial function to be very close to the eigenvalue. In this particular case of $_{\Lambda}Be^{9}$, we believe that the difference between the upper bound and the eigenvalue is, in all likelihood, less than 0.1 MeV, which is smaller than the experimental uncertainty in the value of B_Λ.[19]

From Table II, it is seen that potential B yields a value of B_Λ which agrees very well with the experimental value of 6.50±0.16 MeV.[10] This is interesting, since potential B also gives the best agreement with the experimentally determined α-α $^1S_0$ phase shifts in the low-energy region.[20] In addition, we note that B_Λ is rather sensitive to the variation in the α-α potential. Although potential A and potential C yield phase shifts which are not too different, the values of B_Λ

[14] Except for a minor difference in $V_c(r)$, potentials *A* and *C* are the same as potentials *q* and *r* of Bodmer and Ali (Ref. 12).

[15] N. P. Heydenburg and G. M. Temmer, Phys. Rev. **104**, 123 (1956); C. W. Reich, J. L. Russell and G. C. Phillips, *ibid.* **104**, 135 (1956); R. Nilson, W. K. Jentschke, G. R. Briggs, R. O. Kerman, and J. N. Snyder, *ibid.* **109**, 850 (1958); C. M. Jones, G. C. Phillips, and P. D. Miller, *ibid.* **117**, 525 (1960); N. Berk, F. E. Steigert, and G. L. Salinger, *ibid.* **117**, 531 (1960); J. R. Dunning, A. M. Smith, and F. E. Steigert, *ibid.* **121**, 580 (1961).

[16] E. W. Schmid, Nucl. Phys. **32**, 82 (1962); E. W. Schmid, Y. C. Tang, and R. C. Herndon, *ibid.* **42**, 95 (1963).

[17] The computation on $_{\Lambda}Be^{9}$ was done on the IBM 7094 computer at the Brookhaven National Laboratory.

[18] F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959).

[19] For this reason, we have simply identified the total binding energy as the negative of the upper bound in both $_{\Lambda}Be^{9}$ and $_{\Lambda\Lambda}Be^{10}$.

[20] The same conclusion has also been reached by Bodmer and Ali (Ref. 12) in their analysis of $_{\Lambda}Be^{9}$.

TABLE III. Results for $_{\Lambda\Lambda}Be^{10}$.[a]

$v_{\Lambda\Lambda}$ (MeV)	α_f (F^{-1})	β_f (F^{-1})	e_f (MeV)	d_f (F)	α_g (F^{-1})	β_g (F^{-1})	e_g (MeV)	d_g (F)	α_h (F^{-1})	β_h (F^{-1})	e_h (MeV)	d_h (F)	E_U (MeV)	$B_{\Lambda\Lambda}$ (MeV)	$\langle r_{12}^2\rangle^{1/2}$ (F)
910	0.49	2.0	−7.0	3.0	0.29	5.5	−8.0	1.5	0.035	1.8	−23.0	1.1	−16.96 ±0.22	17.06 ±0.22	3.24
980	0.50	2.0	−5.5	3.0	0.29	5.0	−8.0	1.5	0.060	1.3	−18.0	1.1	−17.97 ±0.23	18.07 ±0.23	3.20
1056	0.52	2.2	−4.0	3.0	0.29	5.0	−8.5	1.5	0.10	1.3	−12.0	1.1	−19.24 ±0.26	19.34 ±0.26	3.14

[a] The statistical uncertainty in this table is achieved with 60 000 estimates in the Monte Carlo calculation.

obtained with these potentials differ by almost 1 MeV.

We wish to point out that the arbitrariness in the choice of the spatial dependence of the α-α potential does not affect the conclusion mentioned above, since it has been shown by Bodmer and Ali[12] that, for two α-α potentials of different shape which yield nearly the same $^1S_0$ phase shifts, the values of B_Λ in $_\Lambda Be^9$ are also nearly the same. Thus, this analysis supports strongly the validity of the $2\alpha+\Lambda$ model for $_\Lambda Be^9$, which, in turn, means that the results we shall obtain for $_{\Lambda\Lambda}Be^{10}$ using the $2\alpha+2\Lambda$ model will similarly be quite reliable.

