Frequency Shift in High-Intensity Compton Scattering*†

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(Received 12 November 1964)

The frequency shift predicted by Brown and Kibble, and by Goldman, for a photon scattered out of an intense beam by a free electron is re-examined. It is shown that the effect has a very simple classical interpretation, as a Doppler shift arising from the nonzero average velocity of the electron in the beam. The discrepancy between this prediction and the recent perturbation calculation of Fried and Eberly is shown to arise from the use, in the latter, of a pure monochromatic beam rather than a wave train of finite length. It is shown that the effect should arise for a quantized photon beam as well as for a classical one. The question of energy-momentum conservation is discussed. With the help of a one-dimensional model which exhibits all the essential features of the effect, it is shown that the extra energy and momentum which are generated in the scattering process are taken up by the beam itself in the form of an extremely small shift in the average momentum of a photon in the beam. The possibility of experimental detection of the effect is briefly discussed.

1. INTRODUCTION

THE interaction of intense laser beams with matter exhibits many interesting and unusual features.[1] It is therefore of considerable interest to examine their interaction with a simple system for which the effects to be expected can be calculated with precision. In a previous paper,[2] here referred to as BK, the interaction with a single free electron was examined, using a semiclassical treatment in which the laser beam was represented as a classical plane-wave field. It was shown that the frequency of a photon scattered out of the beam should be shifted by an amount depending on the beam intensity, though quite small for presently available intensities. The same result was obtained independently by Goldman,[3] using similar methods.

In this paper, we shall examine certain aspects of this effect in greater detail. In particular, we hope to elucidate an apparent paradox concerned with energy-momentum conservation, and to show that, contrary to a recent assertion of Fried and Eberly,[4] the effect should occur for a quantized beam as well as for a classical one.

We begin by presenting, in Sec. 2, a simple discussion of the origin of the effect. We show that the frequency shift is simply a Doppler shift due to the fact that an electron initially at rest acquires a nonzero average velocity in the direction of the beam. This velocity arises from the circumstance that, when the amplitude is varying, the phase difference between the electron velocity $\mathbf{v}$ and the electric field $\mathbf{E}$ is not exactly $\frac{1}{2}\pi$, so that there is a nonzero force $\mathbf{v}\times\mathbf{B}$ in the beam direction. This provides an acceleration when the amplitude is increasing, and a deceleration when it is decreasing. Quantum-mechanically, the effect arises, as was shown in BK, from the mass shift of an electron propagating in an intense beam. It may also be regarded as an effect complementary to the refraction of a photon passing through a cloud of electrons.

In Sec. 3, we shall show explicitly that a quantum-mechanical calculation of this process is necessarily equivalent (so long as radiative corrections are neglected) to a semiclassical calculation in which the field is treated as an unquantized external field. This equivalence is essentially a statement of the correspondence principle for such processes. The differences between our calculation and that of Fried and Eberly are analyzed in Sec. 4, and the discrepancy is shown to arise from the use, in the latter work, of a monochromatic beam from the outset of the calculation. Such a beam must necessarily occupy the whole of space, and it is therefore inconsistent to regard the electron as free even in the remote past. Instead, we should use a wave train of finite length, and treat the monochromatic beam as a limiting case obtained when the length of the beam tends to infinity. This limiting procedure is unusually delicate for problems involving beams of finite density. We shall show that a calculation using a beam of finite length must yield a frequency shift, which persists even in the limit. Classically, it is clear that the effect arises from the acceleration of the electron during the process of switching the beam on and off. The velocity of the electron inside the beam depends only on the beam intensity. If the beam is switched on more gradually, the acceleration is slower, but the final velocity is the same. Thus, the precise way in which the amplitude varies with time is not important.

There is one reservation to be made at this point. It is essential that the amplitude should be a slowly varying function of time, or equivalently, that the frequency

* The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research under Grants AF EOAR 62-87 and 64-46 through the European Office of Aerospace Research, (OAR), U. S. Air Force.

† A preliminary version of this work was presented at the Conference on Quantum Electrodynamics of High-Intensity Photon Beams held at Durham, North Carolina on 26–27 August 1964, under the auspices of the U. S. Army Research Office.

[1] See, for example, P. A. Franken and J. F. Ward, Rev. Mod. Phys. **35**, 23 (1963); Z. Fried and W. M. Frank, Nuovo Cimento **27**, 218 (1963).

[2] L. S. Brown and T. W. B. Kibble, Phys. Rev. **133**, A705 (1964).

[3] I. I. Goldman, Phys. Letters **8**, 103 (1964). See also A. I. Nikishov and V. I. Ritus, Zh. Eksperim. i Teor. Fiz. **46**, 776 (1963) [English transl.: Soviet Phys.—JETP **19**, 529 (1964)].

[4] Z. Fried and J. H. Eberly, Phys. Rev. **136**, B871 (1964).

spread in the beam should be small. If the beam possesses a sharp cutoff there are edge effects which modify the conclusions, and which could be quite large.[5] These effects are discussed in Sec. 5, and shown to be small in physically interesting situations, except conceivably for high-velocity electrons and sharply focused beams.

In Sec. 6 we examine the nature of the limiting process involved in going to the monochromatic limit. By considering the case of an exponentially varying amplitude, we show explicitly that in this limit we recover the results of BK.

Because the use of coherent photon states makes it rather difficult to discuss in detail any changes in the energy and momentum of the beam, we shall consider in Sec. 7 a simplified one-dimensional model which exhibits all the important features of the effect. For this model, a calculation involving states with definite photon number can be carried through even when the photons are not in pure momentum eigenstates. The calculation is considerably simplified by employing coherent states as generating functions for the N-particle states. It is possible in this model to see explicitly where the energy corresponding to the frequency shift goes. In fact, it is taken up by the beam itself in the form of a very small shift in the average momentum of a photon in the beam.

The conclusions are summarized in Sec. 8, and some remaining unsolved problems are discussed. We also include a discussion of the possibility of experimental detection of the effect.

For simplicity, we shall treat the electron as a scalar particle throughout. The spin is in any case unimportant in the optical region of frequencies, and does not affect the frequency shift, but only the cross section.

