$\Lambda\Lambda$ Hypernucleus $_{\Lambda\Lambda}Be^{10}$ and the Λ-Λ Interaction*

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The $\Lambda\Lambda$ hypernucleus $_{\Lambda\Lambda}Be^{10}$ has been analyzed by use of a four-body α-α-Λ-Λ model which allows for distortion of the core by the Λ particles. In particular, the dependence of the internal energy of the core on the rms separation of the α particles is required. This was obtained from three-body α-α-Λ calculations for $_{\Lambda}Be^{9}$. Several types of α-α potentials, whose s-wave phase shifts had been previously obtained, were considered. Calculations for $_{\Lambda\Lambda}Be^{10}$ were made for a singlet Λ-Λ Yukawa potential (I) of intrinsic range $b=1.48$ F, appropriate to the exchange of two pions, and for a hard-core Yukawa potential (II) with a hard-core radius $r_c=0.42$ F and $b=2.66$ F, appropriate to a range corresponding to two pion masses for the attractive Yukawa part. Results are also given for a hard-core meson-theory potential (III) which has $r_c=0.42$ F and $b=1.48$ F. Calculations for III were made for $_{\Lambda\Lambda}He^{6}$, and the results were adapted to $_{\Lambda\Lambda}Be^{10}$. For α-α potentials which give s-wave phase shifts consistent with experiment, it is found that (almost independently of the details of the Λ-Λ potential) the effects of core distortion account for rather more than a third of the experimental additional binding energy of 4.5 ± 0.5 MeV which is obtained after the Λ separation energy of $_{\Lambda}Be^{9}$ has been allowed for. Slightly more than half the contribution due to core distortion comes from the core energy of $_{\Lambda}Be^{9}$. The remainder is due to the further distortion of the core by the second Λ, which causes approximately a 10% decrease in the rms α-α separation relative to the value for $_{\Lambda}Be^{9}$. The effects of core distortion weaken the resulting Λ-Λ potential quite appreciably. For $b=1.48$ F, one obtains the scattering length $a_{\Lambda\Lambda}\approx-1\pm0.3$ F and the effective range $r_{0\Lambda\Lambda}\approx3.3\pm0.6$ F, approximately independent of the shape of the Λ-Λ potential. For II, one gets $a_{\Lambda\Lambda}=-2.3_{-0.5}^{+0.8}$ F and $r_{\Lambda\Lambda}=4.9_{-0.7}^{+1.1}$ F. The well-depth parameters are 0.45 ± 0.08, 0.675 ± 0.065, and 0.77 ± 0.04 for I, II, and III, respectively. These values are about 35%, 20%, and 12%, respectively, less than the values obtained for a rigid core with a three-body Be^{8}-Λ-Λ model. The Σ-Λ-π coupling constant, obtained with III, is close to the value obtained from the singlet Λ-N interaction for the same hard-core radius.

1. INTRODUCTION

DISTORTION of the core nucleus by the Λ hyperons in a $\Lambda\Lambda$ hypernucleus will give an apparent binding between the two Λ particles. This must be known if reliable information about the Λ-Λ interaction is to be deduced from the experimental separation energy of both Λ particles, with respect to the ground-state energy of the core nucleus. Because of the exclusion principle, the relevant Λ-Λ interaction for the ground state is the singlet one. If B_Λ is the Λ separation energy for the Λ hypernucleus, then with a completely rigid core nucleus the "additional" binding energy $\Delta B_{\Lambda\Lambda}=B_{\Lambda\Lambda}-2B_\Lambda$ may be rather directly related to the strength of the Λ-Λ interaction. This has been done by Dalitz and Rajasekaran[1] for a Gaussian interaction with the intrinsic range $b=1.48$ F, appropriate to the two-pion-exchange mechanism, and by Tang *et al.*[2] for a hard-core interaction of almost the same intrinsic range (1.5 F). However, if the core can be distorted by the Λ particles, then, even if there were no Λ-Λ interaction, one might still get an appreciably positive value of $\Delta B_{\Lambda\Lambda}$. The core distortion will be a compression because this is energetically favored; for a rigid core, the binding due to given Λ-N and Λ-Λ interactions will increase as the core size is reduced.

The most favored candidate for the event reported by Danysz *et al.*[3] is $_{\Lambda\Lambda}Be^{10}$ with $\Delta B_{\Lambda\Lambda}=4.5\pm0.5$ MeV [$B_{\Lambda\Lambda}=17\pm0.5$ MeV, $B_\Lambda(_{\Lambda}Be^{9})=6.5\pm0.15$ MeV]. However, $_{\Lambda\Lambda}Be^{11}$ is also a possible candidate. For $_{\Lambda\Lambda}Be^{10}$, the effects of core distortion are expected to be particularly important, since the core nucleus Be^{8} is not even bound. The hypernucleus $_{\Lambda\Lambda}Be^{10}$ has been considered in Refs. 1 and 2, in which a rigid core was assumed. The effects of distortion of the Be^{8} core have been considered by Deloff[4] and by Nakamura[5] for an α-particle model. These authors reach contradictory results. Thus Nakamura finds a large effect due to core distortion, whereas Deloff obtains only a small effect.

Our approach is also based on an α-particle model for the core, but uses a better trial wave function and, in particular, improved values for the core energy. Furthermore, we consider a variety of α-α interactions and also consider both soft- and hard-core Λ-Λ interactions.

We thus consider a four-body model of $_{\Lambda\Lambda}Be^{10}$, consisting of two α particles and the two Λ particles.

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[1] R. H. Dalitz and G. Rajasekaran, Nucl. Phys. **50**, 450 (1964).

[2] Y. C. Tang, R. C. Herndon, and E. W. Schmid, Phys. Letters **10**, 358 (1964).

[3] M. Danysz *et al.*, Nucl. Phys. **49**, 121 (1963).

[4] A. Deloff, Phys. Letters **6**, 83 (1963).

[5] H. Nakamura, Phys. Letters **6**, 207 (1963).

In view of the large binding and incompressibility of an α particle, the Λ particles are assumed to affect only the relative motion of the α particles and not these individually. This model allows a dynamical treatment of the Be^8 core, and therefore also of distortion effects, and is a natural extension of a three-body (α-α-Λ) model of $_{\Lambda}Be^9$. This has been treated in detail by the authors.[6] In particular, this model is remarkably consistent with α-α potentials which give excellent agreement with the experimental s-wave α-α phase shifts. This success gives confidence in the use of an α-α-Λ-Λ model for $_{\Lambda\Lambda}Be^{10}$. The total binding energy of the α and Λ particles relative to each other will be very nearly the separation energy $B_{\Lambda\Lambda}$, since the ground-state energy of Be^8 is only 0.1 MeV.

Our calculations are based on the use of the trial function

$$\Psi=\Phi_c(R)g_{\Lambda c}(r_{c1})g_{\Lambda c}(r_{c2})g_{\Lambda\Lambda}(r_{\Lambda\Lambda})^1\chi_0. \quad (1)$$

Here r_{c1}, r_{c2}, and $r_{\Lambda\Lambda}$ are triangular coordinates: r_{c1} and r_{c2} are the Λ-core separations, and $r_{\Lambda\Lambda}$ is the Λ-Λ separation. The function $^1\chi_0$ is the singlet spin function for the Λ particles. The functions $g_{\Lambda c}(r)$ and $g_{\Lambda\Lambda}(r)$ are trial functions. As emphasized by Dalitz and Rajasekaran,[1] it is essential to include the function $g_{\Lambda\Lambda}(r)$ in order to allow for the effect of Λ-Λ correlations resulting from the short-ranged and possibly strong Λ-Λ interaction. As justified below, the trial wave function $\Phi_c(R)$ of the core may be considered as effectively depending only on one variational parameter R, which may be taken as the rms separation between the α particles.