For potential B, $\langle r_{12}^2\rangle^{1/2}$ has a value of 3.65 F which is larger than twice the rms radius of the alpha particle. This indicates that the two alpha particles do not overlap appreciably, thus strongly supporting the α-particle model for the Be^8 core.

III. ANALYSIS OF $_{\Lambda\Lambda}Be^{10}$

The trial wave function used for $_{\Lambda\Lambda}Be^{10}$ is

$$\Psi=F(r_{12})\Big[\prod_{\substack{i=1,2\\k=3,4}} G(r_{ik})\Big]H(r_{34})\chi_s, \tag{8}$$

where 1, 2 denote the α particles and 3, 4 denote the Λ particles. The function χ_s is a singlet spin function describing the coupling of the Λ particles into a $^1S_0$ state. For the functions $F(r)$ and $G(r)$, we adopt the same forms as those used for $_\Lambda Be^9$. Also, a similar procedure is used to define the function $H(r)$; one only needs to note that, in Eq. (7), μ_f is to be replaced by μ_h, the reduced mass of the two Λ particles, and the potential function $V_{\alpha\alpha}(r)$ is to be replaced by the potential function $V_{\Lambda\Lambda}(r)$. There is a total of twelve parameters in the trial function Ψ, namely, α_f, β_f, d_f, e_f, α_g, β_g, e_g, d_g, α_h, β_h, e_h, and d_h. Although the number of variational parameters is rather large, we have found that it was not overly difficult to find their optimum values, since the upper bound is rather insensitive to the variation in about half of these parameters.

The upper bounds to the eigenvalues of $_{\Lambda\Lambda}Be^{10}$ are calculated with α-α potential B for three values of $v_{\Lambda\Lambda}$, namely, 1056, 980, and 910 MeV. The results are given in Table III.[21] From this table, we see that, for $v_{\Lambda\Lambda}=1056$ MeV, the four-body model yields an upper bound of −19.24 MeV which is 1.84 MeV lower than the corresponding value obtained with the three-body model.[2] This indicates clearly that the effect of the distortion of the Be^8 core plays an important role in the determination of the strength of the Λ-Λ interaction.

To have some idea about how close the upper bound is to the eigenvalue, we have also computed the lower bound for $v_{\Lambda\Lambda}=1056$ MeV, using a formula of Temple,[22] i.e.,

$$E_L=\langle H\rangle-(\langle H^2\rangle-\langle H\rangle^2)/(E_1-\langle H\rangle), \tag{9}$$

where H is the Hamiltonian of $_{\Lambda\Lambda}Be^{10}$, and E_1 is the energy of the first excited state, which, in this case, is equal to −11.08 MeV, the energy of the double hypernucleus $_{\Lambda\Lambda}He^6$ computed with this value of $v_{\Lambda\Lambda}$.[2] The result obtained with the optimum parameters of the upper bound is $E_L=-37.4$ MeV, which is about 18 MeV lower than the upper bound. Using the experience we gained from calculations in nuclear three- and four-body problems with hard-core potentials,[23] we believe that a gap of this size between the two bounds means that the upper bound is about 0.3 MeV away from the eigenvalue. Since the latter magnitude is only about half of the experimental uncertainty in $B_{\Lambda\Lambda}$, we shall, in the following, simply identify the total binding energy of $_{\Lambda\Lambda}Be^{10}$ as the negative of the value of the upper bound.

From $B_{\Lambda\Lambda}$ of 17.5±0.5 MeV for $_{\Lambda\Lambda}Be^{10}$, we get, with the results of Table III,

$$v_{\Lambda\Lambda}=944_{-44}^{+35}\text{ MeV}, \tag{10}$$

where the error quoted is that due to the uncertainty in $B_{\Lambda\Lambda}$ and the statistical uncertainty arising from the Monte Carlo calculation.

The comparison between the results of this calculation for the well-depth parameter $s_{\Lambda\Lambda}$, scattering length $a_{\Lambda\Lambda}$, and effective range $(r_0)_{\Lambda\Lambda}$ with those of our previous study with the three-body model,[2] and those of Dalitz and Rajasekaran,[2] is made in Table IV. From this table, we note that the values of these quantities are quite different in the three- and four-body model, thus indicating strongly that a reliable value for $v_{\Lambda\Lambda}$

[21] The computation on $_{\Lambda\Lambda}Be^{10}$ was done on the CDC 3600 computer at the Lawrence Radiation Laboratory, Livermore.