2. ORIGIN OF THE FREQUENCY SHIFT

The predicted frequency shift is a purely classical effect (though, as we shall show explicitly later, it does not disappear in a quantum-mechanical calculation). We shall therefore begin by discussing its origin in classical terms.

Let us consider a circularly polarized wave propagating in the z direction, with an amplitude which increases slowly to its maximum value, and after some interval decreases again to zero. As the electron enters the beam, it will start to oscillate with increasing amplitude. Now, if the amplitude were constant, the velocity $\mathbf{v}$ would follow $\mathbf{E}$ exactly $\frac{1}{2}\pi$ out of phase. However, when the amplitude is increasing, the phase difference is less than $\frac{1}{2}\pi$, and there is a nonzero accelerating force $\mathbf{v}\times\mathbf{B}$ in the z direction. Similarly, when the amplitude is decreasing, there is a decelerating force. If the electron is initially at rest, then, while in the beam, it has a nonzero average velocity in the direction of propagation. The frequency shift is simply a Doppler shift produced by this mean velocity.

To be specific, let us consider the field represented by the vector potential

$$A_x=a(\tau)\cos\omega\tau\,,\quad A_y=a(\tau)\sin\omega\tau\,,\quad \tau=t-z\,,$$

where $a(\tau)$ is a smooth function of τ vanishing at $\tau=\pm\infty$. The electric and magnetic fields are then

$$E_x=B_y=a\omega\sin\omega\tau-a'\cos\omega\tau\,,$$
$$-E_y=B_x=a\omega\cos\omega\tau+a'\sin\omega\tau\,.$$

To find the electron momentum, we have to integrate the Lorentz force equation. It is not hard to verify that, if the electron is initially at rest at the origin, then its momentum and energy at proper time[6] τ are (with $c=1$)

$$\begin{aligned} p_x(\tau)&=-ea\cos\omega\tau\,,\\ p_y(\tau)&=-ea\sin\omega\tau\,,\\ p_z(\tau)&=e^2a^2/2m\,,\\ p_0(\tau)&=m+e^2a^2/2m\,. \end{aligned}\tag{1}$$

The mean velocity of the electron in the beam is therefore in the z direction, and of magnitude[7]

$$v=\tfrac{1}{2}\mu^2/(1+\tfrac{1}{2}\mu^2)\,,\quad \mu^2=e^2a^2/m^2\,.\tag{2}$$

Thus for low frequencies, $\omega\ll m$, the wavelength of light scattered at an angle ϑ is shifted by the amount

$$(\lambda'-\lambda)/\lambda=\mu^2\sin^2(\tfrac{1}{2}\vartheta)\,.\tag{3}$$

In a quantum-mechanical calculation, the origin of the frequency shift may be traced to the fact that the mass of an electron propagating in an intense electromagnetic field is increased by the amount[8]

$$\Delta m^2=\langle -e^2A_\mu A^\mu\rangle_{\mathrm{av}}=\mu^2m^2\,,\tag{4}$$

where the brackets denote a time average over many oscillation periods. This may be seen either from the Klein-Gordon equation or, in terms of Feynman diagrams, by noting that the e^2A^2 term in the interaction yields a constant (or slowly varying) mass correction as well as an oscillatory term.[9]

To acquire this mass, the electron as it propagates into the beam must take up some energy and momentum from it. If the energy-momentum is initially p^μ, then inside the beam it must be $\bar{p}^\mu=p^\mu+\alpha k^\mu$, where k^μ is the momentum of a photon in the beam. (Classically, $k^\mu=\omega n^\mu$, where n^μ is a null vector in the direction of propagation, and ω is the angular frequency.) The constant α

[5] Because these effects were neglected, the discussion of a beam with sharp cutoff in an earlier (unpublished) version of this paper was incorrect.

[6] It is characteristic of propagation in a unidirectional field that the proper time is linearly related to $t-z$. This may easily be verified by integrating (1) to find the coordinates as functions of τ.

[7] The parameter μ^2 was denoted by ν^2 in BK. This was, however, an unfortunate choice of notation, since ν might be confused with the frequency.

[8] We use natural units with $c=\hbar=1$, and a metric with signature $(1-1-1-1)$ and scalar product $a\cdot b=a_0b_0-\mathbf{a}\cdot\mathbf{b}$.

[9] This is also evident from the structure of the electron Green's function. See BK, Appendix A.

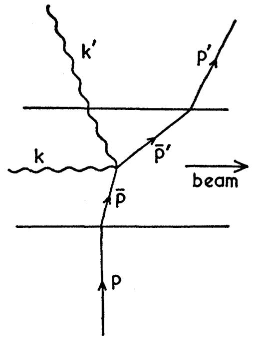

FIG. 1. Kinematics of the high-intensity Compton scattering process.

may be determined from the condition

$$\bar{p}^2=p^2+\Delta m^2.$$

This yields

$$\bar{p}^\mu=p^\mu+(\Delta m^2/2p\cdot k)k^\mu. \qquad (5)$$

This momentum $\bar{p}$ is simply the average value of the classical momentum (1).

It is interesting to note that this relation may be obtained in yet another way, by considering the propagation of a photon through a cloud of electrons. It is well known that the optical-path length l in a medium with electron density n_e is increased by the amount

$$\Delta l/l=n_e\lambda^2 r_0/2\pi,$$

where $r_0=e^2/4\pi m$ is the classical electron radius. This change in optical-path length must be reflected in a corresponding decrease in the photon momentum. Hence, if the photon density is n_{ph}, the total transfer of momentum per unit volume is

$$n_e\Delta p=-n_{ph}\Delta k=\lambda r_0 n_e n_{ph}. \qquad (6)$$

It is easy to verify that this agrees with (5). For an electron initially at rest (which is of course implicit in the discussion of optical-path length), (5) yields

$$\Delta p=\Delta m^2/2m=\tfrac{1}{2}\mu^2 m.$$

But, using the relation between amplitude and photon density, we easily find from (2)

$$\mu^2=2\lambda r_0 n_{ph}/m, \qquad (7)$$

and this clearly reproduces (6).