2. CORE ENERGY

The core energy as a function of R is given by

$$E_{\alpha\alpha}(R)=\langle\Phi_c(R)\,|\,T_{\alpha\alpha}+V_{\alpha\alpha}\,|\,\Phi_c(R)\rangle\,, \quad (2)$$

where $V_{\alpha\alpha}$ denotes the α-α potential and $T_{\alpha\alpha}$ is the kinetic-energy operator. The energy $E_{\alpha\alpha}$ (R) was obtained from three-body calculations for the α-α-Λ model of $_{\Lambda}Be^9$ by use of the procedures described in Ref. 6. These are based on the use of the trial function $f_{\alpha\Lambda}(r_{\alpha_1\Lambda})f_{\alpha\Lambda}(r_{\alpha_2\Lambda})f_{\alpha\alpha}(r_{\alpha\alpha})$, in an obvious notation. Thus for a given $V_{\alpha\alpha}$ and a definite function $f_{\alpha\Lambda}(r)$ (which is taken to be a superposition of exponentials), a Schrödinger eigenvalue problem is obtained for the α-α motion which is solved exactly. This solution then gives the best variational solution for $f_{\alpha\alpha}(r)$ and the corresponding *three-body* binding energy $B_\Lambda[f_{\alpha\Lambda}]$ appropriate to the assumed function $f_{\alpha\Lambda}(r)$. The associated values of $E_{\alpha\alpha}$ and of $R=\langle r_{\alpha\alpha}^2\rangle^{1/2}$ may then also be obtained and from the latter also the rms radius $\bar{R}$ of the density distribution of the core. For a wide range of values of R, one finds the linear relation

$$\bar{R}=\sqrt{2}a=0.381(2.48+R)\,, \quad (3)$$

where a is the oscillator size parameter, defined by Eq. (7), and where R, $\bar{R}$, and a are in fermis. It is found that, to a very good approximation, $E_{\alpha\alpha}$ depends only on R. Thus for a given R, the energy $E_{\alpha\alpha}$ is almost independent of $f_{\alpha\Lambda}(r)$ and of the strength of the α-Λ interaction—for both of which a wide range was considered. For a given α-Λ interaction, the binding energy $B_\Lambda(_{\Lambda}Be^9)$ and the associated values of $R=R_\Lambda$ and of $E_{\alpha\alpha}$ which are realized for $_{\Lambda}Be^9$ are, of course, only obtained for that function $f_{\alpha\Lambda}(r)$ that maximizes the three-body energy $B_\Lambda[f_{\alpha\Lambda}]$. It thus seems very well justified to consider $E_{\alpha\alpha}=E_{\alpha\alpha}(R)$ as a function only of R and thus to consider the trial function $\Phi_c(R)$ as effectively depending only on R and, furthermore, to use for $E_{\alpha\alpha}(R)$ the results obtained from our calculations for $_{\Lambda}Be^9$. In view of Eq. (3), it is clear that $\bar{R}$ or a could equally well be used, instead of R, for the variational parameter characterizing the core.

It is convenient to expand $E_{\alpha\alpha}(R)$ about the value $R=R_\Lambda$ to obtain

$$E_{\alpha\alpha}(R)=E_{\alpha\alpha}(R_\Lambda)-\epsilon_1(R-R_\Lambda)+\epsilon_2(R-R_\Lambda)^2-\epsilon_3(R-R_\Lambda)^3. \quad (4)$$

A cubic polynomial was found to give satisfactory fits to the results for $E_{\alpha\alpha}(R)$ for all our potentials $V_{\alpha\alpha}$ and for the relevant values of R. The coefficients ϵ_1, ϵ_2, and ϵ_3 are all positive. This is a reflection of the fact that $E_{\alpha\alpha}$ increases more rapidly as R is reduced. It is to be noted that, because Be^8 is not bound, $E_{\alpha\alpha}(R)$ does not have a minimum for any value of R. For a given $V_{\alpha\alpha}$, one may obtain R_Λ as a function of B_Λ by varying the strength of the α-Λ interaction used in the three-body calculations[7] for $_{\Lambda}Be^9$. The value of R_Λ for any particular $V_{\alpha\alpha}$ was then chosen so as to give the experimental value $B_\Lambda(_{\Lambda}Be^9)=6.5$ MeV. The core energy in $_{\Lambda}Be^9$, which is effectively also the rearrangement energy, is then $E_{\alpha\alpha}(R_\Lambda)$.

The results obtained for $E_{\alpha\alpha}(R)$ are shown in Table I. The α-α potentials and the symbols used to label these are the same as those in Ref. 6. Thus the hard-core potentials a to g are defined by

$$\begin{aligned} V_{\alpha\alpha}&=\infty\,, && r<c;\\ V_{\alpha\alpha}&=-V_0\,, && c<r<d;\\ V_{\alpha\alpha}&=4e^2/r\,, && r>d. \end{aligned} \quad (5)$$

The soft repulsive-core potentials p to s are given by

$$V_{\alpha\alpha}(r)=V_R\exp(-\mu_R^2r^2)-V_A\exp(-\mu_A^2r^2)+4e^2/r. \quad (6)$$

All the potentials have a shape reasonably consistent with theoretical expectations and have all been chosen so as to give the ground-state energy (effectively zero) of Be^8 correctly. In particular, the hard-core potentials [Eq. (5)] are thus characterized by the two parameters c and d, since, if these are given, V_0 is then determined

[6] A. R. Bodmer and S. Ali, Nucl. Phys. **56**, 657 (1964).

[7] The α-Λ interaction is obtained as described in Ref. 6. The (spin-averaged) Λ-N interaction, having a Yukawa shape with a range $\mu^{-1}=0.7$ F, is folded into the α-particle density distribution.

TABLE I. Results for the core energy $E_{\alpha\alpha}(R)$.

Label	Hard-core potentials [Eq. (5)] c(F)	d(F)		V_0(MeV)	R_Λ (F)	a_Λ (F)	$E_{\alpha\alpha}(R_\Lambda)$ (MeV)	ϵ_1 (MeV F^{-1})	ϵ_2 (MeV F^{-2})	ϵ_3 (MeV F^{-3})	K (MeV)
a	0.6	4		3.68	2.816	1.427	1.385	3.225	3.525	2.40	24.71
b	1.7	4		7.20	3.627	1.646	0.836	2.634	4.536	4.656	42.29
d	2.6	4		17.24	4.17	1.792	0.580	2.381	4.973	5.325	54.98
e	1.7	3.5		11.10	3.432	1.594	0.753	2.493	4.895	5.776	42.77
f	1.7	4.5		5.19	3.792	1.691	0.890	2.766	3.974	3.013	39.08
g	1.7	5.0		4.04	3.932	1.728	0.932	2.821	3.424	2.125	35.19
	Soft-core potentials [Eq. (6)]										
	μ_A(F^{-1})	V_A(MeV)	μ_R(F^{-1})	V_R(MeV)							
p	0.341	30	0.637	96	3.423	1.591	0.598	2.825	3.205	4.304	27.92
q	0.475	160	0.6	300	3.480	1.606	1.216	2.914	4.547	6.342	40.38
r	0.475	160	0.7	750	3.737	1.676	1.035	2.765	4.692	5.395	45.34
s	0.420	165	0.6	700	4.162	1.790	0.594	2.502	2.833	0.865	31.25

by the condition that the ground-state energy of Be^8 be given correctly. The s-wave α-α phase shifts for all our potentials are given in Ref. 6, up to about 12 MeV.

The coefficient K, whose values are given in Table I, is defined (in analogy with the usual compressibility coefficient) by $K=A^{-1}\bar{R}^2(d^2E_{\alpha\alpha}/d\bar{R}^2)_{R=R_\Lambda}$, where $A=8$ is the mass number of the core. Thus K has been obtained from ϵ_2 with the help of Eq. (3). It is interesting that the values of K are comparable to, although somewhat smaller than, the values ($\gtrsim 100$ MeV) typically quoted for the compressibility coefficient of nuclear matter. Of course, our K is not really comparable with this, in particular, because $E_{\alpha\alpha}(R)$ does not have a minimum since Be^8 is unbound.