[22] G. Temple, Proc. Roy. Soc. (London) **119**, 276 (1928).

[23] Y. C. Tang, E. W. Schmid, and R. C. Herndon, Nucl. Phys. (to be published).

TABLE IV. Well-depth parameter, scattering length, and effective range of the Λ-Λ potential.

	Model for $_{\Lambda\Lambda}Be^{10}$	$s_{\Lambda\Lambda}$	$a_{\Lambda\Lambda}$ (F)	$(r_0)_{\Lambda\Lambda}$ (F)
This analysis	$2\alpha+2\Lambda$	$0.732^{+0.027}_{-0.034}$	$-\left(1.04^{+0.20}_{-0.22}\right)$	$2.91^{+0.46}_{-0.27}$
Tang, Herndon, and Schmid[a]	$Be^8+2\Lambda$	0.817 ± 0.029	$-\left(1.93^{+0.51}_{-0.38}\right)$	2.21 ± 0.17
Dalitz and Rajasekaran[a]	$Be^8+2\Lambda$	0.599 ± 0.048	$-(1.76\pm0.33)$	2.10 ± 0.12

[a] Reference 2.

can be obtained only when the effect of the distortion of the Be^8 core is taken into account.

We wish to point out also that the Be^8 core is quite compressed. With $v_{\Lambda\Lambda}=944$ MeV, the value of $\langle r_{12}^2\rangle^{1/2}$ is only 3.22 F, which is about 12% less than the corresponding value in $_{\Lambda}Be^9$. Fortunately, however, it is still more than twice the rms radius of the nucleon distribution in the alpha particle, which means that the α-particle model for the Be^8 core in $_{\Lambda\Lambda}Be^{10}$ is still a rather good one.

The value of $B_{\Lambda\Lambda}$ as a function of $v_{\Lambda\Lambda}$ for the double hypernucleus $_{\Lambda\Lambda}He^6$ considered as $\alpha+2\Lambda$ has been calculated previously.[24] The results are listed in Table V. Using $v_{\Lambda\Lambda}=944$ MeV obtained from the analysis of $_{\Lambda\Lambda}Be^{10}$, the value of $B_{\Lambda\Lambda}$ is 9.33 MeV.

IV. CONCLUSION

In this investigation, we determine the strength of the Λ-Λ interaction by analyzing the double hypernucleus $_{\Lambda\Lambda}Be^{10}$ considered as a four-body system of $2\alpha+2\Lambda$. The Λ-Λ potential used as a hard core of radius 0.4 F and an attractive well of exponential shape. The intrinsic range is chosen as 1.5 F, corresponding to the mechanism of 2-pion exchange. Using the observed separation energy of the two Λ particles in $_{\Lambda\Lambda}Be^{10}$, we find that the well-depth parameter of the Λ-Λ potential in the $^1S_0$ state has a value of $0.732_{-0.034}{}^{+0.027}$. This value is about 10% smaller than that obtained when $_{\Lambda\Lambda}Be^{10}$ is considered as a three-body system of $Be^8+2\Lambda$,[2] which indicates that the effect of the distortion of the Be^8 core is quite essential in the determination of the strength of the Λ-Λ interaction from the binding energy of $_{\Lambda\Lambda}Be^{10}$.

The value of the well-depth parameter determined here is still somewhat overestimated due to two factors. First, the distortion of the two alpha particles in $_{\Lambda\Lambda}Be^{10}$ is not considered. Second, our trial wave function yields and upper bound which is estimated to be about 0.3 MeV higher than the eigenvalue. If both of these factors are properly taken into account, there will probably be a small reduction of about 3 to 4% in the value of the well-depth parameter, which is not too significant, however, since this reduction is even smaller than the uncertainty in the well-depth parameter arising from the experimental uncertainty in the value of $B_{\Lambda\Lambda}$ of $_{\Lambda\Lambda}Be^{10}$.

