Very Short-Range Interaction in Pion-Nucleon Scattering*

A. DONNACHIE
Department of Physics, University College London, London, England

AND

J. HAMILTON
Nordita, Copenhagen, Denmark
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The dispersion relations for $F_l(s)=f_l(s)/q^{2l}$, where $f_l(s)$ is the usual partial-wave amplitude, are required by unitarity to obey a high-energy boundary condition. It is shown that this gives rise to a unitary sum rule. This sum rule can be used to estimate the short-range parts of the pion-nucleon interaction, and that makes it possible to give accurate predictions of the nonresonant P-, D-, and F-wave π-N amplitudes up to around 650 MeV. The results are in good agreement with a recent analysis of the experimental data.

1. INTRODUCTION

The Peripheral Method

A PERIPHERAL method for predicting pion-nucleon partial-wave amplitudes at low and moderate energies has been developed by Donnachie *et al.*[1] (DHL). The method depends on using the dispersion relations for the amplitudes $F_{l\pm}(s)=f_{l\pm}(s)/q^{2l}$, where $f_{l\pm}(s)$ are the partial-wave amplitudes as usually defined. The orbital angular momentum is l and $J=l\pm\frac{1}{2}$ is the total angular momentum; q is the momentum in the c.m. system and s is the square of the total energy in the c.m. system.

The factor q^2 is large on those parts of the unphysical cuts which lie far away from the physical threshold, so for $l\geqslant 1$ the role of the shorter range parts of the π-N interaction is very much suppressed when we use the dispersion relations for $F_{l\pm}(s)$.

The longer range parts of the pion-nucleon interaction are associated with those parts of the unphysical cuts which lie nearest to the physical threshold. These longer range parts of the interaction are produced by the four basic processes: N exchange, N^* exchange, $(\pi\pi)_0$ exchange, and ρ exchange.[1] The coupling constants and other data necessary for calculating the contributions of these four basic processes to the dispersion relations are known, and DHL have computed these contributions for P, D, and F waves (i.e., $l=1,2,3$). The results are as follows:

(a) In the range of energies for which the peripheral method is valid, four amplitudes ($P_{33},D_{13},F_{15},F_{37}$) should resonate.[2]

(b) Because of special circumstances, precise predictions of the amplitude P_{11} cannot be made above a few hundred MeV.[3]

(c) Reasonably precise predictions of the remaining amplitudes can be given up to around 400 MeV, and their general behavior up to 1.0–1.5 BeV can also be predicted.

A Unitary Sum Rule and an Improved Peripheral Method

In order to improve the accuracy of the peripheral method and to extend its use to higher energies, it is necessary to have some estimate of the short-range parts of the π-N interaction which have so far been ignored. Such information is given by unitarity. This comes about as follows. The dispersion relation for $F_{l\pm}(s)$ expresses $\mathrm{Re}F_{l\pm}(s)$, for physical s, in terms of an integral over the unphysical cuts plus an integral over the physical cut (the rescattering integral). Now on account of the factor q^{-2l} which occurs in $F_{l\pm}(s)$, it follows that for $l\geqslant 1$, $\mathrm{Re}F_{l\pm}(s)$ tends to zero faster than s^{-1} as $s\to+\infty$. This imposes a condition on the asymptotic behavior of the sum of the two integrals. To satisfy this condition, it is necessary that the integrated weight function for $F_{l\pm}(s)$ should vanish. This is the unitary sum rule. (For $l\geqslant 2$ there are additional unitary sum rules.)

Now it is readily seen that the sum rule imposes a condition on the short-range part of the π-N interaction, provided we know the long-range part of the interaction. In practice, we represent the short-range part of the interaction by a single pole, for simplicity. The sum rule then enables us to estimate the residue of this pole. There remains the problem of where to place the pole. For D and F waves it will be seen that the structure of the unphysical cuts and the behavior of $|q|^{-2l}$ on these cuts suggest a region where the pole should be placed, and, moreover, variations of the position of the pole over this region should not introduce noticeable errors. For P waves the suppression factor $|q|^{-2l}$ does not have so strong an effect, and we can only guess a position for the pole. In order to do this we have to use the known P-wave π-N scattering lengths. It follows that our final predictions for P_{31} may contain appreciable errors. For

* This research has been supported in part by the U. S. Air Force Office of Scientific Research, OAR and the European Office of Aerospace Research, USAF.
[1] A. Donnachie, J. Hamilton, and A. T. Lea, Phys. Rev. **135**, B515 (1964). This paper will be referred to as DHL.
[2] See Sec. 9 of DHL for further details. Also, A. Donnachie and J. Hamilton, Ann. Phys. (N. Y.) (to be published).
[3] See Sec. 7 of DHL.

P_{13} the residue of the short-range pole is very small and the short-range interaction is negligible.[4]

The Applications

We shall be concerned here mainly with the amplitudes in category (c) above, that is, the nonresonant P, D, and F waves, excluding P_{11}. Using the data on the long-range parts of the π-N interactions as calculated by DHL, we can readily evaluate the sum rule and find the residue of the short-range pole for these amplitudes. This enables us to correct the DHL predictions of the real part of the phase shifts and extend them up to 600 or 700 MeV. In the DHL predictions for these amplitudes the rescattering term was neglected, since on general grounds DHL believe it would be small. A recent analysis of the experimental data by Auvil *et al.*[5] gives values of the inelasticity coefficient η up to 700 MeV for the partial waves in which we are interested. These results and other data on inelasticity at higher energies enable us to give some rough values of the rescattering correction for the amplitudes in category (c). It will be seen in the later sections of this paper that when we correct the DHL predictions for the short-range interaction and add the rescattering correction, the predictions of the real parts of the phase shifts for the amplitudes in category (c) are in reasonably good agreement with the values deduced from the experiments by ADLL[5] up to 600–700 MeV.

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In Sec. 2 we derive the unitary sum rule and examine some of its consequences. One of these is that, for $l\geqslant 1$, $F_{l\pm}'(s)$, which is the total contribution to the dispersion relation from the unphysical cuts, must behave like A/s as $s\rightarrow+\infty$, where A is a positive constant. In other words, at very high energies the "interaction" must become attractive. This means, for example, that for $l\geqslant 1$ the unphysical cuts cannot be represented by a single repulsive pole. A simple N/D example which illustrates this point is given. It is seen that a ghost necessarily occurs, and the ghost ensures that the sum rule is obeyed.

In Sec. 3 we examine the general method of using the sum rule for π-N scattering and in Sec. 4 we derive the corrections to the DHL predictions for D and F waves. In Sec. 5 the corrections for P_{31} and P_{13} are considered, and also the method is used to confirm that there is only a very small short-range interaction in the case of the resonant amplitude P_{33}. Section 6 contains a general discussion of the results.

[4] If, instead of $F_{l\pm}(s)$, we had used the dispersion relation for $\mathfrak{F}_{l\pm}(s)=(s+s_0)^l F_{l\pm}(s)$ (where $s_0>0$), there would be no sum rule. We do not favor this as a *practical* method because the dispersion relation would then have large contributions from large negative values of s. Using $F_{l\pm}(s)$ avoids this serious practical difficulty.

[5] P. Auvil, A. Donnachie, A. T. Lea, and C. Lovelace, Phys. Letters **12**, 76 (1964). This paper will be referred to as ADLL.

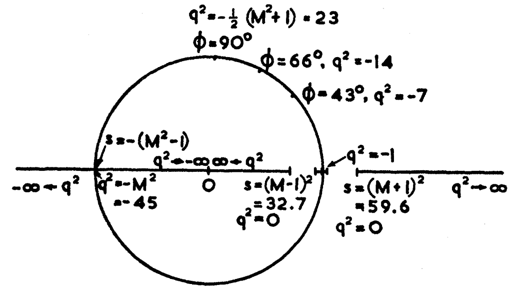

FIG. 1. The singularities of the partial-wave π-N amplitudes in the complex s plane. Values of q^2 at various positions on the cuts are shown. The units are $\mu=c=\hbar=1$.