3. ΛΛ-BINDING ENERGY EXCLUSIVE OF THE CORE ENERGY

For a given value of R, the binding energy $b_{\Lambda\Lambda}$, which is specifically due to the interactions of the Λ particles with each other and with the Be^8 core, was obtained with the trial function of Eq. (1). The total binding energy is $B_{\Lambda\Lambda}=b_{\Lambda\Lambda}-E_{\alpha\alpha}$, and the energy $b_{\Lambda\Lambda}$ is thus the binding energy exclusive of the internal core energy and would be the actual separation energy for a rigid core whose size is characterized by R. The three-body trial function $g_{\Lambda c}(r_{c1})g_{\Lambda c}(r_{c2})g_{\Lambda\Lambda}(r_{\Lambda\Lambda})$ is thus used to obtain $b_{\Lambda\Lambda}$, the size parameter R of the core entering only through the Λ-core potential $V_{\Lambda c}$. This was obtained by folding a Gaussian Λ-N interaction, with an intrinsic range $b=1.48$ F appropriate to the two-pion-exchange mechanism, into the core-density distribution.

For this, an harmonic-oscillator density distribution was used appropriate to four $1s$ and four $1p$ nucleons. Normalized to unity, this is

$$\rho(r)=\frac{1}{2\pi^{3/2}a^3}\left[1+\frac{2}{3}\frac{r^2}{a^2}\right]\exp\left[-\frac{r^2}{a^2}\right], \qquad (7)$$

where a is the oscillator-size parameter and the corresponding value of R is given by Eq. (3). The potential $V_{\Lambda c}$ is then

$$V_{\Lambda c}(r)=-\frac{U_8}{2\pi^{3/2}a'^3}\left[1+\frac{(a'^2-a^2)}{a'^2}+\frac{2}{3}\frac{a^2r^2}{a'^4}\right]\exp\left(-\frac{r^2}{a'^2}\right), \qquad (8)$$

with $a'^2=a^2+x^2$, where x is the Gaussian range of the Λ-N interaction ($x=0.697$ b) and the relevant volume integral U_8 is eight times the spin-averaged volume integral of the Λ-N interaction.

By numerical solution of the Λ-core eigenvalue problem, the value $U_8=1990.1$ MeV F^3 was chosen to give a binding energy of 6.5 MeV for a single Λ to the core for $a=1.65$ F ($R=3.64$ F). This value of a is close to the values obtained from three-body calculations for $_\Lambda Be^9$, with potentials $V_{\alpha\alpha}$ which give a satisfactory fit to the α-α phase shifts. As will become clear below, the precise value of a is not important so long as $V_{\Lambda c}$ gives approximately the correct binding for a single Λ. For other (neighboring) values of a ($a=1.5$ and 1.6 F), the potential $V_{\Lambda c}$ is given by Eq. (8) with the above value of U_8 which is obtained for $a=1.65$ F. This corresponds to a fixed Λ-N interaction as is appropriate for our subsequent calculation of $b_{\Lambda\Lambda}$ as a function of R.

Calculations for $_{\Lambda\Lambda}Be^{10}$ were made for two types of Λ-Λ interactions. The first, denoted by I, is a purely attractive Yukawa potential

$$V_{\Lambda\Lambda}(r)=-W_{\Lambda\Lambda}(e^{-\mu r}/\mu r), \qquad (I)$$

with $\mu^{-1}=\mu_{2\pi}^{-1}=0.7$ F, appropriate to the two-pion-exchange mechanism. Its intrinsic range is 1.48 F. The interaction I may also be characterized by its volume integral $U_{\Lambda\Lambda}$. The second interaction II is a hard-core Yukawa potential

$$\begin{aligned} V_{\Lambda\Lambda}(r)&=\infty && \text{for } r<r_c \\ &=-W_{\Lambda\Lambda}(e^{-\mu r}/\mu r) && \text{for } r>r_c, \end{aligned} \qquad (II)$$

also with $\mu^{-1}=0.7$ F and with $r_c=0.3\,\mu_\pi^{-1}=0.42$ F. The intrinsic range of II is 2.66 F.

Some results will also be given for the Λ-Λ potential obtained from meson theory for even ΣΛ parity. A hard core of radius $r_c=0.3\mu_\pi^{-1}$ was used. For the attractive part of the potential, which is due to the exchange of two pions, the static meson-theory expressions given by de Swart[8] and by de Swart and Iddings[9] were used. We have neglected the coupling with the ΣΣ channel. This is well justified for the *singlet* Λ-Λ interaction, if $f_{\Sigma\Sigma}$ is not too large ($\lesssim 0.1$). The attractive part is then proportional to $f_{\Sigma\Lambda}{}^4$, where $f_{\Sigma\Lambda}$ is the Σ-Λ-π coupling constant. This hard-core potential, denoted by III, has an intrinsic range which is very nearly the same as that of the Yukawa potential I.

The numerical calculation of $b_{\Lambda\Lambda}$ was made with the three-body method of Ref. 6 in a manner analogous to the one described there for $_{\Lambda}Be^9$. Its application to the present problem therefore needs only to be briefly sketched. Thus, for any given $V_{\Lambda\Lambda}$ and $V_{\Lambda c}$, one obtains a Schrödinger equation for the Λ-Λ motion with the effective Λ-Λ potential $V_{\Lambda\Lambda}+V_{\Lambda\Lambda}{}^{(3)}[g_{\Lambda c},V_{\Lambda c}]$, where $V_{\Lambda\Lambda}{}^{(3)}$ is a functional of $g_{\Lambda c}$ and $V_{\Lambda c}$ and represents the effects due to the presence of the third particle, i.e., the core. Then the eigenvalue obtained as a solution of this Schrödinger eigenvalue problem is the value of $b_{\Lambda\Lambda}[g_{\Lambda c},V_{\Lambda c}]$ corresponding to the best variational function $g_{\Lambda\Lambda}$ appropriate to a *given* function $g_{\Lambda c}$.

The potentials $V_{\Lambda c}$, obtained from Eq. (8), are fitted by a superposition of exponentials. For $B_\Lambda=6.5$ MeV, the Λ-core eigensolutions for the fitted potentials are then found to give almost the same values of U_8 (within 0.05%) as are obtained from the eigensolutions for the original potentials $V_{\Lambda c}$. For $g_{\Lambda c}$, the three-parameter trial function used is

$$g_{\Lambda c}(r)=e^{-\alpha r}+se^{-\beta r}. \qquad (9)$$

This is expected to be an excellent trial function. Thus a variational calculation for the *two-body* Λ-core problem with the fitted potential and with the trial function of Eq. (9) gives an energy which is within 1% of the exact value obtained by numerical solution of the eigenvalue problem for the original potential $V_{\Lambda c}$.

With the fitted potentials and with $f_{\Lambda c}(r)$ of the form of Eq. (9), one then gets an algebraic expression for $V_{\Lambda\Lambda}{}^{(3)}$, which, in particular, depends on the variational parameters α, β, and *s*. Numerical solution of the Λ-Λ eigenvalue problem with the potential $V_{\Lambda\Lambda}+V_{\Lambda\Lambda}{}^{(3)}$ then gives the three-body binding energy $b_{\Lambda\Lambda}(\alpha,\beta,s;R,W_{\Lambda\Lambda})$. The maximum of this as a function of the variational parameters α, β, *s* then gives the required result $b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})$, which now depends only on *R* (through the potential $V_{\Lambda c}$ for a given Λ-*N* interaction) and on the strength $W_{\Lambda\Lambda}$ for a given shape of $V_{\Lambda\Lambda}$. The method is very accurate not only for a "soft" Λ-*N* potential but also for one with a hard core, since it treats the Λ-Λ correlation exactly.

The results obtained for $b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})$ as a function of *R* may be represented by the quadratic expression

$$b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})=b_{\Lambda\Lambda}-b_1(R-R_\Lambda)+b_2(R-R_\Lambda)^2, \qquad (10)$$

where $b_{\Lambda\Lambda}$, b_1, and b_2 are functions of $W_{\Lambda\Lambda}$.