Due to the relative weakness of the average Λ-nucleon interaction, it is generally believed that there is no $I=1$ bound state for $_{\Lambda}H^3$.[25] Since, in the double hypernucleus $_{\Lambda\Lambda}H^3$ or $_{\Lambda\Lambda}n^3$, the average Λ-nucleon interaction is the same as that in the $I=1$ configuration of $_{\Lambda}H^3$ and the Λ-Λ well-depth parameter is smaller than the nucleon-nucleon well-depth parameter in the $^1S_0$ state, we would expect that there is no bound system of $_{\Lambda\Lambda}H^3$ or $_{\Lambda\Lambda}n^3$.[26] For the double hypernucleus $_{\Lambda\Lambda}H^4$, the situation is not so simple, and a detailed calculation using the Λ-Λ potential determined here and the Λ-nucleon potential determined previously[27] is really required in order to decide whether or not there exists a bound state for $_{\Lambda\Lambda}H^4$.

Finally, we wish to mention that Danysz *et al.*[1] have pointed out that the double hypernuclear event which they discovered might be interpreted as $_{\Lambda\Lambda}Be^{11}$ with separation energy 19.0 ± 0.6 MeV relative to $Be^9+2\Lambda$. If this interpretation is adopted, then a modified analysis is required, which, however, would not be expected to change appreciably the value of $v_{\Lambda\Lambda}$ arrived at in this investigation.[28]

TABLE V. Values of $B_{\Lambda\Lambda}$ for $_{\Lambda\Lambda}He^6$.[a]

$v_{\Lambda\Lambda}$ (MeV)	$B_{\Lambda\Lambda}$ (MeV)
750	7.27 ± 0.07
950	9.40 ± 0.09
1056	11.08 ± 0.11
1120	12.28 ± 0.13

[a] Number of estimates used in the Monte Carlo calculation is 40 000.

[24] The value of $B_{\Lambda\Lambda}$ for $v_{\Lambda\Lambda}=1056$ MeV has already been given in Ref. 2. In that calculation, the α-Λ potential is slightly different from that used here. Also, the trial wave function used has a different form. However, we believe that the results for $B_{\Lambda\Lambda}$ are very nearly the same as those which would be obtained if we had used the α-Λ potential and the type of trial function of this investigation.

[25] B. W. Downs and R. H. Dalitz, Phys. Rev. **114**, 593 (1959).

[26] H. Nakamura, Progr. Theoret. Phys. (Kyoto) **30**, 84 (1963).

[27] R. C. Herndon, Y. C. Tang, and E. W. Schmid, Phys. Rev. (to be published).

[28] For a discussion on this point, see R. H. Dalitz, in Proceedings of the International Conference on Hyperfragments, St. Cergue, Switzerland, 1963 (unpublished).

ACKNOWLEDGMENTS

The authors wish to thank Dr. A. R. Bodmer for sending us a preprint on hypernucleus $_{\Lambda}Be^{9}$ which has been very helpful in this investigation.

APPENDIX A: TESTS ON THE TRIAL FUNCTION

To show that the type of trial function used in this investigation is capable of yielding an upper bound very close to the eigenvalue E_0, we present here the results of a calculation with purely attractive two-body potentials for the nuclear three- and four-body problems. In this calculation, both the upper bound E_U and the lower bound E_L are computed. For the upper bound, we employ the usual Rayleigh-Ritz method, while, for the lower bound, Temple's formula given by Eq. (9) is adopted.[29]

Two-body potentials of both Gaussian and exponential dependence have been used. They have the forms

$$V_g(r)=-V_{0g}\exp(-\kappa_g r^2) \tag{A1}$$

and

$$V_e(r)=-V_{0e}\exp(-\kappa_e r), \tag{A2}$$

with $V_{0g}=51.5$ MeV, $\kappa_g=0.3906$ F^{-2}, $V_{0e}=96.937$ MeV, and $\kappa_e=1.156$ F^{-1}. The potential $V_g(r)$ was used in a calculation by Baker *et al.*[30] on the nuclear three-body problem, while the potential $V_e(r)$ is an average of the potential in the triplet-even and singlet-even states used by Rarita and Present in a study on the nuclear two-, three-, and four-body systems.[31]