If the electron passes right through the beam without scattering, its momentum reverts from $\bar{p}$ to p on emerging from the other side. However, if some process occurs within the beam to change its momentum, this will not be the case. In particular, if the electron scatters a photon of energy-momentum k' out of the beam, as shown in Fig. 1, then, looking at the process from inside the beam, we find that the energy-momentum conservation equation takes the form

$$\bar{p}'+k'=\bar{p}+k. \qquad (8)$$

Equivalently, in terms of the momenta outside the beam,

$$p'+k'=p+k-\zeta k, \qquad (9)$$

where the dimensionless constant ζ is given by

$$\zeta=\frac{\Delta m^2}{2}\left(\frac{1}{p'\cdot k}-\frac{1}{p\cdot k}\right). \qquad (10)$$

It is the ζk term in (9) which gives rise to the frequency shift. From (8) or (9) we easily find

$$p\cdot k=p\cdot k'+(\Delta m^2/2p\cdot k)k\cdot k'+k\cdot k'. \qquad (11)$$

For $\omega\ll m$, the last term on the right is negligible, and we recover the expression (3) for the wavelength shift. (The general expression is given in Sec. 8.)

From (9) we may conclude that the total energy-momentum of the beam must change in this process by the amount $(1-\zeta)k$. So long as we regard the beam as a classical electromagnetic wave, this conclusion poses no particular problems. However, when we consider a quantized photon beam, we are immediately faced with an apparent difficulty. Since the number of photons in the beam can only change by an integer, it seems at first sight impossible to change the momentum of the beam by any amount less than k. These considerations have led Fried and Eberly[4] to conclude that the frequency shift is an exclusively classical effect which cannot occur for a quantized beam.

On the other hand, there is a general correspondence between the semiclassical and quantum-mechanical descriptions of a radiation field[10] which would lead one to believe that the two methods of calculation ought to lead to the same answer, so long as radiative corrections are neglected. Indeed, we shall prove that this is true for our problem in the next section, and exhibit the equivalence explicitly in Sec. 4. Any failure of the equivalence would in fact be tantamount to a violation of the correspondence principle.

We are still faced, of course, with the problem of what happens to the momentum ζk. However, there is another way in which the momentum of the beam can be changed, apart from removing photons from it. This is to change the average momentum of the remaining photons. We shall try to show in the following sections that this is indeed what happens. This conclusion may seem less surprising when it is realized that the momentum of the beam is in any case decreased, albeit by a minute amount, during the presence of the electron. (If many electrons were present, the amount could even be large and lead to refraction of the beam.) If the electron scatters inside the beam, the momentum does not revert precisely to its original value at the end of the process.

A simple analogy may perhaps help to explain the mechanism of this process. Consider a potential which has the effect of splitting a degenerate energy level into two components with separation ΔE. Let us slowly switch on the potential. Then, if the system is in the

[10] See E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

upper state, we can wait till it decays emitting a photon of energy ΔE, and then switch off again. The amount of energy extracted from the potential ΔE is clearly independent of the rate at which it is switched on and off. In this case, it is of course clear where the energy has come from. The work done in switching on the potential exceeds that recovered when it is switched off by just this amount. It is our contention that a similar mechanism operates to produce the frequency shift. The work done by the beam in accelerating the electron as it enters is not quite equal to the energy recovered when it is decelerated on emerging from the other side.

It is clear that to see an effect of this kind, we must use a beam of limited extent. The important feature is the acceleration of the electron while the amplitude is increasing, and its deceleration when it falls again to zero. The calculation of Fried and Eberly was based on a description of the beam as a state of definite momentum. However, such a state must necessarily occupy the whole of space, and it is therefore inconsistent to assume that the electron is initially outside the beam. To avoid this problem, it is necessary to use a wave packet to describe the beam, and only go to the limit of a monochromatic beam at the end of the calculation. For a beam of finite intensity, this limiting procedure is unusually delicate. In most scattering problems, it is equally satisfactory to use a pure momentum eigenstate normalized in a box. The reason for this is that, as the volume of the box is increased to infinity, the particle density tends everywhere to zero. Thus, although the particles are not localized, they are on the average infinitely far apart and may consistently be treated as free. The situation in our problem is quite different however. If we let the number of photons tend to infinity with the size of the box, then the electron is always moving in a beam of finite density and cannot be regarded as free. Therefore, any results obtained using a pure monochromatic beam from the outset of the calculation must refer to momenta measured inside the beam, not to free-particle momenta.

A similar situation obtains in the example of the potential quoted above. If we do not switch off the potential, but consider a constant potential which is always on, then the initial and final states of the system are not the unperturbed energy levels, but have energies which include the perturbation. It is only with this interpretation that energy can be conserved in the process.

It is actually unnecessary for our purposes to limit the extent of the beam in transverse directions. It is sufficient to take it to be a wave train of finite length, with infinite plane-wave fronts. It is then consistent to assume that the electron is free before the arrival of the beam and after it has passed. Such a beam possesses a unique propagation direction, which may be covariantly characterized by a null vector n^μ lying on the forward light cone. It is described classically by a vector potential A_μ which is a function only of $\tau=n\cdot x(=t-z$, say). Quantum-mechanically it is described as a superposition of many-photon states in which each photon has a momentum parallel to n^μ, but with a spread of frequencies.

It is not impossible that the results would be affected by limiting the extent of the beam in transverse directions also. This is a point which deserves further study. However, it is very unlikely that the effect would be to remove the frequency shift altogether.

3. EQUIVALENCE OF SEMICLASSICAL AND QUANTUM-MECHANICAL CALCULATIONS

The semiclassical method of BK seems adequate to describe scattering processes occurring in the intense, coherent electromagnetic field produced by a laser. However, its validity has recently been questioned by a number of authors,[4,11,12] who assert that qualitatively different results are obtained when the field is quantized. It is therefore necessary to show that a fully quantum-mechanical calculation leads to the same results, so long as radiative corrections are neglected. This result, which is of quite general validity, is essentially a verification that the quantum theory possesses the correct classical limit.