The energy difference

$$\Delta b_{\Lambda\Lambda}=b_{\Lambda\Lambda}(R_\Lambda,W_{\Lambda\Lambda})-2b_\Lambda(R_\Lambda) \qquad (11)$$

is the energy difference $\Delta B_{\Lambda\Lambda}$ appropriate to a rigid core. Results for $\Delta b_{\Lambda\Lambda}$ for $_{\Lambda\Lambda}Be^{10}$ with $a=1.65$ F are shown in Tables II and III for the interactions I and II, respectively. The associated values of the scattering length $a_{\Lambda\Lambda}$ are also shown. For obtaining $\Delta b_{\Lambda\Lambda}$ from our results for $b_{\Lambda\Lambda}$, we used the value $b_\Lambda=6.455$ MeV obtained from a two-body variational calculation of the Λ-core binding energy. This calculation used the trial function of Eq. (9) for $a=1.65$ F and the value of U_8 given above (which gives 6.5 MeV with the exact eigenvalue solution). The results of the variational calculations of $b_{\Lambda\Lambda}$ and b_Λ should then be strictly comparable. In particular, the value of $\Delta b_{\Lambda\Lambda}(V_{\Lambda\Lambda}=0)=0.14$ MeV $[b_{\Lambda\Lambda}(V_{\Lambda\Lambda}=0)=13.05$ MeV$]$ will be given quite accurately, since for $V_{\Lambda\Lambda}=0$ the optimum function $g_{\Lambda c}(\alpha=0.725\ \text{F}^{-1}, \beta=1.60\ \text{F}^{-1}, s=-0.594)$ is quite close to the optimum function for the two-body Λ-core problem ($\alpha=0.75\ \text{F}^{-1}$, $\beta=1.80\ \text{F}^{-1}$, $s=-0.6285$).

As pointed out in Ref. 1, the small positive value of $\Delta b_{\Lambda\Lambda}$ for $V_{\Lambda\Lambda}=0$ arises because the Λ particles are correlated as a result of the finite mass of the core.[10] Thus, roughly, the reduced mass effective for each Λ is increased by the presence of the other Λ, with a consequent reduction in the kinetic energy. (The value of $\Delta b_{\Lambda\Lambda}$ obtained for $V_{\Lambda\Lambda}=0$ is, in fact, roughly consistent with an estimate in which the reduced mass used for the second Λ corresponds to a core of mass M_c+M_Λ and the kinetic energy of a single Λ is taken to be $T_\Lambda=8.4$ MeV. This estimate gives $\Delta b_{\Lambda\Lambda}=0.24$ MeV.)

Tables II and III also show the rigid-core results $\Delta b_{\Lambda\Lambda}$ for $_{\Lambda\Lambda}He^6$. These were obtained in the same way as for $_{\Lambda\Lambda}Be^{10}$ for a Gaussian α-particle density distribution with rms radius $\bar{R}_\alpha=1.44$ F and for a Yukawa Λ-*N* interaction of range $\mu^{-1}=0.7$ F. For a Λ separation energy $B_\Lambda(_{\Lambda}He^5)=3.1$ MeV, the corresponding variational result with a trial function of the form (9) is $b_\Lambda=3.05$ MeV, and with this one gets $\Delta b_{\Lambda\Lambda}(V_{\Lambda\Lambda}=0)=0.28$ MeV. This is, as expected, about twice the value for $_{\Lambda\Lambda}Be^{10}$.

The results for $\Delta b_{\Lambda\Lambda}$ are seen to be quite similar for $_{\Lambda\Lambda}Be^{10}$ and $_{\Lambda\Lambda}He^6$. This is in agreement with the conclusions of Dalitz and Rajasekaran.[1] Furthermore, very nearly the same results as those in Table I were obtained in our calculation of $b_{\Lambda\Lambda}$ for both $_{\Lambda\Lambda}He^6$ and $_{\Lambda\Lambda}Be^{10}$ with different core radii ($a=1.5$ F for Be^8 and $\bar{R}_\alpha=1.54$ F for He^4) and with $V_{\Lambda c}$ of the corresponding strengths to give the correct values of B_Λ. It is clear that $\Delta b_{\Lambda\Lambda}$ is

[8] J. J. de Swart, Phys. Letters **5**, 58 (1963).
[9] J. J. de Swart and C. Iddings, Phys. Rev. **128**, 2810 (1962).

[10] If the core is infinitely massive then one has, as expected, the relation $b_{\Lambda\Lambda}=2b_\Lambda$ for $V_{\Lambda\Lambda}=0$, as has been shown in Ref. 6. Numerical calculations for $_{\Lambda\Lambda}He^6$ with $M_c=\infty$ reproduce this result.

rather insensitive to the details of the core size as well as to the value of B_Λ.

Our results for the Yukawa interaction I are in good agreement with those obtained by Dalitz and Rajasekaran[1] for a Gaussian interaction of the same intrinsic range. In particular, the results for $a_{\Lambda\Lambda}$ as a function of $b_{\Lambda\Lambda}$ are in good agreement. However, for a given value of $\Delta b_{\Lambda\Lambda}$, the values of $a_{\Lambda\Lambda}$ for the hard-core Yukawa potential II are considerably larger than for I. This is a reflection of the larger intrinsic range of II.

Calculations for ${}_{\Lambda\Lambda}He^6$ have also been made with the meson-theory potential III. The results are in good agreement with those of Tang *et al.*[2] who used a hard-core potential with only a slightly greater intrinsic range and a slightly smaller hard-core radius than for our potential III. Furthermore, the results for $a_{\Lambda\Lambda}$ as a function of $\Delta b_{\Lambda\Lambda}$ are quite similar to those obtained for I. Thus for a given intrinsic range the *scattering lengths* as a function of $\Delta b_{\Lambda\Lambda}$ are insensitive to the shape of $V_{\Lambda\Lambda}$.

The results for III are shown in Table II, which gives the values of $f_{\Sigma\Lambda}$ appropriate to the relevant values of $\Delta b_{\Lambda\Lambda}({}_{\Lambda\Lambda}He^6)$. The resulting relation between $U_{\Lambda\Lambda}$ and $f_{\Sigma\Lambda}$ is very nearly the same as is obtained for ${}_{\Lambda\Lambda}Be^{10}$, if the relation between $f_{\Sigma\Lambda}$ and $\Delta b_{\Lambda\Lambda}({}_{\Lambda\Lambda}He^6)$ is adjusted to ${}_{\Lambda\Lambda}Be^{10}$ with the aid of the results for I which were obtained for both ${}_{\Lambda\Lambda}He^6$ and ${}_{\Lambda\Lambda}Be^{10}$. Table II also shows the values of $a_{\Lambda\Lambda}$ for the corresponding values of $f_{\Sigma\Lambda}$ for the potential III.

The coefficient b_1 in Eq. (10) is positive, since the binding increases as the core size decreases. For the Yukawa interaction I, the coefficient b_1 as a function of $U_{\Lambda\Lambda}$ (in MeV F^3) is given by

$$b_1(U_{\Lambda\Lambda}) = b_1^{(0)}[1+5.54\times10^{-4}U_{\Lambda\Lambda} - 2.55\times10^{-7}U_{\Lambda\Lambda}^2]. \quad (12a)$$

For the interaction II, the expression obtained for b_1 as a function of the strength $W_{\Lambda\Lambda}$ (in MeV) is

$$b_1(W_{\Lambda\Lambda}) = b_1^{(0)}[1+3.31\times10^{-4}(W_{\Lambda\Lambda}-W_{\Lambda\Lambda}^{(0)}) + 3.66\times10^{-6}(W_{\Lambda\Lambda}-W_{\Lambda\Lambda}^{(0)})^2], \quad (12b)$$

where $W_{\Lambda\Lambda}^{(0)} = 140$ MeV is the strength which gives the value $\Delta b_{\Lambda\Lambda} = 0.14$ MeV appropriate to $V_{\Lambda\Lambda} = 0$. This value of $W_{\Lambda\Lambda}^{(0)}$ is close to that which gives $a_{\Lambda\Lambda} = 0$. The precise value which is used for $W_{\Lambda\Lambda}^{(0)}$ is, in fact, not very important. Both (12a) and (12b) thus correspond to expansions in the strength of the interaction about the strength appropriate to $V_{\Lambda\Lambda} = 0$. The value of $b_1^{(0)}$ which, as expected, is very nearly the same for both (12a) and (12b), is $b_1^{(0)} = 7.2$ MeV F^{-1}.