2. THE UNITARY SUM RULES

We consider the partial wave amplitude $F_{l\pm}(s)$ for the process $\pi+N\rightarrow\pi+N$. It is defined by

$$F_{l\pm}(s)=\frac{\exp(2i\delta_{l\pm}(s))-1}{2iq^{2l+1}}, \qquad (1)$$

where l is the orbital angular momentum and $l\pm$ refers to the states having total angular momentum $J=l\pm\frac{1}{2}$. Also q is the momentum in the c.m. system and

$$s=[(M^2+q^2)^{1/2}+(\mu^2+q^2)^{1/2}]^2,$$

where M and μ are the nucleon and pion masses. The phase shift $\delta_{l\pm}$ is given by

$$\delta_{l\pm}=\alpha_{l\pm}+i\beta_{l\pm},$$

where $\alpha_{l\pm}$, $\beta_{l\pm}$ are real, and $\beta_{l\pm}\geqslant 0$. We consider only the cases $l\geqslant 1$.

The singularities of $F_{l\pm}(s)$ as an analytic function of the complex variable s are well known.[6] The positions of the various cuts are shown in Fig. 1. The amplitude $F_{l\pm}(s)$ obeys the dispersion relation

$$\mathrm{Re}F_{l\pm}(s)=\frac{1}{\pi}P\int_{s_0}^{\infty}\frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}ds'+\frac{1}{\pi}\int_{-\infty}^{s_1}\frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}ds' \qquad (2)$$

for physical values of s. For convenience of writing, the unphysical cuts have been represented by the line $-\infty\leq s\leq s_1$, and we write $s_0=(M+\mu)^2$ for the physical threshold.

We consider the limiting form of Eq. (2) for infinite physical energies ($s\rightarrow+\infty$). Clearly, for physical s,

$$\mathrm{Re}F_{l\pm}(s)=\frac{\eta_{l\pm}(s)}{2q^{2l+1}(s)}\sin(2\alpha_{l\pm}(s)),$$

where $\eta_{l\pm}(s)=\exp(-2\beta_{l\pm}(s))$. Thus

$$|\mathrm{Re}F_{l\pm}(s)|\leq 1/(2q^{2l+1}). \qquad (3)$$

Also

$$q^2=(1/4s)[s-(M+\mu)^2][s-(M-\mu)^2]. \qquad (4)$$

[6] See, for example, J. Hamilton and T. D. Spearman, Ann. Phys. (N. Y.) **12**, 172 (1961).

Therefore, the magnitude of the left-hand side of Eq. (2) is bounded by $2^{2l-1}/s^{l+\frac{1}{2}}$ for large physical s.

Next we look at the right-hand side of Eq. (2) as $s \to +\infty$. We shall assume, as is reasonable,[7] that for $l \geqslant 1$, $\mathrm{Im}F_{l\pm}(s)$ decreases faster than $(-s)^{-1-\epsilon}$ (where $\epsilon > 0$) as $s \to -\infty$. With this assumption,

$$\frac{1}{\pi}\int_{-\infty}^{s_1} \frac{\mathrm{Im}F_{l\pm}(s')}{s'-s} ds' = -\frac{1}{\pi s}\int_{-\infty}^{s_1} \mathrm{Im}F_{l\pm}(s')ds' + o\left(\frac{1}{s}\right) \quad (5)$$

as $s \to +\infty$.

The first term on the right of Eq. (2) is a principal value integral, and it is not so easy to find its asymptotic form. Sufficient conditions[8] for

$$\frac{1}{\pi} P\int_{s_0}^{\infty} \frac{\mathrm{Im}F_{l\pm}(s')}{s'-s} ds'$$

to behave like

$$-\frac{1}{\pi s}\int_{s_0}^{\infty} \mathrm{Im}F_{l\pm}(s')ds'$$

as $s \to +\infty$ are that

(i) $$\int^{\infty} \mathrm{Im}F_{l\pm}(s')ds' \text{ exists},$$

and

(ii) given $\epsilon > 0$, there exists an S such that

$$\left|\frac{s' \,\mathrm{Im}F_{l\pm}(s') - s\,\mathrm{Im}F_{l\pm}(s)}{s'-s}\right| < \frac{\epsilon}{(ss')^{1/2}}$$

for all $s' \geqslant s > S$.

Now

$$\mathrm{Im}F_{l\pm}(s) = \frac{1-\eta_{l\pm}(s)\cos(2\alpha_{l\pm}(s))}{2q^{2l+1}}. \quad (6)$$

As $\eta_{l\pm} \leq 1$, condition (i) is obeyed for $l \geqslant 1$. Condition (ii) is equivalent to requiring that, given $\epsilon > 0$, there exists an S such that

$$\left|\frac{[\eta_{l\pm}(s')/s'^{(l-\frac{1}{2})}]\cos(2\alpha_{l\pm}(s')) - [\eta_{l\pm}(s)/s^{(l-\frac{1}{2})}]\cos(2\alpha_{l\pm}(s))}{s'-s}\right| < \frac{\epsilon}{(ss')^{1/2}} \quad (7)$$

for all $s' \geqslant s > S$. However, we cannot apply condition (7) without detailed knowledge[9] of how $\alpha_{l\pm}$ and $\eta_{l\pm}$ behave for large s.

In order to avoid this difficulty we proceed as follows. By Eqs. (2), (3), and (5),

$$\lim_{s\to+\infty}\left[\frac{s}{\pi} P\int_{s_0}^{\infty} \frac{\mathrm{Im}F_{l\pm}(s')}{s'-s} ds'\right] = \frac{1}{\pi}\int_{-\infty}^{s_1} \mathrm{Im}F_{l\pm}(s')ds' \quad (8)$$

(for $l \geqslant 1$). Thus the limit on the left-hand side exists. Moreover, by Eqs. (4) and (6), $\mathrm{Im}F_{l\pm}(s) \geqslant 0$ on $s_0 \leq s \leq \infty$, and $\mathrm{Im}F_{l\pm}(s)$ goes to zero at least as quickly as $s^{-(l+\frac{1}{2})}$ when $s \to +\infty$. Thus, for $l \geqslant 1$, it is a very reasonable conjecture that the limit equals

$$-\frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{Im}F_{l\pm}(s')ds'.$$

Equation (8) then gives the *unitary sum rule*[10] for $l \geqslant 1$,

$$\int_{s_0}^{\infty} \mathrm{Im}F_{l\pm}(s')ds' + \int_{-\infty}^{s_1} \mathrm{Im}F_{l\pm}(s')ds' = 0. \quad (9)$$

For $l \geqslant 2$ the left-hand side of Eq. (2) must not exceed $8/s^{5/2}$ as $s \to +\infty$. In a similar fashion this gives rise to the unitary sum rule for $l \geqslant 2$,

$$\int_{s_0}^{\infty} s'\,\mathrm{Im}F_{l\pm}(s')ds' + \int_{-\infty}^{s_1} s'\,\mathrm{Im}F_{l\pm}(s')ds' = 0, \quad (10)$$

and so on for higher values of l.