The trial function employed is of the type described in Sec. II. Its spatial part is given by

$$\phi=\prod_{i<j=1}^{N} f(r_{ij}), \tag{A3}$$

with N equal to three for the triton and four for the alpha particle. For the function $f(r)$, we use either $f_1(r)$ or $f_2(r)$, which are

$$\begin{aligned} f_1(r)&=u(r)/r, && (r<d)\\ &=Ar^{-1/(N-1)}[\exp(-\alpha r)\\ &\qquad +B\exp(-\beta r)], && (r>d) \end{aligned} \tag{A4}$$

and

$$\begin{aligned} f_2(r)&=u(r)/r, && (r<d)\\ &=Ar^{-1/(N-1)}[\exp(-\alpha r)+B\exp(-\beta r)\\ &\qquad +C\exp(-\gamma r)], && (r>d), \end{aligned} \tag{A5}$$

where $u(r)$ is a solution of the equation

$$-(\hbar^2/m)(d^2/dr^2)u+[V(r)-e]u=0, \tag{A6}$$

with $V(r)$ representing either $V_g(r)$ or $V_e(r)$. When the function $f_1(r)$ is used, the constants A and B are adjusted such that the function $f_1(r)$ and its first derivative are continuous at the separation distance d. There are a total of four variational parameters in this function, namely, α, β, e, and d. When $f_2(r)$ is used, the additional constant C is utilized to insure that its second derivative is also continuous at the separation distance. In this latter function, there are five variational parameters, namely, α, β, γ, e, and d.

The significance of the factor $r^{-1/(N-1)}$ has been explained previously[23,27]; hence, we shall not go into it further here. It suffices to say that, if our trial function represents the eigenfunction closely, then the variationally determined value of α should be approximately equal to the value of α' given by

$$\alpha'=\{[2m/N(N-1)\hbar^2](E_1-E_0)\}^{1/2}, \tag{A7}$$

where (E_1-E_0) is the separation energy of a single nucleon from the rest of the system.

A.1. The Three-Body System (Triton)

In Table VI, we show the results obtained for the three-body system. To get the values of E_L, values of E_1 equal to the eigenvalues of the two-body system are used. With $V_g(r)$ and $V_e(r)$, these values are calculated to be -0.40 and -0.35 MeV, respectively.

TABLE VI. Results for the three-body system.[a]

Two-body potential	Trial function	Optimum parameters of E_U: α (F^{-1})	β (F^{-1})	γ (F^{-1})	e (MeV)	d (F)	Optimum parameters of E_L: α (F^{-1})	β (F^{-1})	γ (F^{-1})	e (MeV)	d (F)	E_U (MeV)	E_L (MeV)
V_g	f_1[b]	0.288	5.15		−2.0	1.5	0.292	2.55		−0.6	1.8	$-9.74_0\pm0.05$	-10.04 ± 0.06
V_g	f_2	0.270	0.80	1.18	−0.3	1.5	0.265	0.80	1.20	−0.6	1.5	$-9.74_5\pm0.04$	-9.99 ± 0.05
V_e	f_1[b]	0.260	1.60		0	1.2	0.250	1.60		−1.0	1.2	$-7.65_0\pm0.05$	-7.84 ± 0.08
V_e	f_2	0.243	0.62	1.24	−0.1	1.2	0.253	0.62	1.26	−0.5	1.2	$-7.65_4\pm0.04$	-7.79 ± 0.07

[a] The numbers of estimates used are 80 000 and 40 000 for the cases with $V_g(r)$ and $V_e(r)$, respectively.
[b] These results have been given previously [Y. C. Tang, R. C. Herndon, and E. W. Schmid, Phys. Rev. **134**, B743 (1964)]. Here, we have corrected a small mistake made before in computing the statistical uncertainty of E_L.

[29] For a discussion on the variational methods, see Ref. 23.
[30] G. A. Baker, Jr., J. L. Gammel, B. J. Hill, and J. G. Wills, Phys. Rev. **125**, 1754 (1962).
[31] W. Rarita and R. D. Present, Phys. Rev. **51**, 788 (1937).

From the Table VI, it is seen that the difference in the values of the upper bound obtained with $f_1(r)$ and $f_2(r)$ is rather unimportant, being much smaller than the statistical uncertainty arising from the Monte Carlo calculation. In the case of the lower bound, the difference is small but more significant. For both potentials, the improvement in the lower bound calculated with $f_2(r)$ over that with $f_1(r)$ is about 10 times the corresponding improvement in the upper bound.

The above-mentioned results lead us to conclude that if the sole purpose is to obtain a good upper bound, then the function $f_1(r)$ is flexible enough for most problems. On the other hand, if one is interested in a trial function which can give a very good approximation to the eigenfunction, then it might be worthwhile to consider the function $f_2(r)$. From Table VI, it is evident that $f_2(r)$ represents the eigenfunction even better than $f_1(r)$, since the optimum parameters of the upper and lower bound are closer to each other for $f_2(r)$ than they are for $f_1(r)$.