In discussing the classical limit of the quantized radiation field it is convenient to employ the "coherent" states,[13] defined as eigenstates of the positive-frequency part of the vector potential operator $\hat{A}_\mu^{(+)}(x)$,[14]

$$\hat{A}_\mu^{(+)}(x)|a\rangle=|a\rangle a_\mu(x).$$

These states are the analogs of the classical-limit states of a quantum oscillator. Each coherent state corresponds uniquely to a classical solution of the wave equation, $A_\mu(x)=2\,\mathrm{Re}[a_\mu(x)]$. It is essentially the quantum state which most closely approximates this classical field. (The classical-limit states of the oscillator may be defined as those states which minimize the product $\Delta q\Delta p$, and simultaneously, for given $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$, minimize the energy expectation value $\langle\hat{H}\rangle$.) Because these states form an overcomplete set, all matrix elements of any operator $\hat{B}$ may be obtained from its diagonal matrix elements $\langle a|\hat{B}|a\rangle$.

Now, to be specific, let us consider a calculation of the Compton scattering process by standard Feynman-Dyson perturbation theory, representing the beam as a coherent state $|a\rangle$ of the radiation field. In evaluating the contribution of the diagrams of any particular order, we have to calculate the matrix element of a time-ordered product,

$$\langle\mathbf{p}',\mathbf{k}'\epsilon',a|T[\hat{V}(x_1)\cdots\hat{V}(x_n)]|\mathbf{p},a\rangle,$$

[11] P. J. Redmond, paper presented at the Conference on Quantum Electrodynamics of High-Intensity Photon Beams, Durham, North Carolina, August 1964 (unpublished).

[12] P. Stehle and P. G. deBaryshe (unpublished).

[13] See J. Schwinger, Phys. Rev. **91**, 728 (1953); S. S. Schweber, J. Math. Phys. **3**, 831 (1962); R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[14] We use a circumflex to denote an operator.

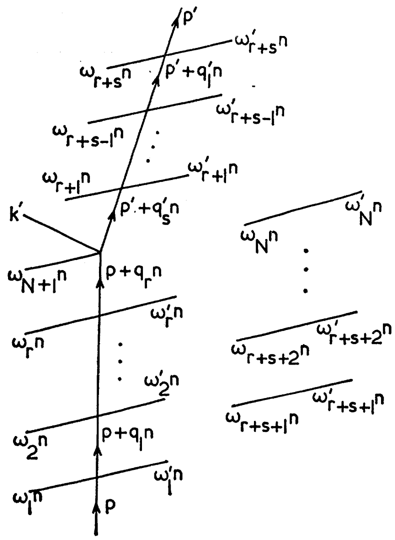

FIG. 2. A typical Feynman diagram for the one-dimensional model calculation.

where $\hat{V}$ is the interaction operator of scalar electrodynamics,

$$\hat{V}=ie\hat{\varphi}^*\overleftrightarrow{\partial}_\mu\hat{\varphi}\hat{A}^\mu-e^2\hat{A}_\mu\hat{A}^\mu\hat{\varphi}^*\hat{\varphi}.$$

To do this, we have to rewrite the time-ordered product as a sum of normal-ordered products. However, the approximation of neglecting radiative corrections is equivalent to omitting all internal photon lines, that is, all contractions between $\hat{A}_\mu$ operators. Hence for these operators (but not for $\hat{\varphi}$), we may immediately replace the time ordering by normal ordering. Now the matrix element of a normal-ordered product between coherent states is very simple. Each $\hat{A}_\mu^{(+)}$ yields a factor of a_μ, and each $\hat{A}_\mu^{(-)}$ yields a_μ^*, except for one which is attached to the scattered photon line. Hence, except at this one vertex, we may replace the operator $\hat{A}_\mu(x)$ by the classical field $A_\mu(x)$. The net result is that the scattering amplitude has the form

$$\langle\mathbf{p}',\mathbf{k}'\epsilon',a|\hat{S}|\mathbf{p},a\rangle=\langle\mathbf{p}',\mathbf{k}'\epsilon'|\hat{S}(A)|\mathbf{p}\rangle, \tag{12}$$

where $\hat{S}(A)$ is the scattering operator in the presence of the classical external field $A_\mu(x)$.

4. SCATTERING FROM A FINITE PARALLEL WAVE TRAIN

The general discussion of the preceding section is sufficient to demonstrate the equivalence of the two methods of calculation but, in order to see where the differences between our calculation and that of Fried and Eberly originate, it will be useful to examine the perturbation calculation in more detail.

It is clear from the structure of the Feynman diagrams that

$$\langle\mathbf{p}',\mathbf{k}'\epsilon'|\hat{S}(A)|\mathbf{p}\rangle=-i\int d^4x\, e^{ik'\cdot x}\epsilon'^\mu \times\Phi_{p'}{}^{\text{out}}(x)^*[ie\overleftrightarrow{\partial}_\mu-2e^2A_\mu(x)]\Phi_p{}^{\text{in}}(x), \tag{13}$$

where $\Phi_p{}^{\text{in}}(x)$ represents the sum of all diagrams containing a single electron path which begins with a free electron line of momentum p, and ends at the vertex x, and $\Phi_{p'}{}^{\text{out}}(x)^*$ is defined similarly. This expression simplifies enormously if we make the restriction to a unidirectional beam, for which $A_\mu(x)$ depends only on $\tau=n\cdot x$. We can then perform three of the four integrations over x at each vertex, and obtain δ functions which tell us that the momenta of successive electron lines can differ only by a multiple of n_μ. (Compare Fig. 2, which actually refers to the one-dimensional model discussed later, and therefore contains no one-photon vertices.) Writing $p_\mu+q_jn_\mu$ for the momentum vector of the jth line, we obtain

$$\Phi_p{}^{\text{in}}(x)=\sum_n\frac{1}{(2\pi)^n}\int d\tau_1\cdots d\tau_2dq_1\cdots dq_n \times e^{-ip\cdot x}\prod_{j=1}^n\exp[-iq_j(\tau_{j+1}-\tau_j)]\frac{I_p(\tau_j)}{q_j+i\epsilon}, \tag{14}$$

where $\tau_{n+1}=\tau=n\cdot x$, and

$$I_p(\tau)=[2ep\cdot A(\tau)-e^2A^2(\tau)]/2n\cdot p. \tag{15}$$

We can then perform the integrations over q_j and recognize (14) as the expansion of the expression obtained in BK, Eq. (2.6), by solving the Klein-Gordon equation,

$$\Phi_p{}^{\text{in}}(x)=e^{-ip\cdot x}\exp\left[-i\int_{-\infty}^{\tau}d\tau' I_p(\tau')\right]. \tag{16}$$