However, the values which we have used for $b_1^{(0)}$ are, in fact, somewhat different from this; they are smaller and give rise to smaller distortion effects. They include allowance for the fact that the dependence of $b_{\Lambda\Lambda}(R, W_{\Lambda\Lambda})$ on R should be consistent with an α-particle description for the core. This will modify $b_1^{(0)}$ from the above value, which is obtained on the assumption that the core has no structure and is represented merely by a density distribution of the appropriate mass.

Thus if one uses for ${}_\Lambda Be^9$ the wave function $\Phi_c(R) \times F(r_{\Lambda c})$, which is the analog of Eq. (1) for ${}_{\Lambda\Lambda}Be^{10}$,

TABLE II. Results for the Yukawa interaction I. All energies are in MeV, the lengths $a_{\Lambda\Lambda}$ and δR are in F.

	Rigid-core results[a]						α-α potential[a]									
$U_{\Lambda\Lambda}$ (MeV F^3)	$a_{\Lambda\Lambda}$	$\Delta b_{\Lambda\Lambda}$ (${}_{\Lambda\Lambda}He^6$)	$\Delta b_{\Lambda\Lambda}$ (${}_{\Lambda\Lambda}Be^{10}$)	$f_{\Sigma\Lambda}$	$a_{\Lambda\Lambda}$		*a*	*b*	*d*	*e*	*f*	*g*	*p*	*q*	*r*	*s*
−100	0.2	−0.43	−0.51	0.2386	0.1	$\Delta B_{\Lambda\Lambda}$	1.69	0.67	0.32	0.52	0.86	1.06	0.63	1.10	0.88	0.84
						$E_{\Lambda\Lambda}$	0.82	0.34	0.25	0.27	0.48	0.64	0.54	0.39	0.355	0.76
						$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	4.01	1.905	1.355	1.595	2.425	3.02	2.20	2.39	2.13	3.30
						δR	0.48	0.26	0.21	0.22	0.34	0.44	0.36	0.26	0.255	0.60
0	0.02	0.28	0.14	0.2466	−0.1	$\Delta B_{\Lambda\Lambda}$	2.54	1.405	1.03	1.23	1.63	1.87	1.41	1.84	1.62	1.69
						$E_{\Lambda\Lambda}$	1.01	0.43	0.31	0.34	0.60	0.80	0.67	0.48	0.44	0.95
						$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	4.38	2.06	1.47	1.72	2.65	3.32	2.41	2.55	2.29	3.74
						δR	0.52	0.28	0.23	0.24	0.38	0.48	0.385	0.28	0.28	0.66
100	−0.23	1.27	1.005	0.2551	−0.41	$\Delta B_{\Lambda\Lambda}$	3.65	2.37	1.97	2.175	2.64	2.93	2.42	2.81	2.58	2.81
						$E_{\Lambda\Lambda}$	1.26	0.53	0.38	0.42	0.74	0.99	0.82	0.59	0.54	1.21
						$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	4.88	2.24	1.60	1.855	2.93	3.73	2.67	2.74	2.47	4.40
						δR	0.58	0.31	0.255	0.26	0.41	0.53	0.42	0.31	0.305	0.75
200	−0.62	2.56	2.205	0.2638	−0.89	$\Delta B_{\Lambda\Lambda}$	5.17	3.68	3.245	3.46	4.01	4.38	3.80	4.135	3.90	4.38
						$E_{\Lambda\Lambda}$	1.58	0.64	0.46	0.50	0.91	1.24	0.99	0.71	0.66	1.58
						$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	5.59	2.465	1.755	2.02	3.30	4.30	3.00	2.96	2.69	5.43
						δR	0.65	0.34	0.28	0.29	0.46	0.60	0.46	0.34	0.335	0.88
300	−1.43	4.50	3.86	0.2723	−1.73	$\Delta B_{\Lambda\Lambda}$	7.24	5.465	4.99	5.21	5.88	6.36	5.67	5.93	5.69	6.57
						$E_{\Lambda\Lambda}$	2.00	0.77	0.55	0.60	1.13	1.57	1.21	0.86	0.79	2.12
						$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	6.61	2.75	1.94	2.22	3.80	5.11	3.44	3.25	2.97	7.07
						δR	0.73	0.38	0.31	0.32	0.52	0.68	0.51	0.37	0.37	1.025

[a] To obtain the values of $B_{\Lambda\Lambda}$ from those for $\Delta B_{\Lambda\Lambda}$ the value $B_\Lambda = 6.455$ MeV should be used. For the results for ${}_{\Lambda\Lambda}He^6$ the value $b_\Lambda = 3.05$ MeV should be used.

TABLE III. Results for the hard-core Yukawa interaction II. Energies are in MeV, the difference δR is in F.

$W_{\Lambda\Lambda}$ (MeV)	$a_{\Lambda\Lambda}$ (F)	Rigid-core results[a] $\Delta b_{\Lambda\Lambda}(_{\Lambda\Lambda}He^5)$	$\Delta b_{\Lambda\Lambda}(_{\Lambda\Lambda}Be^{10})$		α-α potential[a] *a*	*b*	*d*	*e*	*f*	*g*	*p*	*q*	*r*	*s*
100	0.06	−0.75	−0.58	$\Delta B_{\Lambda\Lambda}$	1.62	0.63	0.28	0.475	0.82	1.00	0.58	1.06	0.84	0.74
				$E_{\Lambda\Lambda}$	0.815	0.37	0.275	0.30	0.50	0.65	0.565	0.42	0.39	0.72
				$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	3.74	1.89	1.36	1.60	2.35	2.83	2.12	2.37	2.12	2.87
				δR	0.45	0.26	0.21	0.22	0.33	0.41	0.34	0.26	0.25	0.53
140	−0.33	0.00	0.14	$\Delta B_{\Lambda\Lambda}$	2.42	1.375	1.01	1.21	1.58	1.78	1.35	1.81	1.59	1.54
				$E_{\Lambda\Lambda}$	0.89	0.40	0.29	0.32	0.545	0.71	0.61	0.45	0.41	0.81
				$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	3.97	1.96	1.41	1.65	2.46	3.01	2.23	2.445	2.19	3.16
				δR	0.475	0.27	0.22	0.23	0.35	0.44	0.36	0.27	0.26	0.58
150	−0.46	0.21	0.33	$\Delta B_{\Lambda\Lambda}$	2.63	1.57	1.21	1.41	1.78	1.99	1.55	2.01	1.79	1.76
				$E_{\Lambda\Lambda}$	0.92	0.41	0.30	0.33	0.56	0.73	0.62	0.46	0.42	0.835
				$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	4.03	1.98	1.42	1.66	2.495	3.06	2.26	2.47	2.21	3.24
				δR	0.48	0.27	0.22	0.23	0.35	0.45	0.365	0.27	0.27	0.59
250	−2.03	2.58	2.62	$\Delta B_{\Lambda\Lambda}$	5.26	4.00	3.595	3.805	4.265	4.55	4.05	4.45	4.22	4.395
				$E_{\Lambda\Lambda}$	1.26	0.55	0.40	0.435	0.76	1.00	0.83	0.61	0.565	1.185
				$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	4.72	2.235	1.61	1.86	2.89	3.62	2.63	2.73	2.465	4.10
				δR	0.56	0.31	0.26	0.26	0.41	0.52	0.415	0.31	0.305	0.71
365	−21.51	6.63	6.35	$\Delta B_{\Lambda\Lambda}$	9.61	8.03	7.56	7.79	8.40	8.79	8.20	8.51	8.26	8.71
				$E_{\Lambda\Lambda}$	1.87	0.845	0.625	0.68	1.16	1.50	1.24	0.94	0.87	1.76
				$E_{\alpha\alpha}(R_{\Lambda\Lambda})$	5.49	2.63	1.91	2.18	3.42	4.28	3.11	3.14	2.86	4.92
				δR	0.64	0.365	0.30	0.31	0.475	0.60	0.47	0.36	0.36	0.81

[a] See footnote a to Table II.

then, for the Λ separation energy as a function of the variational parameter R, one has

$$B_\Lambda(R)=b_\Lambda(R)-E_{\alpha\alpha}(R)\,, \qquad (13)$$

where $b_\Lambda(R)$ is obtained by solving the two-body Λ-core Schrödinger equation for the wave function $F(r)$ with the potential $V_{\Lambda c}$ (which depends on R). For $R=R_\Lambda$, the value of $B_\Lambda(R)$ must then be a maximum and one must have $(dB_\Lambda/dR)_{R=R_\Lambda}=0$ and thus $(db_\Lambda/dR)_{R=R_\Lambda}=(dE_{\alpha\alpha}/dR)_{R=R_\Lambda}$ which, by Eq. (4), is equal to $-\epsilon_1$.