The Physical Nature of the Unitary Sum Rules

If we are given the discontinuity $\mathrm{Im}F_{l\pm}(s)$ on the complete unphysical cut $-\infty \leq s \leq s_1$, and the inelasticity coefficient $\eta_{l\pm}(s)$, then, by the N/D method or otherwise, the dispersion relation (2) can be solved. The solution gives $\mathrm{Im}F_{l\pm}(s)$ on the physical cut $s_0 \leq s \leq \infty$, so it follows that the unitary sum rule of Eq. (9) provides a relation governing the values which $\mathrm{Im}F_{l\pm}(s)$ can have on the unphysical cut $-\infty \leq s \leq s_1$. Further relations of a similar nature are given by the other sum rules, Eqs. (10), etc.

In practice, the long and fairly long-range parts of the π-N interaction are reasonably well known.[11] That is, we know $\mathrm{Im}F_{l\pm}(s)$ on the portion of the unphysical cut $-\infty \leq s \leq s_1$ which is closest to the physical threshold s_0. Then, for $l \geqslant 1$, the unitary sum rule Eq. (9) gives us some information about the value of $\mathrm{Im}F_{l\pm}(s)$ on the remainder of the unphysical cut. In other words, using our knowledge of the long and fairly long-range parts of the π-N interaction, the unitary sum rule enables us

[7] If this assumption is not valid, the high-energy behavior of the dispersion relation must be very pathological.

[8] For further details see J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963) [Sec. 2 (ii)].

[9] Letting $s' \to s$ we see that *at least* condition (7) requires that $s^{(3/2-l)}(d/ds)\,(\eta\cos 2\alpha)$ tends to zero as $s \to +\infty$.

[10] The unitary sum rules have been known to several other authors, including J. Rothleitner and B. Stech (private communication), and A. P. Balachandran and F. von Hippel, Ann. Phys. (N. Y.) **30**, 446 (1964).

[11] For details see DHL (Ref. 1).

to get some information about the short range part of the π-N interaction.

It is not surprising that this should be so. An interesting analogy is provided by Blankenbecler *et al.*[12] in their treatment of the double variable dispersion relations for nonrelativistic scattering by a potential of finite range. Over a certain domain of the (s,t) plane—which is not far from the physical region—the spectral function $\rho(s,t)$ is given exactly by the second Born approximation. The unitary relation then makes it possible to determine $\rho(s,t)$ over larger and larger domains using an iterative procedure. A similar iteration scheme based on unitarity has been outlined in principle in the field theory case.[13]

Consequences of the Unitary Sum Rules

Because $\mathrm{Im}F_{l\pm}(s)>0$ on some segment of the physical cut $s_0\leq s\leq\infty$, if there is any scattering, it follows from Eq. (9) that for $l\geqslant 1$

$$\int_{-\infty}^{s_1}\mathrm{Im}F_{l\pm}(s')ds'<0\,. \tag{11}$$

Let

$$F_{l\pm}'(s)=\frac{1}{\pi}\int_{-\infty}^{s_1}\frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}ds'\,, \tag{12}$$

so $F_{l\pm}'(s)$ is the total contribution of the unphysical cuts in Eq. (2). It was seen in DHL that a positive $F_{l\pm}'(s)$ corresponds to an attractive interaction (provided there are no bound states). Now Eq. (11) shows that at high energies the asymptotic form of $F_{l\pm}'(s)$ is

$$F_{l\pm}'(s)\sim A_{l\pm}/s\,,\quad (s\rightarrow+\infty)\,, \tag{13}$$

where $A_{l\pm}$ is a *positive* constant. This is a significant result since it might have been thought that $F_{l\pm}'(s)$ would decrease as $s^{-(l+\frac{1}{2})}$ when $s\rightarrow+\infty$.

Another consequence of Eq. (11) or (13) is that the discontinuity across the unphysical cut $(-\infty\leq s\leq s_1)$ cannot reduce to a single pole having *negative* residue. That is, if

$$\mathrm{Im}F_{l\pm}(s)=-\pi\Gamma\delta(s-\bar{s})\,,\quad(-\infty<\bar{s}\leq s_1)\,,$$

then we must have $\Gamma>0$. In other words, for $l\geqslant 1$, a single repulsive pole is not permissible.

Consequences of the higher unitary sum rules, such as Eq. (10), are also easily deduced. For example, statements can be made about the minimum number of changes of sign of $\mathrm{Im}F_{l\pm}(s)$ on the unphysical cut.[14]

An N/D Example

As an illustration of the unitary sum rule Eq. (9) we consider the N/D solution for a P wave having a single repulsive pole. For simplicity, we use the nonrelativistic kinematics with

$$q^2=s-s_0\,,$$

and we assume that the scattering process is purely elastic, so that the simple form of the N/D method is applicable.

The basic equations for the amplitude $F(s)$, defined as in Eq. (1) are[15]:

$$\begin{aligned}F(s)&=N(s)/D(s)\,,\\ N(s)&=\frac{1}{\pi}\int_{-\infty}^{s_1}\frac{D(s')\,\mathrm{Im}F(s')}{s'-s}ds'\,,\\ D(s)&=D(s_0)-\frac{s-s_0}{\pi}\int_{s_0}^{\infty}\frac{N(s')q'^3ds'}{(s'-s_0)(s'-s)}\,.\end{aligned} \tag{14}$$

The single repulsive pole is given by the spectral function

$$\mathrm{Im}F(s)=-\pi\Gamma\delta(s-\bar{s})\,, \tag{15}$$

where $-\infty<\bar{s}<s_0$ and $\Gamma<0$. Obviously this spectral function violates Eq. (11).

Substituting Eq. (15) in Eqs. (14) gives[15]

$$N(s)=\Gamma\bar{D}/(s-\bar{s})\,, \tag{16a}$$

where $\bar{D}\equiv D(\bar{s})$, and

$$D(s)=D(s_0)+\Gamma\bar{D}\frac{s_0-s}{s-\bar{s}}[a-(s_0-s)^{1/2}]\,,\qquad -\infty\leq s\leq s_0\,, \tag{16b}$$

$$D(s)=D(s_0)-\frac{\Gamma\bar{D}}{s-\bar{s}}[(s-s_0)a+iq^3]\,,\quad s_0\leq s\leq\infty\,. \tag{16c}$$

Here we have written $a=(s_0-\bar{s})^{1/2}$, and we take $D(s_0)>0$. From Eq. (16b) we deduce

$$\bar{D}=D(s_0)/(1-\tfrac{1}{2}\Gamma a)\,. \tag{17}$$

Remembering that $\Gamma<0$, it is easy to see that $D(s)$ has no zero for $\bar{s}\leq s\leq s_0$. However, $D(s)$ has one zero on $-\infty<s<\bar{s}$; let this occur at s_g. This is a ghost. Near s_g

$$D(s)\simeq K(s-s_g)\,,$$

where $K>0$. Also

$$N(s_g)=-\bar{D}\Gamma/(\bar{s}-s_g)\,,$$

so $N(s_g)>0$. Thus, near s_g,

$$F(s)\simeq N(s_g)/K(s-s_g)\,, \tag{18}$$

that is, the ghost is a pole in $F(s)$ having positive residue. Hence, for $-\infty\leq s\leq s_0$ we actually have

$$\mathrm{Im}F(s)=-\pi\Gamma\delta(s-\bar{s})-\pi(N(s_g)/K)\delta(s-s_g)\,. \tag{19}$$

Equation (11) can be satisfied because the residue at the ghost is positive.

[12] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).
[13] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).
[14] See, for example, P. Beckmann, Z. Physik **179**, 379 (1964).
[15] See A. Donnachie and J. Hamilton, Phys. Rev. **133**, B1053 (1964) for the details of the calculation.