Since we are considering a wave train of finite length, the electromagnetic field $F_{\mu\nu}(\tau)$ must tend to zero as $\tau\to\pm\infty$. Hence we can certainly choose the gauge so that

$$A_\mu(\tau)\to 0,\quad \tau\to-\infty. \tag{17}$$

This choice is implicit in the discussion above, for otherwise the integral in the exponent of (16) would not be convergent. There is, however, a possible complication (unimportant for the original argument of BK) which may arise when we come to calculate $\Phi_{p'}{}^{\text{out}}(x)^*$. For in general we cannot simultaneously impose the analogous condition at $\tau=+\infty$. In fact, if

$$F_{\mu\nu}(\tau)=(\epsilon_\mu n_\nu-\epsilon_\nu n_\mu)F(\tau), \tag{18}$$

then

$$A_\mu(+\infty)=\epsilon_\mu\int_{-\infty}^{+\infty}d\tau\, F(\tau), \tag{19}$$

and this integral will be nonzero unless the zero-frequency component of F vanishes. In that case, it is clear that we cannot expect to find a solution of the Klein-Gordon equation satisfying the boundary condition

$$\Phi_{p'}{}^{\text{out}}(x)^*\sim e^{ip'\cdot x},\quad x^0\to+\infty,$$

unless p' is interpreted as the *canonical* momentum rather than the *physical* momentum $p'-eA(\infty)$. It will be more convenient to retain the usual physical significance for p', and write instead the boundary condition

$$\Phi_{p'}{}^{\text{out}}(x)^* \sim e^{i[p'+eA(\infty)]\cdot x}, \quad x^0 \to +\infty .$$

The corresponding solution is then easily found to be

$$\Phi_{p'}{}^{\text{out}}(x)^* = e^{i[p'+eA(\infty)]\cdot x} \exp\left[-i\int_\tau^{+\infty} d\tau' I_{p'}(\tau')\right], \quad (20)$$

where $I_{p'}(\tau)$ is defined exactly as in (15) but with $A(\tau)$ replaced by $A(\tau)-A(\infty)$. The factor involving $A(\infty)$ clearly exhibits the dependence of the wave function on the chosen gauge. Its physical significance will be discussed further in the next section.

We can now see clearly where this calculation differs from one involving an infinite wave train. If $A(\tau)$ were chosen from the start to be strictly monochromatic, then the integrals over τ_j in (14) would yield δ functions which have the effect of reducing the integrations over q_j to discrete sums. Many of the resulting terms would involve vanishing Feynman denominators, arising from intermediate states in which equal numbers of photons have been absorbed and re-emitted. Such infinite terms were encountered by Fried and Eberly, and were removed by a process of normalization of the wave functions. The normalization constant involved is infinite, and this is not a purely formal infinity, but the mathematical expression of the fact that the Klein-Gordon equation in the presence of a truly monochromatic field possesses no solution with the prescribed boundary condition

$$\Phi_p{}^{\text{in}}(x) \sim e^{-ip\cdot x}, \quad x^0 \to -\infty . \quad (21)$$

Physically, this is because the electron cannot be a free particle even in the remote past. In the form (16) this fact makes its appearance in the circumstance that the integral in the exponent fails to converge. For a beam of limited extent, we could drop the contribution from the lower limit of integration, replacing the integral from $-\infty$ to τ by one from 0 to τ, say. The only effect is to multiply the scattering amplitude by an unobservable phase factor. For an infinitely long beam, however, this is impermissible because the phase factor is infinite. Indeed, the wave function obtained in this way, though a solution of the Klein-Gordon equation, would not satisfy the boundary condition (21), and would therefore not represent an asymptotically free electron.

It is convenient, as in BK, to utilize the gauge invariance of the scattering amplitude by introducing the gauge-invariant polarization vector

$$\bar{\epsilon}' = \epsilon' - (n\cdot\epsilon'/n\cdot k')k' , \quad (22)$$

which satisfies $n\cdot\bar{\epsilon}'=0$. Then, substituting the wave functions (16) and (20) into the expression (13) for the scattering amplitude, we obtain

$$\langle \mathbf{p}',\mathbf{k}'\epsilon' | \hat{S}(A) | \mathbf{p}\rangle$$

$$= -i(2\pi)^3 \int dq\, \delta_4[p'+eA(\infty)+k'-p-qn] f(q) , \quad (23)$$

where

$$f(q) = \int d\tau\, e^{iq\tau} \exp\left(i\int_0^\tau d\tau' [I_{p'}(\tau') - I_p(\tau')]\right) \times[2e\bar{\epsilon}'\cdot p - 2e^2\bar{\epsilon}'\cdot A(\tau)] . \quad (24)$$

This expression agrees with that found in BK, except for the neglect of electron spin and the appearance of the extra term $eA(\infty)$ in (23). We shall discuss the origin of this term in the next section.

5. EDGE EFFECTS

We have already seen that it is necessary to consider a wave train of limited duration, but we have not examined the question of whether the actual shape of the amplitude function is important. We shall see that within reasonable limits it is not. It is essential, however, to consider a smoothly varying amplitude, for the use of a sharp cutoff introduces some additional and unwanted effects.[15] We shall discuss these effects in this section, and show that for any physically reasonable choice of the amplitude function they will be negligibly small.

In the discussion at the beginning of Sec. 2 we made the implicit assumption that the average value of A_μ over several periods is always zero. Now, if $F_{\mu\nu}$ has the form (18), then

$$A_\mu(\tau) = \epsilon_\mu \int_{-\infty}^\tau d\tau' F(\tau') , \quad (25)$$

and for an amplitude function with a sharp cutoff this quantity can have a nonzero average value. For example, suppose that

$$F(\tau) = (a/\omega)\sin\omega\tau , \quad 0<\tau<n\pi/\omega ,$$
$$= 0 , \text{ otherwise.}$$

Then clearly

$$A_\mu(\tau) = 0 , \qquad \tau<0 ,$$
$$= a\epsilon_\mu(1-\cos\omega\tau) , \quad 0<\tau<n\pi/\omega ,$$
$$= a\epsilon_\mu[1-(-1)^n] , \quad \tau>n\pi/\omega .$$

In this case, as is easy to verify, the average velocity of the electron in the beam will have a component in the transverse direction defined by ϵ_μ. Thus the Doppler shift will be altered by the corresponding amount. Physically, this average velocity arises from the initial acceleration of the electron during the first quarter period. Its direction is that of the electric field during this time.