Further, one expects that $b_1^{(0)}=-2(db_\Lambda/dR)_{R=R_\Lambda}$ to a very good approximation. This has, in fact, been checked by explicit calculation. Thus, with the aid of Eq. (3) one obtains $(db_\Lambda/dR)_{R=R_\Lambda}=-3.63$ MeV F^{-1} for $a=1.65$ F. It will then be seen that the values of ϵ_1 in Table I are somewhat, although not much, less than $\frac{1}{2}b_1^{(0)}$. This difference is then to be understood as due to the α-particle structure of the core. For any given $V_{\alpha\alpha}$, we have therefore used

$$b_1^{(0)}=2\epsilon_1\,. \qquad (14)$$

The values of $b_1^{(0)}$ now, of course, depend somewhat on the potential $V_{\Lambda\Lambda}$.

This modification then ensures that the Λ-core system stabilizes at the appropriate value of R_Λ, which is obtained from the three-body calculations for $_\Lambda Be^9$. This is now consistent with our use of the trial wave function (1) and in particular with our core energies $E_{\alpha\alpha}(R)$. Use of Eq. (14) thereby allows for the effect of the α-particle structure of the core on the R dependence of $b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})$.

For $b_2^{(0)}=b_2(V_{\Lambda\Lambda}=0)$, we obtain 2.63 MeV F^{-2} for I and 2.10 MeV F^{-2} for II; for the corresponding ratios $b_2^{(0)}/b_1^{(0)}$, the values are 0.36 F^{-1} and 0.285 F^{-1}. We then use these ratios together with Eq. (14), i.e., we use $b_2^{(0)}=0.72\,\epsilon_1$ and $0.57\,\epsilon_1$ for I and II, respectively. The dependence on the strength of $V_{\Lambda\Lambda}$ is given by

$$b_2=b_2^{(0)}[1+5.54\times10^{-4}U_{\Lambda\Lambda}-2.55\times10^{-7}U_{\Lambda\Lambda}^2] \qquad (15a)$$

for I, and by

$$b_2=b_2^{(0)}[1+3.19\times10^{-3}(W_{\Lambda\Lambda}-W_{\Lambda\Lambda}^{(0)}) -1.28\times10^{-5}(W_{\Lambda\Lambda}-W_{\Lambda\Lambda}^{(0)})^2] \qquad (15b)$$

for II. In fact, the curvature of $b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})$ with respect to R turns out to be fairly small, and, consequently, the core distortion is rather little affected by b_2. Hence, the precise values used for b_2 are not too important. Thus our final results are not much changed if the dependence of b_2 on $W_{\Lambda\Lambda}$ is neglected, or even if b_2 is neglected entirely.

4. ΛΛ BINDING ENERGY AND THE Λ-Λ INTERACTION

The binding energy of both Λ particles obtained with the trial function of Eq. (1) is finally given as a function of the remaining variational parameter R by

$$B_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})=b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})-E_{\alpha\alpha}(R)\,. \qquad (16)$$

Maximization of this with respect to R then gives the value of $B_{\Lambda\Lambda}=B_{\Lambda\Lambda}(R_{\Lambda\Lambda},W_{\Lambda\Lambda})$ together with the core radius $R_{\Lambda\Lambda}$ and the core energy $E_{\alpha\alpha}(R_{\Lambda\Lambda})$ for $_{\Lambda\Lambda}Be^{10}$. The energy difference $\Delta B_{\Lambda\Lambda}$, which is of principal interest, is given by

$$\Delta B_{\Lambda\Lambda}=B_{\Lambda\Lambda}(R_{\Lambda\Lambda},W_{\Lambda\Lambda})-2B_\Lambda(R_\Lambda)\,, \qquad (17)$$

where $B_\Lambda(R_\Lambda)$ is given by Eq. (13) with the values of the core energy $E_{\alpha\alpha}(R_\Lambda)$ for $_\Lambda Be^9$ given in Table I.

It is instructive to write $\Delta B_{\Lambda\Lambda}$ in the form

$$\Delta B_{\Lambda\Lambda}=\Delta b_{\Lambda\Lambda}+E_{\alpha\alpha}(R_\Lambda)+E_{\Lambda\Lambda}. \quad (18)$$

The energy difference $\Delta b_{\Lambda\Lambda}$, given by Eq. (11), has already been discussed. It is the only nonvanishing contribution for a rigid core. The contribution to $\Delta B_{\Lambda\Lambda}$ due to core distortion is thus $E_{\Lambda\Lambda}(R_\Lambda)+E_{\Lambda\Lambda}$, where

$$E_{\Lambda\Lambda}=[b_{\Lambda\Lambda}(R_{\Lambda\Lambda},W_{\Lambda\Lambda})-b_{\Lambda\Lambda}(R_\Lambda,W_{\Lambda\Lambda})] -[E_{\alpha\alpha}(R_{\Lambda\Lambda})-E_{\alpha\alpha}(R_\Lambda)]. \quad (19)$$

The energy $E_{\Lambda\Lambda}$ is thus the gain in binding energy which arises from the *additional* core distortion due to the second Λ. It is to be noted that even if there were no such further distortion, i.e., if $R_{\Lambda\Lambda}=R_\Lambda$ and thus $E_{\Lambda\Lambda}=0$, then, in addition to $\Delta b_{\Lambda\Lambda}$, there would still be the contribution from the core energy $E_{\alpha\alpha}(R_\Lambda)$ of $_\Lambda Be^9$. This is because, from Eq. (13), the term $2B_\Lambda$ in Eq. (17) brings $E_{\alpha\alpha}(R_\Lambda)$ in twice, whereas, if there were no further distortion by the second Λ, the core energy of $_{\Lambda\Lambda}Be^{10}$ would be just $E_{\alpha\alpha}(R_\Lambda)$.

The results obtained by maximizing $B_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})$ are shown in Tables II and III for the interactions I and II, respectively; $\delta R=R_\Lambda-R_{\Lambda\Lambda}$ is the decrease in R when the second Λ is added. The values of $\Delta B_{\Lambda\Lambda}$ differ appreciably, although not too much, between different $V_{\alpha\alpha}$, except for the potential a, which has far too little repulsion and gives α-α phase shifts which are very considerably larger than the experimental ones. In this connection, it is to be remembered that all our potentials give the ground-state resonance energy of Be^8 correctly and that R_Λ for each $V_{\Lambda\Lambda}$ was chosen so as to give $B_\Lambda(_\Lambda Be^9)$ correctly. For the same values of $\Delta b_{\Lambda\Lambda}$, the results for the core distortion are quite similar for I and II, the distortion being slightly smaller for the hard-core interaction II.

For a given $V_{\alpha\alpha}$, the additional distortion energy $E_{\Lambda\Lambda}$ increases only slowly with the strength of $V_{\Lambda\Lambda}$ and is mostly somewhat smaller than $E_{\alpha\alpha}(R_\Lambda)$, although the core energy $E_{\alpha\alpha}(R_{\Lambda\Lambda})$ is mostly substantially larger than $E_{\alpha\alpha}(R_\Lambda)$. The total distortion energy $E_{\Lambda\Lambda}+E_{\alpha\alpha}(R_\Lambda)$ thus also increases only slowly with the strength of $V_{\Lambda\Lambda}$ and is in the region of 1.5−2 MeV, i.e., it is rather more than one-third of the experimental value $\Delta B_{\Lambda\Lambda}\approx 4.5$ MeV.

The second Λ causes a quite appreciable radial compression of the core. Thus the decrease δR is in the region of 10% of R_Λ, except for potentials a and s for which it is considerably larger. Clearly, if the values of R_Λ and/or $R_{\Lambda\Lambda}$ are less than about twice the rms radius of the α particle (as is the case for potential a and to a lesser extent for potential p, both of which have too little repulsion), then an α-particle model for the core cannot be expected to have too much validity because of the large overlap of the two α particles.