A Very Weak Repulsive Pole

In the special case $|\Gamma|a \ll 1$, i.e., for a very weak repulsive pole, we can easily see that the sum rule Eq. (9) is satisfied. Now $\bar{D} \simeq D(s_0)$, and by Eq. (16b) the approximate solution of $D(s_g)=0$ is

$$(s_0-s_g)^{1/2} \simeq -1/\Gamma. \tag{20a}$$

Also,

$$\left.\frac{dD(s)}{ds}\right|_{s_g} \simeq -\frac{\Gamma\bar{D}}{2(s_0-s_g)^{1/2}}.$$

Thus,

$$\begin{aligned} K &\simeq \Gamma^2\bar{D}/2, \\ N(s_g) &\simeq -\Gamma^3\bar{D}. \end{aligned} \tag{20b}$$

From Eq. (18) it follows that near s_g,

$$F(s) \simeq -2\Gamma/(s-s_g). \tag{20c}$$

This pole together with the original pole does not satisfy the sum rule (9). The reason is that in this elastic model the phase shift goes to $(\pi/2)$ as $s \to +\infty$, and there is a contribution of first order in Γ from the integral over the physical range in Eq. (9).

Again assuming $|\Gamma|a \ll 1$, Eqs. (16a) and (16c) give for $s_0 \leq s \leq \infty$,

$$\mathrm{Im}F(s) = -\frac{N(s)\,\mathrm{Im}D(s)}{|D(s)|^2} \simeq \frac{(\Gamma\bar{D})^2(s-s_0)^{1/2}}{\bar{D}^2+\Gamma^2\bar{D}^2(s-s_0)}.$$

This yields

$$\frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{Im}F(s)ds \simeq -\Gamma. \tag{21}$$

Equations (19), (20b), and (21) now check that the unitary sum rule is obeyed to the highest order in Γ. More laborious calculations would show that it is obeyed to all orders in Γ.

Comment

In the present paper and in DHL the N/D method is not used. Instead the second integral on the right of Eq. (2) is evaluated directly; this integral gives the "interaction" for the partial wave concerned. The first integral in Eq. (2) (i.e., the rescattering integral) is evaluated by various approximations or empirical methods. It is of interest to compare the contributions to the second integral in Eq. (2) from the original weak repulsive pole and from the ghost. The former is $\Gamma/(s-\bar{s})$, and by Eqs. (20a) and (20c) the latter gives $-2\Gamma^3$ at low and moderate energies. Thus, at low energies the ratio of the contributions is of order $2(\Gamma a)^2$, which is very small. The ratio rises slowly with increasing energy until for $s-s_0 \geq \Gamma^{-2}$ the contribution from the ghost dominates over that of the original pole. However, at all energies the absolute magnitude of the ghost contribution is very small.

3. APPLICATIONS TO π-N SCATTERING

In the predictions of π-N partial waves by DHL[1] the values of $\mathrm{Im}F_{l\pm}(s)$ on those parts of the unphysical cuts which are closest to the physical threshold were determined by the four simple exchange processes shown in Fig. 2. These are: (a) N exchange, (b) N^* exchange, (c) $(\pi\pi)_0$ exchange, and (d) ρ exchange. (a) gives the cut $(M-\mu^2/M)^2 \leq s \leq M^2+2\mu^2$ (cf., Fig. 1), (b) gives the cut $0<s\leq(M-\mu)^2$, while (c) and (d) give the portion $0\leq|\arg s|\leq 66°$ of the circle $|s|=M^2-\mu^2$. These exchange processes exhaust the long or fairly long-range parts of the π-N interaction.

Short-range parts of the interaction can arise from the following: (i) iterations of the processes (a)–(d), (ii) small pion pair exchange processes coming from the arc of the circle with $|\arg s|>66°$, (iii) exchange processes involving higher mass mesons, such as $F°$ exchange, (iv) nucleon pair processes. We shall represent the sum of these short-range parts by a pole placed at a suitable position on the line $-\infty \leq s \leq 0$ (cf., Fig. 1), and we shall use the unitary sum rule Eq. (9) to determine the residue of the pole. In this way we correct the predictions of DHL so as to include the effect of the short-range interactions.

Size of the Short-Range Effects

It should be emphasized that for $l \geq 1$ the value of $\mathrm{Im}F_{l\pm}(s)$ on the faraway parts of the unphysical cuts is expected to be small because of the factor q^{-2l} which appears in the definition of $F_{l\pm}(s)$ [cf., Eq. (1)]. Thus, we expect that the corrections to the results of DHL will be small in absolute magnitude. It will be seen that this is indeed true. The corrections are only of importance for the class of amplitudes for which DHL predict small phase shifts (and only in some of these cases).[16] The behavior of the resonant amplitudes is dominated by the strong fairly long-range attractions. In the relevant energy regions, the contributions of these to $F_{l\pm}'(s)$ [Eq. (12)] are orders of magnitude larger than the short-range interactions we are discussing here.

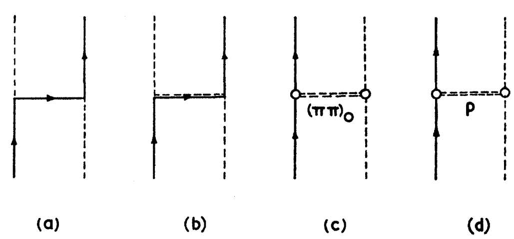

FIG. 2. The four processes which give the longer range parts of the π-N interaction. They are: (a) N exchange, (b) N^* exchange, (c) exchange of the $T=0$ $J=0$ pion pair $(\pi\pi)_0$, and (d) exchange of the $T=1$ $J=1$ ρ meson.

[16] We do not discuss the amplitude P_{11} in this paper. As was pointed out by DHL, it may have a moderate-sized short-range interaction.

Accurate Prediction of Phase Shifts

It was pointed out in Sec. 8 of DHL that lack of knowledge of the short-range interactions was one of the factors which limited the accurate predicton of phase shifts (as distinct from the general features such as resonant behavior, etc.) to the range up to 400 MeV. The other limiting factor was lack of knowledge of the inelasticity coefficient $\eta(s)$. Using recent estimates of $\eta(s)$ by Auvil and Lovelace[17] and by ADLL,[5] and our short-range corrections, the range of accurate prediction for most of the P, D, and F waves can be pushed up to around 600–700 MeV.

4. D AND F WAVES

It is convenient to start with the D- and F-wave amplitudes, since for them the factor q^{-2l} ensures that the relative importance both of the short-range interactions and of the high-energy rescattering is much less than for the P waves.

We first examine the D waves and we shall here give a method which is suitable for D waves and F waves. The P waves have to be treated somewhat differently as we shall see in Sec. 5. We illustrate the method in the typical case of D_{33}.

D_{33}

The contributions to $F_{l\pm}'(s)$ [Eq. (12)] from the four fairly long-range interactions, N exchange, N^* exchange, $(\pi\pi)_0$ exchange, and ρ exchange, were computed by DHL up to $s=200$ (i.e., a lab energy of about 1.5 BeV). We denote the sum of these contributions to $F_{l\pm}'(s)$ by $F_{\text{L.R.}}'(s)$. The data of DHL show that in the case of D_{33} the asymptotic behavior of $F_{\text{L.R.}}'(s)$ is

$$F_{\text{L.R.}}'(s)\sim -4.3\times10^{-2}/s\,,\quad (s\rightarrow+\infty)\,. \tag{22}$$

As usual the units are $\hbar=c=\mu=1$. If we denote by

$$\frac{1}{\pi}\int_{\text{L.R.}} \text{Im}F(s)ds\,,$$

the corresponding part of the integral

$$\frac{1}{\pi}\int_{-\infty}^{s_1} \text{Im}F(s)ds\,,$$

Eq. (22) gives

$$\frac{1}{\pi}\int_{\text{L.R.}} \text{Im}F(s)ds=+4.3\times10^{-2}\,. \tag{22a}$$

Obviously if there were no short-range interaction Eq. (11) would be violated.