[15] See footnote 5.

If n above is odd, then $A_\mu(\infty)$ is also nonzero, and the electron velocity is changed even in the absence of any scattering. This is because of the acceleration produced by the excess half-period. It is just this net change in momentum which is described by the additional term $eA(\infty)$ in the δ function of (22).

It is clear that these effects are a consequence of the sharp cutoff. In fact, it is easy to see that they will normally be small if the amplitude function is smooth and slowly varying. Let us introduce the Fourier transform of $F(\tau)$,

$$\tilde{F}(\omega)=\int d\tau\, e^{i\omega\tau}F(\tau). \qquad (26)$$

Then $A_\mu(\infty)=\epsilon_\mu\tilde{F}(0)$, so that there can be an appreciable transverse momentum transfer only if the zero-frequency component of F is significantly different from zero. If the amplitude function is smooth, this can happen only if the spread of frequencies γ is of the same order of magnitude as the frequency ω itself, that is for very short pulses. Moreover, the average value of $A_\mu(\tau)$ over many periods is also related to the behavior of $\tilde{F}(\omega)$ near $\omega=0$. In fact,

$$\frac{1}{2T}\int_{t-T}^{t+T} d\tau\, A_\mu(\tau)=\epsilon_\mu\int\frac{d\omega}{2\pi}e^{-i\omega t}\frac{\sin\omega T}{\omega T}\frac{\tilde{F}(\omega)}{-i\omega+\epsilon}.$$

The contribution to this integral from the neighborhood of $\omega=0$ is of the same order of magnitude as $\tilde{F}(0)$, and therefore very small. If T is chosen to be an integral number of periods, then the contribution from near the peak ω_0 of the spectrum $\tilde{F}(\omega)$ is also small—in fact, smaller than the maximum value of A_μ by at least a factor $(\gamma/\omega_0)^2$.

In a laser beam γ is of course many orders of magnitude smaller than ω_0, so that these edge effects will certainly be negligible at least as long as the approximation of treating the beam as unidirectional is valid. At first sight, one might think that they could be important for a focused beam, since the amplitude changes rapidly in the vicinity of the focus. However, for nonrelativistic electrons they should still be negligible. The significant quantity is the amplitude "seen" by the electron, and this will not vary rapidly unless the electron velocity is large. The relevant parameter is the square of the ratio of the period of oscillation to the time spent in the beam, which is of order $(v/c)^2$ or less.

6. THE MONOCHROMATIC LIMIT

To illustrate the nature of the limiting process involved in going to an infinitely long wave train, we shall consider a specific example, in which the cutoff function has an exponential form,

$$A_\mu(\tau)=2\,\mathrm{Re}(a_\mu e^{-i\omega\tau-\frac{1}{2}\gamma|\tau|}), \qquad (27)$$

where a_μ is a constant complex vector describing the amplitude and polarization of the beam.[16]

In one respect, this is a rather unphysical example, since the electromagnetic field changes discontinuously at $\tau=0$. This difficulty could be avoided by choosing slightly different amplitudes for positive and negative values of τ. If we replace a_μ in (27) by

$$a_\mu\pm(i\gamma/2\omega)\,\mathrm{Re}a_\mu$$

for $\tau>0$ and $\tau<0$, respectively, then both A_μ and its first derivative are continuous at $\tau=0$. However, this replacement does not affect the analysis in any significant way, and we shall be content with the simpler form (27).

For simplicity, we shall restrict the discussion to the case of a circularly polarized wave, for which $a^2=0$, and choose the phase of a_μ so that the quantity

$$\xi=2e\left(\frac{p'\cdot a}{p'\cdot k}-\frac{p\cdot a}{p\cdot k}\right) \qquad (28)$$

is real. We also define

$$\zeta=-e^2a\cdot a^*\left(\frac{1}{p'\cdot k}-\frac{1}{p\cdot k}\right), \qquad (29)$$

in agreement with (10) and (4).

We can now evaluate the exponential factor appearing in (24) explicitly. Since we are assuming that $\gamma\ll\omega$, we shall make the simplifying approximation of replacing denominators of the form $\omega\pm\frac{1}{2}i\gamma$ by ω. (This is not strictly necessary. These factors may be carried through the calculation, but do not significantly affect the results.) We then obtain for the exponent the expression

$$i\xi\sin\omega\tau\, e^{\mp\frac{1}{2}\gamma\tau}\mp i\zeta(\omega/\gamma)(e^{\mp\gamma\tau}-1)$$

according as $\tau>0$ or $\tau<0$. Expanding the oscillatory term in a series of Bessel functions and inserting in (23) and (24), we find that the integral from $-\infty$ to 0 is just the complex conjugate of that from 0 to ∞. Hence we obtain

$$f(q)=\sum_r 2\,\mathrm{Re}\int_0^\infty d\tau D_r(\xi e^{-\frac{1}{2}\gamma\tau},ae^{-\frac{1}{2}\gamma\tau})\times\exp[i(q-r\omega)\tau-(i\zeta\omega/\gamma)(e^{-\gamma\tau}-1)], \qquad (30)$$

where

$$D_r(\xi,a)=2e\bar{\epsilon}'\cdot pJ_r(-\xi)-2e^2[\bar{\epsilon}'\cdot aJ_{r-1}(-\xi)+\bar{\epsilon}'\cdot a^*J_{r+1}(-\xi)]. \qquad (31)$$

Apart from the exponential cutoff factors, the functions D_r are identical with the emA_r of BK, Eq. (3.24).