For the hard-core α-α potentials [Eq. (5)], our results may be considered as a function of only the hard-core radius c, the outer square-well radius d being kept fixed—or conversely. Thus for the sequence of potentials a, b, d, the outer radius $d=4$ F is kept fixed and c increases from 0.6 to 2.6 F, while for g, f, b, and e the hard-core radius $c=1.7$ F is fixed and d decreases from 5 to 3.5 F. For both these sequences, there is thus an effective increase in the repulsive part with a corresponding decrease in the s-wave α-α phase shifts.

Consequently, both sequences correspond to a decreasing core compressibility and, correspondingly, as may be seen from Table I, ϵ_2, ϵ_3, and R_Λ increase along these sequences, while $E_{\alpha\alpha}(R_\Lambda)$ and ϵ_1, and therefore also b_1 and b_2, decrease. If b_1 and b_2 decrease, while at the same time ϵ_2 and ϵ_3 increase, then core distortion will become less. This is, in fact, the case for both sequences and, in particular, our results for a given $V_{\Lambda\Lambda}$ have a reasonable behavior as a function of c or d.

It is interesting to observe that, for a given $V_{\Lambda\Lambda}$, the energy difference $\Delta B_{\Lambda\Lambda}$ is reasonably sensitive, in particular, to c for fixed d. This would allow, in principle, a test of the α-α-Λ-Λ model of $_{\Lambda\Lambda}Be^{10}$, if $V_{\Lambda\Lambda}$ were reasonably well known, for example, from $_{\Lambda\Lambda}He^6$, for which distortion effects are expected to be much less important than for $_{\Lambda\Lambda}Be^{10}$.

Of course, experimental errors in $B_{\Lambda\Lambda}$ would make any such conclusion correspondingly uncertain. Thus if, for example, one had $\Delta B_{\Lambda\Lambda}(_{\Lambda\Lambda}He^6)=2.5\pm0.5$ MeV, then one would get for the interaction I the value $U_{\Lambda\Lambda}=200\pm35$ MeV F^3. With $\Delta B_{\Lambda\Lambda}(_{\Lambda\Lambda}Be^{10})=4.5\pm0.5$ MeV, one then gets $c=1.0_{-0.5}^{+1.0}$ F for $d=4$ F. The errors are too large for this value of c to be a significant result, and it would certainly be consistent with a $V_{\alpha\alpha}$ which gives reasonable phase shifts. If there were no errors in the value $\Delta B_{\Lambda\Lambda}(_{\Lambda\Lambda}He^6)=2.5$ MeV, then one would get $c=1.0\pm0.4$ F. This would be more significant since it is somewhat, although not too much, on the small side and would correspond to rather too little repulsion for $V_{\alpha\alpha}$. The corresponding results for II are quite similar.

We now discuss the results obtained for the Λ-Λ interaction. For the hard-core α-α potentials, an acceptable fit to the experimental α-α phase shifts is obtained with $d\approx4.0\pm0.5$ F, $c=1.7$ F, and with $c\approx1.7\pm0.3$ F, $d=4$ F; the potential b (that of Van der Spuy and Pienaar[11]) gives about the best over-all agreement. The considerations of Ref. 6 for $_\Lambda Be^9$, where the Λ is regarded as a nuclear probe (realized by assuming that the α-Λ potential for $_\Lambda Be^9$ is the same as that obtained from $_\Lambda He^5$), give α-α potentials consistent with these but within narrower limits, namely, $c=1.77\pm0.1$ F, $d=4$ F and $d=4.17\pm0.2$ F, $c=1.7$ F. For the soft-core potentials [Eq. (6)] both q and r give reasonable phase shifts, and analysis of $_\Lambda Be^9$ suggests that a best potential would be one with an intermediate range

[11] E. Van der Spuy and H. J. Pienaar, Nucl. Phys. 7, 397 (1958).

TABLE IV. Results for the Λ-Λ interaction.

Λ-Λ potential		Strength[a]	$a_{\Lambda\Lambda}$ (F)	$r_{0\Lambda\Lambda}$ (F)	$s_{\Lambda\Lambda}$	$E_{\alpha\alpha}(R_\Lambda)+E_{\Lambda\Lambda}$ (MeV)
I	rigid core	330±25	$-(1.79_{-0.30}^{+0.44})$	$2.54_{-0.20}^{+0.24}$	0.64 ±0.05	
	with core distortion	233±43	$-(0.89_{-0.28}^{+0.31})$	$3.70_{-0.58}^{+0.82}$	0.45 ±0.08	1.72±0.23
II	rigid core	312±15	$-(4.85_{-1.1}^{+1.65})$	$3.61_{-0.23}^{+0.32}$	0.81 ±0.04	
	with core distortion	260±25	$-(2.30_{-0.54}^{+0.80})$	$4.93_{-0.73}^{+1.07}$	0.675±0.065	1.65±0.25
III	rigid core	0.275±0.002	$-(2.36_{-0.48}^{+0.68})$	$2.18_{-0.18}^{+0.19}$	0.87 ±0.03	
	with core distortion	0.267±0.0035	$-(1.13_{-0.27}^{+0.38})$	$3.08_{-0.45}^{+0.60}$	0.77 ±0.04	as for I

[a] This is $U_{\Lambda\Lambda}$(MeV F^3) for I, $W_{\Lambda\Lambda}$(MeV) for II, and $f_{\Sigma\Lambda}$ for III.

($\mu_R \approx 0.65\ F^{-1}$) for the repulsive core. The best soft-core potentials give only slightly larger values of $\Delta B_{\Lambda\Lambda}$ than the best hard-core potentials. One may then consider the potentials b, f, q, r to span a reasonable range of acceptable potentials.

The corresponding results obtained with $\Delta B_{\Lambda\Lambda}=4.5 \pm 0.5$ MeV are shown in Table IV. The quantities $r_{0\Lambda\Lambda}$ and $s_{\Lambda\Lambda}$ are the effective range and the well-depth parameter, respectively. The uncertainty in the choice of $V_{\alpha\alpha}$ contributes about half as much to the error in the potential strengths as does the error in $\Delta B_{\Lambda\Lambda}$. The results for the hard-core, meson-theory potential III have been obtained from those for I by use of the relation between $f_{\Sigma\Lambda}$ and $U_{\Lambda\Lambda}$, which is given implicitly in Table II. This procedure seems well justified, since I and III have very nearly the same intrinsic range and since associated values of $f_{\Sigma\Lambda}$ and $U_{\Lambda\Lambda}$, which give the same $\Delta b_{\Lambda\Lambda}$, are then also expected to give similar core distortions. Moreover, I and II, which have different intrinsic ranges, give quite similar core distortions for the same value of $\Delta b_{\Lambda\Lambda}$.

The fact that the energy $E_{\alpha\alpha}(R_\Lambda)+E_{\Lambda\Lambda}$ due to distortion seems rather insensitive to the shape of $V_{\Lambda\Lambda}$ for a given value of $\Delta b_{\Lambda\Lambda}$ implies that our results for the core distortion may be used to determine the parameters for any shape of $V_{\Lambda\Lambda}$, if the appropriate rigid-core results for $\Delta b_{\Lambda\Lambda}$ have been calculated.

The inherent error of our four-body calculation of $\Delta B_{\Lambda\Lambda}$ for a given $V_{\alpha\alpha}$ is expected to be fairly small and less than the error due to uncertainties in the choice of $V_{\alpha\alpha}$. This is because, on the one hand, the major part of the distortion energy is $E_{\alpha\alpha}(R_\Lambda)$, which is given reliably by the three-body calculations for $_{\Lambda}Be^9$. On the other hand, for obtaining the additional distortion energy $E_{\Lambda\Lambda}$, reliable values have been used for the internal energy $E_{\alpha\alpha}(R)$ of the core. This is confirmed by the consistency of our results for different α-α potentials. Furthermore, by use of Eq. (14), the α-particle structure of the core has been taken into account consistently. Also, both the slow variation of $E_{\Lambda\Lambda}$ with the strength $W_{\Lambda\Lambda}$ and also explicit calculations imply that the precise dependence of $b_{\Lambda\Lambda}(R,W_{\Lambda\Lambda})$ on $W_{\Lambda\Lambda}$, given through Eqs. (13) and (15), is not crucial.