For the sum rule Eq. (9) we also require the value of the physical integral

$$J=\frac{1}{\pi}\int_{s_0}^{\infty} \text{Im}F_{l\pm}(s)ds\,. \tag{23}$$

This is estimated as follows. By Eq. (6) we require $\eta_{l\pm}(s)$ and $\alpha_{l\pm}(s)$ to get $\text{Im}F_{l\pm}(s)$. We take for $\eta(s)$ the values determined by ADLL up to 700 MeV. For $\alpha(s)$ we use the predictions of DHL up to around 700 MeV; these are shown in Fig. 3. It turns out that in this energy range the value of $\eta(s)$ is much more important than the value of $\alpha(s)$. Because of the factor q^{-4}, contributions to J from energies above 700 MeV are not likely to be important. This is confirmed by using the information about $\text{Im}F_{l\pm}(s)$ obtained by Auvil and Lovelace.[17] Our estimate is

$$J\simeq3\times10^{-3}\,. \tag{23a}$$

Clearly J makes a small but not an important contribution to the sum rule. We write

$$\Gamma=-\frac{1}{\pi}\int_{\text{S.R.}} \text{Im}F(s)ds\,, \tag{24}$$

where

$$\frac{1}{\pi}\int_{\text{S.R.}} \text{Im}F(s)ds+\frac{1}{\pi}\int_{\text{L.R.}} \text{Im}F(s)ds=\frac{1}{\pi}\int_{-\infty}^{s_1} \text{Im}F(s)ds\,.$$

Equations (9), (22a), and (23a) give

$$\Gamma=+4.6\times10^{-2}\,. \tag{24a}$$

If the short-range interaction is to be represented by a single pole, then its residue is Γ. The main problem is where the pole should be placed.

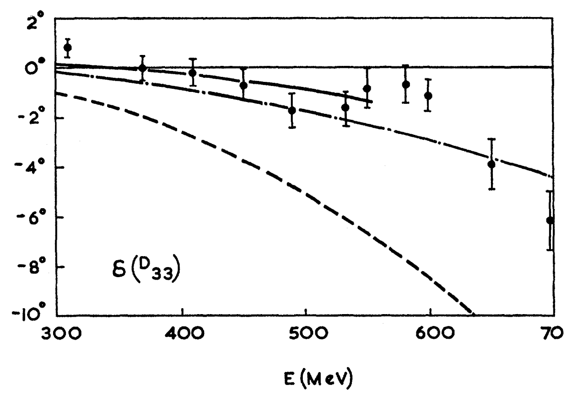

FIG. 3. Predictions for the real part of the phase shift in the state D_{33}. The DHL prediction $\alpha(s)$ is shown by the broken line ‐ ‐ ‐ ‐. The value of $(\alpha(s)+\Delta_s\alpha)$, where $\Delta_s\alpha$ is the correction due to the short-range interaction, is shown by the broken line —·—·—. The solid line is the final prediction $\alpha_F(s)=\alpha(s)+\Delta_s\alpha+\Delta_R\alpha$, where $\Delta_R\alpha$ is the rescattering correction. The points with error flags are the results of ADLL (Ref. 5).

[17] P. Auvil and C. Lovelace, Nuovo Cimento **33**, 473 (1964).

TABLE I. Values of q^2 on the negative axis $-\infty<s<0$.

s	0	-10	-20	-44	-100	-200	-300
q^2	$-\infty$	-74	-52	-45	-52	-75	-100

The Position of the Pole

The short-range interaction comes from the segment of the circle having $|\arg s|>66°$ and the negative axis $-\infty<s<0$. DHL made use of the factor q^{-2l} which appears in the definition of $F_{l\pm}(s)$ in order to get an idea of where $\mathrm{Im}F_{l\pm}(s)$ was large and where it was small. This worked reasonably well, and we shall extend this idea to get a rough notion of where the equivalent short-range pole should be placed.

Values of q^2 are shown in Fig. 1, and Table I gives some values of q^2 on the negative real axis. The minimum value of $(-q^2)$ on this line is $q^2=-M^2$ which occurs at $s=\mu^2-M^2=-44$. It is not reasonable to place the equivalent short-range pole on the positive real axis, since most of the unknown parts of the cuts are further from the physical threshold than is the point $s=0$. On the other hand, for D waves (or F waves) it is not reasonable to place the pole far to the left of $s=\mu^2-M^2$ since $|q^2|$ increases steadily as we go to the left, and q^{-2l} appears in $F_{l\pm}(s)$. We shall place the pole at $s=-M^2$ for D waves (or F waves), so that the short-range term is

$$\Gamma/(s+M^2), \tag{25}$$

Γ being given by Eq. (24).

For D waves a fair amount of uncertainty in the position of the short-range pole can be tolerated (and even more so for F waves). Let $\Delta_s\alpha$ be the change in the phase shift α which is produced by adding the short-range pole. Ignoring the small change in the rescattering, Eq. (2) gives

$$\eta(s)\cos[2\alpha(s)]\Delta_s\alpha\simeq q^{2l+1}[\Gamma/(s+M^2)]. \tag{26}$$

Because of the factor q^{2l+1}, for $l\geqslant 2$ the correction $\Delta_s\alpha$ only becomes appreciable at the higher energies, e.g., for D waves it will be seen below that $\Delta_s\alpha$ is of order of a degree around 400 MeV ($s=98$). Thus we are only interested in fairly large values of s, and the precise value of the additive constant in the denominator of the expression (25) is not too important.

TABLE II. The amplitude D_{33}. The DHL phase shift $\alpha(s)$, the short-range correction $\Delta_s\alpha$, the rescattering correction $\Delta_R\alpha$, and the final prediction α_F at several energies. (The values of $\Delta_R\alpha$, and so of α_F, at 500 and 600 MeV are necessarily rough estimates.)

Lab. energy (MeV)	400	500	600	700
$\alpha(s)$	$-2.6°$	$-5.2°$	$-8.5°$	$-13°$
$\Delta_s\alpha$	$1.7°$	$3.4°$	$5.5°$	$8.5°$
$\Delta_R\alpha$	$0.6°$	$\sim 0.9°$	$\sim 1.0°$	
α_F	$-0.3°$	$\sim -0.9°$	$\sim -2.0°$	

The Results

We use Eq. (26) with Γ given by Eq. (24a). For $\eta(s)$ we use the value of ADLL. $\alpha(s)$ denotes the DHL phase shift which includes no short-range effect and no rescattering. The correction $\Delta_s\alpha$ to $\alpha(s)$ arising from the short-range interaction for the case of D_{33} is given in Table II and Fig. 3.

Now, using $[\alpha(s)+\Delta_s\alpha]$ as an approximate phase shift, we can estimate the rescattering integral

$$I_{l\pm}(s)\equiv\frac{1}{\pi}\int_{s_0}^{\infty}\frac{\mathrm{Im}F_{l\pm}(s')}{s'-s}ds'. \tag{27}$$

Again, the inelasticity coefficient $\eta(s)$ is very important in computing $I(s)$. ADLL only give $\eta(s)$ up to 700 MeV, and for higher energies we only have the qualitative information on $\mathrm{Im}F_{l\pm}(s)$ given by Auvil and Lovelace.[17] Thus the values of $I(s)$ above 400 MeV are necessarily only rough estimates.[18]

The rescattering correction $\Delta_R\alpha$ is given by

$$\eta(s)\cos[2\alpha(s)]\Delta_R\alpha=I(s). \tag{28}$$

The values we get are shown in Table II and Fig. 3.