From the closeness of the two bounds and the way they improve when $f_1(r)$ is changed to $f_2(r)$, we believe strongly that the eigenvalue is equal to -9.77 ± 0.04 MeV when the potential is $V_g(r)$, and equal to -7.67 ± 0.04 MeV when the potential is $V_e(r)$.[32,33]

Using the values of E_0 predicted, we can calculate α' from Eq. (A7), which turns out to be equal to 0.274 and 0.242 F^{-1}, for the cases with $V_g(r)$ and $V_e(r)$, respectively. Comparing with the values of α given in Table VI, we note that the values of α' are almost identical to those of α for the upper bound with the trial function $f_2(r)$, which is another indication that the function $f_2(r)$ with the optimum parameters of the upper bound is a very good approximation to the eigenfunction.

A.2. The Four-Body System (Alpha Particle)

The results obtained for the four-body system with the potential $V_e(r)$ and the trial function $f_1(r)$ are as follows:

Upper bound:

$$\alpha=0.293\ F^{-1},\quad \beta=3.0\ F^{-1},$$
$$e=-4.0\ \text{MeV},\quad d=1.2\ F,$$
$$E_U=-31.45\pm0.14\ \text{MeV},$$
$$\text{number of estimates}=60\,000; \qquad \text{(A8)}$$

Lower bound:

$$\alpha=0.295\ F^{-1},\quad \beta=1.66\ F^{-1},$$
$$e=-2.7\ \text{MeV},\quad d=1.2\ F,$$
$$E_L=-31.81\pm0.18\ \text{MeV},$$
$$\text{number of estimates}=120\,000. \qquad \text{(A9)}$$

To get the values of E_U and E_L, the Coulomb potential between the two protons has been taken into consideration. Also, the value of E_1 used is -7.67 MeV, which is the eigenvalue for the triton predicted in a previous paragraph.

From Eqs. (A8) and (A9), we note that the gap between the two bounds is 0.36 MeV, which is only about 1% of the magnitude of the upper bound. This indicates that, as in the three-body case, the function $f_1(r)$ is capable of yielding a very good upper bound. In fact, even without further calculation with a more flexible function, we can safely predict that the eigenvalue is -31.50 ± 0.14 MeV.

As in the three-body case, the self-consistency condition (A7) is quite well satisfied. The value of α' calculated with the value of E_0 given above is 0.309 F^{-1}, which is rather close to the value of α obtained from the variational calculation.

Our investigation with the nuclear three- and four-body problems shows, therefore, that, with purely attractive potentials, the trial function ϕ of Eq. (A3) with $f_1(r)$ can not only yield a very good upper bound but also give a good representation of the eigenfunction. When the potential has a hard core, the situation is somewhat more complicated, since, from a recent study,[23] we have found that a trial function of this type does not describe the eigenfunction too well when three or more particles are close to each other. However, we have also noted in that same study that the probability of such close-packing of particles is not large enough to affect the upper bound to an appreciable extent, although it is large enough to depress the lower bound quite considerably. Thus, together with the results of this investigation, we conclude that, in the case with hard-core potentials, the type of trial wave function used here is capable of giving a proper description of the eigenfunction except for a rather small region in the configuration space. Since this latter defect does not influence the upper bound very much, it is still safe to assert that the upper bound produced by this trial function is a good approximation to the eigenvalue.

[32] It is interesting to point out that, by improving the independent-pair method of H. J. Mang and W. Wild [Z. Physik **154**, 182 (1959)] for light nuclei, R. Folk [Bull. Am. Phys. Soc. **10**, 112 (1965)] has obtained a value for E_0 with the potential $V_g(r)$ which is almost identical with our value here.

[33] In a previous publication [Phys. Rev. **134**, B743 (1964)], we have made a statement that our result for E_U with $V_g(r)$ is not too consistent with the result obtained by Kalos [M. H. Kalos, Phys. Rev. **128**, 1791 (1962)] using a different method. This statement was made as the consequence of a misinterpretation of Kalos' result. Upon a careful reexamination, we have instead come to the conclusion that the results of Kalos' and our calculations are entirely consistent with each other.