Let us concentrate on a particular value of r. Then the leading term in D_r depends on τ through the factor $e^{-\frac{1}{2}r\gamma\tau}$. So we have to examine integrals of the form

$$f_r(q)=2\,\mathrm{Re}\int_0^\infty d\tau\exp\left[i(q-r\omega)\tau-\tfrac{1}{2}r\gamma\tau-\frac{i\zeta\omega}{\gamma}(e^{-\gamma\tau}-1)\right]. \qquad (32)$$

[16] Note that a_μ here corresponds to $\frac{1}{2}\mathfrak{a}_\mu$ of BK.

Because of the damping factor $e^{-\frac{1}{2}r\gamma\tau}$, only values of $\tau\lesssim 1/r\gamma$ will contribute significantly to the integral. Thus we may expect to be able to expand the coefficient of ζ in powers of $\gamma\tau$. If we retain only the leading (linear) term, then we obtain

$$f_r(q)\approx\frac{r\gamma}{(q-r\omega+\zeta\omega)^2+\frac{1}{4}r^2\gamma^2}. \tag{33}$$

Then we see that the distribution of values of q is peaked around the points $(r-\zeta)\omega$, in agreement (for $r=1$) with the energy-momentum conservation equation (9). There is no peak near the values $r\omega$ which would correspond to scattering at unshifted frequencies. In the limit $\gamma\to 0$ we obtain a sum of δ functions of precisely the form given in BK.

This conclusion is not changed by including higher order terms. When the second-order term is included, we obtain an error function. We write

$$b=-i(q-r\omega+\zeta\omega)+\tfrac{1}{2}r\gamma,$$
$$c=\tfrac{1}{2}i\zeta\omega\gamma,$$
$$x=b/2c^{1/2}.$$

Then we find

$$f_r\approx 2\,\mathrm{Re}[c^{-1/2}e^{x^2}\,\mathrm{Erfc}(x)], \tag{34}$$

where Erfc(x) is the complementary error function.[17] Provided that $q-r\omega+\zeta\omega$ is not precisely zero, x becomes large as $\gamma\to 0$. Thus we may use the asymptotic expansion in inverse powers of x, and obtain

$$f_r\sim 2\sum_n\frac{(2n)!}{n!}\,\mathrm{Re}\left[\frac{1}{b}\left(-\frac{c}{b^2}\right)^n\right]. \tag{35}$$

The first term in this expansion yields precisely (33). The higher terms are peaked around the same point, and serve only to modify the shape of the peak.

These somewhat heuristic arguments can be reinforced by a more detailed examination of the integral in (32), which can be expressed by a simple change of variable in the form

$$f_r=2\,\mathrm{Re}[(1/\gamma)e^z z^{-\rho}\gamma(\rho,z)], \tag{36}$$

where $\gamma(\rho,z)$ is the incomplete gamma function,[18] and the arguments are given by

$$\rho=-(i/\gamma)(q-r\omega)+r/2,$$
$$z=i\zeta\omega/\gamma.$$

Since we are interested in small values of γ, we may use the known asymptotic behavior of this function. Unfortunately, this behavior is rather complicated, particularly when ρ and z are nearly equal in the asymptotic region (that is, when the parameter b is small).[19] A complete discussion would be rather lengthy, and we shall therefore give only a brief indication of the results. One finds that, except in the immediate vicinity of the peak, the leading term is indeed given by (33), except for certain values of q for which additional terms can arise because the imaginary part of the exponent in (32) becomes stationary within the range of integration. However, these terms are rapidly oscillating functions of γ and q, whose contributions are vanishingly small when averaged over a small range of values of q. The precise shape of the peak is hard to determine, but of no importance for our discussion, for which only the position of the peak is relevant.

We may note that if we had chosen a monochromatic beam from the outset of the calculation, and discarded the infinite phase factors, then the term in ζ above would have been absent, and we should have obtained no frequency shift. However, the correct procedure is to take initially a finite value of γ, and only let $\gamma\to 0$ at the end of the calculation. This procedure yields the final result

$$f(q)=\sum_r D_r(\xi,a)2\pi\delta(q-r\omega+\zeta\omega),$$

and hence

$$\langle\mathbf{p}',\mathbf{k}'\epsilon'|\hat{S}(A)|\mathbf{p}\rangle$$
$$=-i\sum_r D_r(\xi,a)(2\pi)^4\delta_4(p'+k'-p-rk+\zeta k), \tag{37}$$

in agreement with BK, Eqs. (3.13) and (3.23). (There is a difference of a factor $2m$, which arises from the use of different normalizations for spin-0 and spin-$\frac{1}{2}$ particles.)

7. ONE-DIMENSIONAL MODEL

In order to substantiate our assertion that the momentum of the beam itself is altered by the interaction with the electron, it is necessary to examine the beam in more detail. The coherent states we used earlier are rather inconvenient for this purpose, and it is better to use states with a definite photon number (though such states are of course a rather poor representation of a laser beam). However, the number of photons in a beam with finite intensity and infinite plane wave fronts is clearly infinite, so that to do this we should have to limit the extent of the beam in transverse directions also. Then the problem would lose its essentially one-dimensional character, and become very difficult to handle. So, instead of examining this problem directly, we shall set up a one-dimensional model which exhibits all the essential features of the effect, and for which the calculations are much easier.

We consider a model with one spatial and one time dimension, in which a scalar "electron" field $\hat{\phi}$ interacts with a scalar "photon" field $\hat{A}$. It is clearly the quadratic term in the interaction which leads to the effect we are interested in, so we shall discard the linear term. More-

[17] See *Bateman Manuscript Project, Higher Transcendental Functions*, edited by H. Erdelyi (McGraw-Hill Publishing Company, Inc., New York, 1953), Vol. 2, p. 147.
[18] See Ref. 17, Vol. 2, Chap. 9.
[19] See F. G. Tricomi, Math. Z. **53**, 136 (1950).

over, it is unnecessary to include vertices at which two photons are emitted or two absorbed. These correspond to the oscillatory terms in A^2, which we eliminated earlier by choosing a circularly polarized beam. The essential vertices are those at which one photon is absorbed, and one re-emitted. So we shall choose the nonlocal interaction which contains just this type of vertex,

$$\hat{V}(x)=2e^2\hat{\varphi}^*(x)\hat{\varphi}(x)\hat{A}^{(-)}(x)\hat{A}^{(+)}(x). \qquad (38)$$

Note that in one dimension the field $\hat{A}$ is dimensionless, while the coupling constant e has the dimensions of mass.