TABLE III. The amplitude D_{35}. The DHL phase shift $\alpha(s)$, the short-range correction $\Delta_s\alpha$, the rescattering correction $\Delta_R\alpha$, and the final prediction α_F at several energies. (The values of $\Delta_R\alpha$ above 400 MeV could have considerable errors.)

Lab energy (MeV)	400	500	600	700
$\alpha(s)$	$-3.2°$	$-4.9°$	$-7.0°$	$-9.3°$
$\Delta_s\alpha$	$0.9°$	$1.7°$	$2.8°$	$4.2°$
$\Delta_R\alpha$	$0.6°$	$\sim 1.5°$	$\sim 3.5°$	
α_F	$-1.7°$	$\sim -1.7°$	$\sim -0.7°$	

There appears to be reasonably good agreement with the phase shifts deduced from the experimental data by ADLL (cf., Fig. 3). The amplitude D_{33} shows a typical short-range effect in which a negative phase shift due to a moderately strong repulsion is appreciably raised. We shall meet this phenomenon in a number of other cases.

D_{35}

We proceed as for D_{33}. The values of DHL give

$$F_{\mathrm{L.R.}}'(s)\sim -1.8\times 10^{-2}/s, \quad (s\to+\infty).$$

Now ADLL find no inelasticity in D_{35} up to 700 MeV, but Auvil and Lovelace[17] suggest that D_{35} is markedly inelastic above 700 MeV. This makes it difficult to determine J. On the basis of Auvil and Lovelace's results we estimate $J\simeq 5\times 10^{-3}$. This gives $\Gamma=2.3\times 10^{-2}$, and again we place the short-range pole at $s=-M^2$. The values of $\Delta_s\alpha$ are shown in Table III and Fig. 4.

[18] The same is true of our calculation of J [Eq. (23)], but we saw that J was not particularly important.

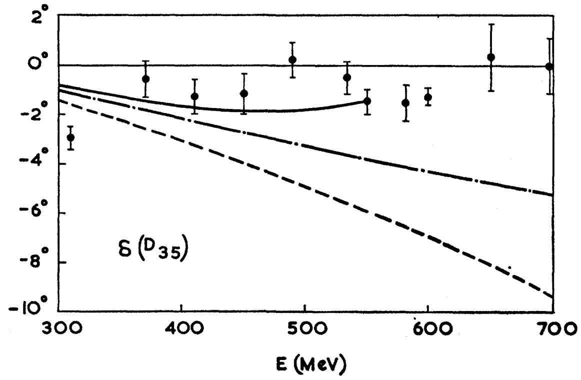

FIG. 4. Predictions for the real part of the phase shift in the state D_{35}, together with the results of ADLL. The notation is the same as in Fig. 3.

The value of $I(s)$ depends markedly on the amount of inelasticity in the region of 800 MeV. Using the data of Auvil and Lovelace[17] and Eq. (28), we get the values of $\Delta_R\alpha$ shown in Table III. They may have large percentage errors. Figure 4 shows that the final predictions α_F are in tolerable agreement with the results of ADLL. However the situation is not so satisfactory as it was in the case of D_{33} where $\Delta_R\alpha$ was determined somewhat more accurately and was also relatively less important.

D_{15}

Again the method is similar to that for D_{33}. The data of DHL give

$$F_{\mathrm{L.R.}}'(s) \sim -0.6\times 10^{-2}/s\,, \quad (s\to +\infty)\,.$$

ADLL find only small inelasticity up to 700 MeV. Auvil and Lovelace[17] do suggest that there may be appreciable inelasticity around 900 MeV. Certainly J is small; we estimate its value to be 2×10^{-3}. Thus the short-range pole is

$$0.8\times 10^{-3}/(s+M^2)\,.$$

This gives $\Delta_s\alpha = 0.3°$, 0.6°, 1.0°, and 1.4°, at 400, 500, 600, and 700 MeV, respectively.

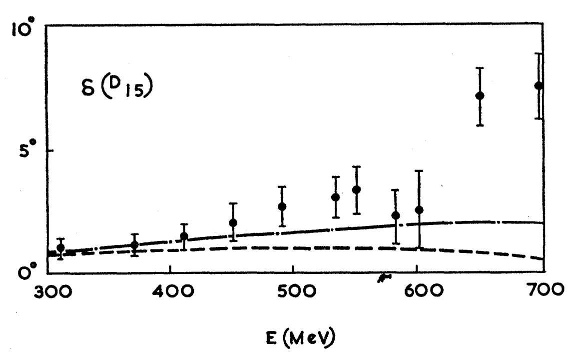

FIG. 5. Predictions for the real parts of the phase shift in the state D_{15}, together with the results of ADLL. The DHL prediction $\alpha(s)$ is shown by – – – – and the value of $[\alpha(s)+\Delta_s\alpha]$ is shown by — · — · —. The points with error flags are the ADLL values.

Rescattering is difficult to estimate accurately from the available information. We get $\Delta_R\alpha = 0.2°$, 0.5°, and 0.9°, at 400, 500, and 600 MeV, respectively. The values of $(\alpha(s)+\Delta_s\alpha)$ are shown in Fig. 5. Allowing for the small rescattering, the agreement with the values of ADLL is reasonably satisfactory except for the two points above 600 MeV.

D_{13}

This partial wave is dominated by the strong attraction due to ρ exchange. Because there is not accurate data on the inelasticity coefficient $\eta(s)$ above 700 MeV, we cannot evaluate J. Rough estimates however show that the short-range interaction is small, and it will not cause much change in the general features of this amplitude, as discussed elsewhere.[2]

F_{17}

The data of DHL give

$$F_{\mathrm{L.R.}}'(s) \sim -9.5\times 10^{-4}/s\,, \quad (s\to +\infty)\,.$$

The integral J can be neglected, and the short-range

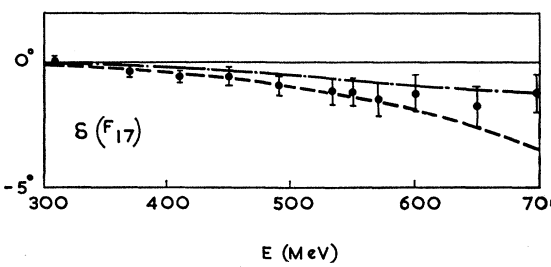

FIG. 6. Predictions for the real part of the phase shift in the state F_{17}, together with the results of ADLL. The DHL value $\alpha(s)$ is shown by – – – –, and our final value $(\alpha(s)+\Delta_s\alpha)$ is shown by — · — · — (rescattering is negligible). The points with error flags are the ADLL values.

pole is

$$9.5\times 10^{-4}/(s+M^2)\,.$$

This gives $\Delta_s\alpha = 0.2°$, 1.2°, and 2.2° at 400, 600, and 700 MeV, respectively. Rescattering is very small in this energy range and can be neglected. In Fig. 6 it is seen that $(\alpha(s)+\Delta_s\alpha)$ is in fairly good agreement with the values of ADLL.

F_{35}

The data of DHL give

$$F_{\mathrm{L\,R.}}'(s) \sim -5.0\times 10^{-4}/s\,, \quad (s\to +\infty)\,.$$

Again J is negligible and the short-range pole is

$$5.0\times 10^{-4}/(s+M^2)\,.$$

The values of $\Delta_s\alpha$ are 0.1°, 0.6°, and 1.1°, at 400, 600, and 700 MeV, respectively Rescattering is very small

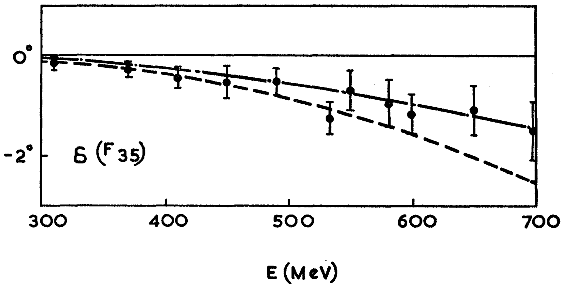

FIG. 7. Predictions for the real part of the phase shift in the state F_{35}, together with the results of ADLL. The notation is the same as in Fig. 6.

and can be neglected. Figure 7 shows that $(\alpha(s)+\Delta_s\alpha)$ is in good agreement with the values of ADLL.