No account has been taken of the possibility that the α particles, individually, may be compressed by the presence of the Λ particles. This effect would lead to larger distortion energies than we have obtained with a four-body model and to correspondingly smaller strengths for $V_{\Lambda\Lambda}$. However, distortion of the individual α particles is expected to be rather small, in view of the expected small compressibility of the α particle. This seems to be confirmed by the quantitative success of the α-α-Λ model for $_{\Lambda}Be^9$.

The rigid-core results for our interactions I, II, and III (Table IV) have already been commented on. The results for I and III, which have the same intrinsic range, show that, also with core distortion, the scattering length and effective range are fairly well determined for a given intrinsic range, independently of the shape of the interaction. The well-depth parameter then increases as the hard-core radius becomes larger. Because of the larger intrinsic range of II, the values of $a_{\Lambda\Lambda}$ and $r_{0\Lambda\Lambda}$ are correspondingly larger than the values for I and III. However, $s_{\Lambda\Lambda}$ is quite similar for II and III, both of which have the same hard-core radius.

Table IV shows that the effect of core distortion weakens the resulting Λ-Λ interaction quite appreciably; in particular, the well-depth parameter is quite substantially reduced. The singlet Λ-Λ interaction then turns out to be considerably weaker than the singlet Λ-N interaction. In fact, the results obtained for the latter[12,13] are quite similar to those obtained for the Λ-Λ interaction, but with a rigid core for $_{\Lambda\Lambda}Be^{10}$.

Clearly, the evidence is strongly against a bound singlet state of the Λ-Λ system. Neither is a bound triplet state to be expected, since it seems likely that the triplet is weaker than the singlet interaction, in view of the fact that this is the case for the closely related Λ-N interaction. Furthermore, the three-body system (either $_{\Lambda\Lambda}H^3$ or $_{\Lambda\Lambda}n^3$) is also not expected to be bound. Thus not only is the relevant Λ-N interaction for $_{\Lambda\Lambda}H^3$ considerably weaker than for the loosely bound hypertriton $_{\Lambda}H^3$, but also the singlet Λ-Λ interaction is much weaker than the triplet n-p interaction which is relevant for $_{\Lambda}H^3$. In fact, the average Λ-N force for $_{\Lambda\Lambda}H^3$ is the same as for the $T=1$ state of $_{\Lambda}H^3$ which is not expected to be bound.[12]

The interpretation of $a_{\Lambda\Lambda}$ in terms of the meson-theory

[12] B. W. Downs and R. H. Dalitz, Phys. Rev. **114**, 593 (1959).
[13] R. C. Herndon, Y. C. Tang, and E. W. Schmid, Phys. Rev. **137**, B294 (1965).

potentials has been discussed, in particular for even $\Sigma\Lambda$ parity, by Dalitz,[14] by de Swart[8] and by Dalitz and Rajasekaran.[1] Our results for the meson-theory interaction III (which is appropriate to even $\Sigma\Lambda$ parity) show that the value of $f_{\Sigma\Lambda}$ is quite close to the comparable value $f_{\Sigma\Lambda}=0.276$ (for $f_{\Sigma\Sigma}=0$) which is obtained from the singlet Λ-N interaction.[9] (The attractive part of this, due to the exchange of two pions, has the same shape as for $V_{\Lambda\Lambda}$ but is proportional to $f_{\Sigma\Lambda}{}^2 f_{NN}{}^2$.) This is in agreement with the conclusions of Dalitz and Rajasekaran[1] which were based on the values of $a_{\Lambda\Lambda}$.

However, as has been emphasized especially by Dalitz,[1,14] any results deduced for $f_{\Sigma\Lambda}$ are very sensitive to the value used for the hard-core radius. (Thus for $r_c=0.35\mu_\pi{}^{-1}$ and $f_{\Sigma\Sigma}=0$, one has $f_{\Sigma\Lambda}=0.30$ for $a_{\Lambda\Lambda}=-1$ F.) This is because of the strong cancellation between the effects of the hard-core repulsion and the short-range attraction due to the exchange of two pions. One must therefore have some understanding of the relation between the hard-core radii for the Λ-N and Λ-Λ potentials if one is to reliably relate the attractive parts of these potentials. Furthermore, for $V_{\Lambda N}$, one can have, for example, exchange of single K mesons, which is not possible for $V_{\Lambda\Lambda}$.

Finally, if the event described in Ref. 3 is interpreted as ${}_{\Lambda\Lambda}Be^{11}$, then the conclusions about the Λ-Λ interaction will have to be modified accordingly. As pointed out by Dalitz,[14] the appropriate value $\Delta B_{\Lambda\Lambda}=4.5\pm1.0$ MeV for ${}_{\Lambda\Lambda}Be^{11}$ is quite similar to the value for the most probable interpretation ${}_{\Lambda\Lambda}Be^{10}$. With a rigid Be^9 core for ${}_{\Lambda\Lambda}Be^{11}$, a Λ-Λ-core model will yield very nearly the same results for $V_{\Lambda\Lambda}$ as are obtained for ${}_{\Lambda\Lambda}Be^{10}$ with a rigid core. Since the odd neutron in Be^9 has a separation energy of only 1.7 MeV, an α-α-n model might be expected to be quite good for Be^9 with a rms separation between the α particles which is rather larger than for ${}_{\Lambda}Be^9$. The contribution to $\Delta B_{\Lambda\Lambda}({}_{\Lambda\Lambda}Be^{11})$ due to distortion may then be expected to be somewhat more than half of that for ${}_{\Lambda\Lambda}Be^{10}$, in view of our results for $E_{\alpha\alpha}(R_\Lambda)$ and for $E_{\Lambda\Lambda}$. The results for the Λ-Λ interaction will then be roughly intermediate between those obtained for ${}_{\Lambda\Lambda}Be^{10}$ with a rigid core, on the one hand, and with core distortion included, on the other.

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Decays of the η Meson*

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The decay modes of the η^0 meson have been investigated in the Lawrence Radiation Laboratory 72-in. hydrogen bubble chamber. The η's were produced in the reaction $\pi^+ + p \to \pi^+ + p + \eta$ at 1225 and 1275 MeV/c and were studied by analysis of all four-prong and two-prong$+\gamma$ events. There appears to be a discrepancy in the measured branching ratio $R=\Gamma_\eta(000)/\Gamma_\eta(+-0)$ as compared with theoretical predictions based on various models relating R to the spectrum of T_{π^0} in $\eta^0\to\pi^+\pi^-\pi^0$. The theoretical predictions calculated from the observed spectrum are uniformly higher than the observed value $R=0.90\pm0.24$. For the Brown and Singer theory of a $T=0$, $S=0$ dipion resonance, we find $m_\sigma=407_{-12}{}^{+25}$ MeV, $\Gamma=117\pm15$ MeV, R(predicted)$=1.49\pm0.07$. The fit to the linear matrix-element expansion, $a=-0.41\pm0.06$, predicts $R=1.63\pm0.02$ (the amplitude $f=1+ay$ where $y=2\,T_{\pi^0}/T_{\pi^0\max}-1$). A fit may be obtained to both the spectrum and the branching ratio with an amplitude $f=(1-0.41y)e^{1.6iy}$, indicating that, although the magnitude of f is essentially smooth, a rapid variation in phase seems to be required to fit the branching ratio.

According to the accepted quantum numbers $0^{-+}=J^{PG}$ for the η meson, the final state reached in the decay $\eta\to\pi^+\pi^-\pi^0$ must have $T=1$, $J^P=0^-$. For this state, the most general decay amplitude is[1]

$$M=(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\boldsymbol{\tau}_3 f(\omega_1,\omega_2,\omega_3)+(\text{c.p.}), \qquad (1)$$

where (c.p.) means cyclic permutation of the indices 1, 2, 3; $\boldsymbol{\tau}_i$ is the isotopic spin vector of pion i; and f is

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