F_{15} and F_{37}

The difficulties mentioned in the case of D_{13} prevent us from making any useful comment in these cases.

5. *P* WAVES

It is more difficult to determine the short-range and rescattering corrections for the small P waves because the factor q^{-2l} is not so effective in suppressing the short-range interactions or in suppressing the high-energy rescattering. We shall discuss the difficulties in relation to the amplitude P_{31}.

P_{31}

The calculations of DHL show that the four basic exchange processes (cf., Sec. 3) give a moderately strong repulsion at high energies, so we expect that there is a short-range attraction. We have

$$F_{\mathrm{L.R.}}'^{(s)}\sim -1.37/s\,,\quad (s\rightarrow +\infty)\,. \tag{29}$$

There is a considerable high-energy contribution to the integral J. ADLL give $\eta(s)$ up to 700 MeV, and the data of Auvil and Lovelace[17] enable us to estimate $\mathrm{Im}F(s)$ at higher energies. We get $J\simeq 0.10$. Fortunately J is only a small term in the sum rule (9) so errors in J are not too serious. Now using Eq. (29) we get $\Gamma\simeq 1.5$.

It is not easy to determine the best position for the short-range pole. A short-range pole

$$1.5/(s+M^2)\,,$$

would give short-range contributions to $F_{1-}'(s)$ of 1.4×10^{-2} and 1.9×10^{-2} at the physical threshold $s=(M+\mu)^2$ and the crossed threshold $s=(M-\mu)^2$, respectively. Using the P-wave scattering lengths, DHL found that the short-range contributions at these thresholds were $(-0.1\pm 0.5)\times 10^{-2}$ and $(-0.3\pm 0.5)\times 10^{-2}$, respectively.

If we are to continue to represent the short-range interaction by a single pole in this case, it is necessary to move the pole to a much larger negative value of s. Remembering that $F_{1-}(s)$ only has the suppression factor q^{-2}, and looking at the values of q^2 in Table I, it appears that it might be reasonable to move the pole as far away as $s=-3M^2$ (i.e., $s=-145$). The short-range pole

$$1.5/(s+3M^2) \tag{30}$$

gives 0.7×10^{-2} and 0.8×10^{-2} at the physical and crossed thresholds, respectively. Allowing for the quoted errors in the DHL values of the short-range contributions at the thresholds, this is perhaps not unreasonable. Certainly it seems undesirable to move the pole to even more negative values of s, since the arc of the circle $|\arg s|>66°$ will make some contribution to the short-range interaction and the pole should not be too far away from this arc. However, it should be remembered in what follows that our estimates of the short-range corrections in the case of P_{31} may contain considerable errors which arise from this difficulty in placing the pole.

In the case of P_{31}, Eq. (26) is not sufficiently accurate to determine $\Delta_s\alpha$. Instead we must use the relation

$$\tfrac{1}{2}\eta(s)\sin[2(\alpha(s)+\Delta_s\alpha)]=q^3\left(F_{\mathrm{L.R.}}'(s)+\frac{\Gamma}{s+3M^2}\right), \tag{30a}$$

where $F_{\mathrm{L.R.}}'(s)$ is the long-range left-hand cut term as calculated by DHL. In discussing the P waves, DHL used the P-wave scattering lengths $a_{2T,2J}$ to get a *rough* estimate of the short-range interaction. The unitary sum rule method is, however, much more reliable. For the P waves the values of $\alpha(s)$ which we use are the values which are given by the calculations of DHL using only the four basic long-range exchange processes.

From Eq. (30a) we get the values of $(\alpha(s)+\Delta_s\alpha)$ in Table IV.

The rescattering correction $\Delta_R\alpha$ is calculated as follows. We evaluate the integral $I(s)$ [Eq. (27)] using $(\alpha(s)+\Delta_s\alpha)$ as an approximation to the phase shift up to 700 MeV, and we use the values of $\eta(s)$ given by ADLL. For the region above 700 MeV we use the information given by Auvil and Lovelace.[17] It is seen that the rescattering correction $\Delta_R\alpha$ is small compared with the short-range correction. Figure 8 shows that the final predicted values $\alpha_F(s)$ are in reasonable agreement with the results of ADLL. However, as was pointed out

TABLE IV. The amplitude P_{31}. The phase shifts $[\alpha(s)+\Delta_s\alpha]$ are calculated from Eqs. (30) and (30a). The phase shift $\alpha(s)$ is the result of DHL using *only* the four basic long-range exchange processes. The rescattering corrections $\Delta_R\alpha$ may be considerably in error at 400 MeV and above. $\alpha_F(s)$ is the final predicted phase shift.

Lab energy (MeV)	300	400	500	600
$\alpha(s)$	$-12.0°$	$-18.6°$	$-28.7°$	⋯
$\alpha(s)+\Delta_s\alpha$	$-8.2°$	$-11.5°$	$-16.8°$	$-20.5°$
$\Delta_R\alpha$	$\sim 0.5°$	$\sim 1.0°$	$\sim 1.5°$	$\sim 2°$
$\alpha_F(s)$	$\sim -7.7°$	$\sim -10.5°$	$\sim -15.3°$	$\sim -18.5°$

above, the uncertainties in our method may be large in the case of P_{31}.

P_{13}

The data of DHL give

$$F_{\text{L.R.}}'(s) \sim +7\times10^{-2}/s\,, \quad (s\to+\infty)\,.$$

This partial wave is elastic or almost elastic up to high energies, according to the results of ADLL and of Auvil and Lovelace.[17] We find $J\simeq1\times10^{-3}$. Thus, $\Gamma=-7\times10^{-2}$. This gives a very small short-range interaction. Even if we place the pole at $s=-M^2$, the term

$$-7\times10^{-2}/(s+M^2)$$

gives only -0.7×10^{-3} at the physical threshold. The values of $\Delta_s\alpha$ are $-0.4°$ and $-0.8°$ at 400 and 600 MeV, respectively. ADLL find that this phase shift is elastic up to 700 MeV, and Auvil and Lovelace[17] find no appreciable value of Im$F(s)$ until above 900 MeV. The rescattering is thus very small up to 600 MeV. Hence, the values $\alpha(s)$ of DHL require negligible correction. It is seen in Fig. 9 that these values are indeed in good agreement with the results of ADLL.

P_{33}

This is the only resonant amplitude for which the short-range interaction can be discussed reasonably accurately at the present time. The data of DHL give

$$F_{\text{L.R.}}'(s) \sim 1.45/s\,, \quad (s\to+\infty)\,. \tag{31}$$

The integral J is of course large in this case. We find

$$J=1.20\pm0.20\,. \tag{32}$$

This value is obtained by using the best Layson type

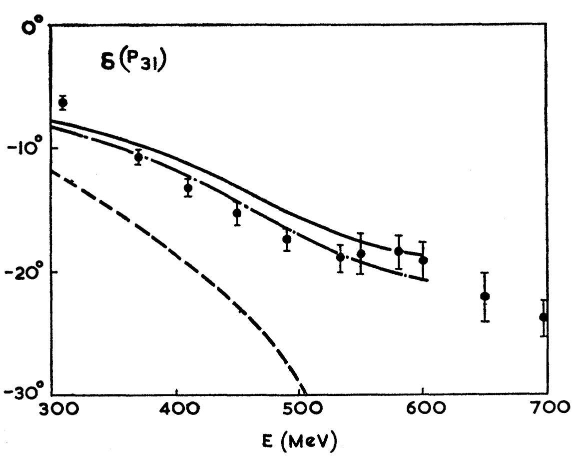

FIG. 8. Predictions for the real part of the phase shift in the state P_{31}. The line ---- shows $\alpha(s)$ which is the value DHL would give using only the four basic long-range interactions (and no rescattering). The line — · — · — shows $(\alpha(s)+\Delta_s\alpha)$ and the solid line is our final value $\alpha_F=\alpha(s)+\Delta_s\alpha+\Delta_R\alpha$. The points with error flags are the ADLL results.

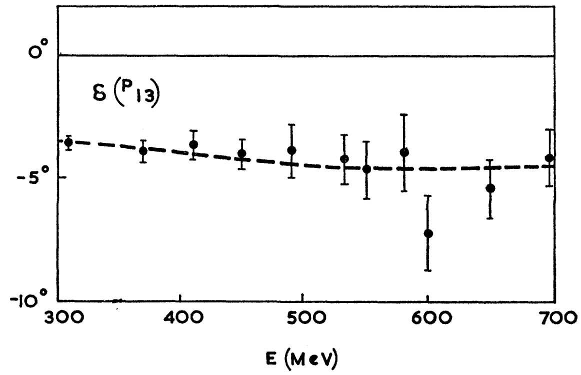

FIG. 9. The broken line shows $\alpha(s)$ which is the DHL value based only on the four basic long-range interactions in the state P_{13}. This is our final value since the short-range correction $\Delta_s\alpha$ and the rescattering correction $\Delta_R\alpha$ are negligible. The points with error flags are the ADLL results.

solution of the authors[19] up to 700 MeV and taking an error which allows for almost complete ignorance as to what happens above 700 MeV. The error is certainly on the pessimistic side, since by using the data of Auvil and Lovelace[17] we could reduce it somewhat. Equations (9), (31), and (32) now give

$$\Gamma=-0.25\pm0.30\,. \tag{33}$$

The short-range pole

$$\Gamma/(s+3M^2)\,,$$

gives $-(1.3\pm1.5)\times10^{-3}$ and $-(1.5\pm1.8)\times10^{-3}$ at the physical and crossed thresholds, respectively. The short-range pole

$$\Gamma/(s+M^2)\,,$$

gives $-(2.4\pm2.9)\times10^{-3}$ and $-(3.2\pm3.8)\times10^{-3}$ at these thresholds. DHL estimate, using the P-wave scattering lengths, that the short-range contributions to $F_{1+}'(s)$ at the physical and crossed thresholds are $(5\pm4)\times10^{-3}$ and $(7\pm5)\times10^{-3}$, respectively; either position for the pole is reasonably consistent with the values of DHL. Further, the short-range contribution to $F_{1+}'(s)$ is, in both cases, very small compared with the large attractive terms arising from N exchange and $(\pi\pi)_0$ exchange, and the effect on the position, width, and shape of the resonance as determined by the variational method[19] is expected to be very small.

6. DISCUSSION AND CONCLUSIONS

We saw in Sec. 2, Eq. (13), that the unitary sum rule Eq. (9) requires that for $l\geqslant1$ the total contribution from the left-hand cuts $F_{l\pm}'(s)$ should be positive at very high energies. The predictions of DHL, which are based on the four long-range exchange processes shown in Fig. 2, violate this high-energy condition in the case of certain partial waves. The most noticeable violations occur in

[19] A. Donnachie and J. Hamilton, Phys. Rev. 133, B1053 (1964).

the amplitudes P_{31}, D_{33}, D_{35}, F_{17}, and F_{35}. Unitarity shows that in these states there must be a very short-range attraction which makes it possible to satisfy Eq. (13). This short-range attraction gives a positive correction $\Delta_s\alpha$ to the phase shifts $\alpha(s)$ predicted by DHL. In addition, there may be a small correction $\Delta_R\alpha$ for rescattering, because rescattering was ignored in the DHL calculations of the nonresonant amplitudes.

Agreement with the Experimental Values

In the cases of F_{17} and F_{35} the rescattering correction $\Delta_R\alpha$ is very small, and our predictions for $\Delta_s\alpha$ account well for the difference between the predictions of DHL and the values deduced by ADLL from the experiments over the range 300 to 700 MeV. For P_{31} and D_{33} the short-range correction $\Delta_s\alpha$ accounts for the greater part of the difference between the DHL and ADLL values over the range 300 to 600 MeV. Our estimates of the rescattering correction (Tables II and IV) show that $\Delta_R\alpha$ is much smaller than $\Delta_s\alpha$ for these amplitudes.

In the case of D_{35} the rescattering correction $\Delta_R\alpha$ may be larger than $\Delta_s\alpha$ above 500 MeV (cf., Table III). Our final result is in fairly good agreement with the values of ADLL up to around 600 MeV.

In the cases of P_{13} and D_{15} the short-range interaction is small. For P_{13}, $\Delta_s\alpha$ is negligible and the rescattering correction $\Delta_R\alpha$ is also small up to around 600 MeV, so we expect the DHL predictions to be close to the experimental values. This is in fact the case. For D_{15} the corrections $\Delta_s\alpha$ and $\Delta_R\alpha$ cannot be ignored, but our final values again show good agreement with ADLL up to 600 MeV.

Comments on the Results

The unitary sum rule has enabled us to estimate the short-range interaction for all the nonresonant P, D, and F waves (except P_{11} which we have not discussed). This gives corrections to the DHL values of the phase shifts which, after allowing for rescattering, in each case bring them close to the results of ADLL. The most uncertain part of our calculations is the choice of the position of the short-range pole in the case of P_{31}.

The predictions of course depend not only on knowledge of the longer range parts of the π-N interaction but also on knowing the inelasticity coefficient $\eta(s)$. The coefficient $\eta(s)$ plays a very important role in the rescattering correction $\Delta_R\alpha$, and it also appears in the integral J [Eq. (23)] which occurs in the sum rule. However in the case of the nonresonant amplitudes, J is only a small term in the sum rule, and large errors in J would have little effect on our results.

It seems that the main limitation to predicting the nonresonant phase shifts at energies above 600–700 MeV is lack of precise knowledge of $\eta(s)$. There is at present no satisfactory theoretical method of predicting $\eta(s)$, so we have to rely entirely on experimental determinations of the amount of inelasticity.

The only resonant amplitude which can be examined accurately at present is P_{33}. It was shown in Sec. 5 that in this state the short-range interaction is very small compared with the dominant long-range attractions [i.e., N exchange and $(\pi\pi)_0$ exchange] which produce the resonance.

The Nature of the Short-Range Interaction

We have seen, for example, in the cases of P_{31} and D_{33} that the attractive short-range interaction tends to reduce appreciably the repulsion produced by the long-range interaction. Ultimately at very high energies the short-range attraction dominates, and furthermore, at moderate energies the net interaction cannot be strongly repulsive. Of course none of these arguments apply to S waves ($l=0$), and it is of interest to note that in the case of S-wave π-N scattering an empirical analysis[20] has shown that there is a very strong short-range *repulsion.*

Finally, we should point out that although the unitary sum rule enables us to estimate the short-range interactions it does not give us a physical picture of their origin; we cannot, for example, identify them as arising from some particular exchange process. The unitary sum rule is in this aspect similar to many other sum rules.

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[20] J. Hamilton, T. D. Spearman, and W. S. Woolcock, Ann. Phys. (N. Y.) **17**, 1 (1962). See particularly Fig. 10 of that paper.