In one dimension there is an invariant distinction between photons traveling to the right and to the left. We shall write $x=(t,z)$, and set

$$\hat{A}^{(+)}(x)=\hat{A}^{(-)}(x)^*$$
$$=\frac{1}{2\pi}\int_0^\infty\frac{d\omega}{2\omega}[\hat{a}_R(\omega)e^{-i\omega(t-z)}+\hat{a}_L(\omega)e^{-i\omega(t+z)}].$$

It will be convenient to introduce the abbreviation

$$(a,b)=\frac{1}{2\pi}\int_0^\infty\frac{d\omega}{2\omega}a(\omega)b(\omega).$$

If $\tilde{\alpha}(\omega)$ is any photon wave function, normalized according to $(\tilde{\alpha}^*,\tilde{\alpha})=1$, then the coherent state of a beam traveling to the right may be defined by

$$|\lambda\alpha\rangle=\exp\lambda(\tilde{\alpha},\hat{a}_R^*)|0\rangle e^{-\frac{1}{2}\lambda^*\lambda}. \qquad (39)$$

We shall consider the scattering of a single photon to the left out of a beam traveling to the right. We examine first the case where the initial and final states of the beam are the coherent states $|\lambda\alpha\rangle$ and $\langle\lambda'\alpha'|$. The calculation is essentially identical with our earlier one, and yields

$$\langle p';k';\lambda'\alpha'|\hat{S}|p;\lambda\alpha\rangle$$
$$=-4\pi ie^2\langle\lambda'\alpha'|\lambda\alpha\rangle\int dq\,\delta_2(p'+k'-p-qn)$$
$$\times\int d\tau\, e^{i\tau q}\lambda\alpha(\tau)e^{\lambda'^*\lambda[B'(\tau)+B(\tau)]}, \qquad (40)$$

where $n=(1,1)$,

$$\alpha(\tau)=\frac{1}{2\pi}\int_0^\infty\frac{d\omega}{2\omega}\tilde{\alpha}(\omega)e^{-i\omega\tau}, \qquad (41)$$

$$B'(\tau)=\frac{-ie^2}{p'\cdot n}\int_\tau^\infty d\tau'\alpha'^*(\tau')\alpha(\tau'), \qquad (42)$$

$$B(\tau)=\frac{-ie^2}{p\cdot n}\int_{-\infty}^\tau d\tau'\alpha'^*(\tau')\alpha(\tau'), \qquad (43)$$

and

$$\langle\lambda'\alpha'|\lambda\alpha\rangle=\exp[\lambda'^*\lambda(\tilde{\alpha}'^*,\tilde{\alpha})-\tfrac{1}{2}\lambda'^*\lambda'-\tfrac{1}{2}\lambda^*\lambda].$$

Now we wish to consider the case in which the initial state of the beam is an $(N+1)$-photon state, and the final state (because photon number is conserved) is an N-photon state. We shall make these states as nearly coherent as possible, given the definite photon number, by placing each photon in the same (wave packet) state. We define the states

$$|N,\alpha\rangle=(N!)^{-1/2}(\tilde{\alpha},\hat{a}_R^*)^N|0\rangle.$$

Because of the relation

$$|\lambda\alpha\rangle=\sum_N(N!)^{-1/2}|N,\alpha\rangle e^{-\frac{1}{2}\lambda^*\lambda}\lambda^N, \qquad (44)$$

the coherent states may be regarded as generating functions for these N-photon states.[20] To obtain the matrix elements between these states, we have only to multiply (40) by $\exp[\frac{1}{2}\lambda'^*\lambda'+\frac{1}{2}\lambda^*\lambda]$, expand in powers of λ'^* and λ, and identify the appropriate term. The result is

$$\langle p';k';(N+1),\alpha'|\hat{S}|p;N,\alpha\rangle$$
$$=-2\pi i\int dq\,\delta_2(p'+k'-p-qn)f(q), \qquad (45)$$

where

$$f(q)=2e^2(N+1)^{1/2}$$
$$\times\int d\tau\, e^{i\tau q}\alpha(\tau)[B'(\tau)+B(\tau)+(\tilde{\alpha}'^*,\tilde{\alpha})]^N. \qquad (46)$$

It may be well to note that this expression can also be obtained by more conventional means, by summing Feynman diagrams. If we expand the square bracket in (46) by the trinomial theorem, we obtain a sum of terms involving products like

$$B'^sB^r(\tilde{\alpha}'^*,\tilde{\alpha})^{N-r-s}.$$

This term corresponds to a diagram like that of Fig. 2, in which there are r vertices preceding the scattering vertex, s vertices after it, and $N-r-s$ straight through lines. Using the relations

$$B'(\tau)=\frac{e^2}{(2\pi)^2p'\cdot n}\int_0^\infty\frac{d\omega'}{2\omega'}\frac{d\omega}{2\omega}\frac{e^{i\tau(\omega'-\omega)}}{\omega'-\omega+i\epsilon}\tilde{\alpha}'^*(\omega')\tilde{\alpha}(\omega),$$

$$B(\tau)=\frac{e^2}{(2\pi)^2p\cdot n}\int_0^\infty\frac{d\omega'}{2\omega'}\frac{d\omega}{2\omega}\frac{e^{i\tau(\omega'-\omega)}}{\omega-\omega'+i\epsilon}\tilde{\alpha}'^*(\omega')\tilde{\alpha}(\omega),$$

it is not difficult to verify that this particular term corresponds to a term in the scattering amplitude

$$\langle p';k';\omega_1'\cdots\omega_N'|\hat{S}|p;\omega_1\cdots\omega_{N+1}\rangle$$

[20] I am indebted to Dr. Lowell S. Brown for pointing out the simplification achieved by employing the coherent states in their role as generating functions.

of the form

$$\frac{-i(2e^2)^{r+s+1}}{r!s!}\delta_2(p'+k'+\sum_{i=1}^{N}\omega_i'n-p-\sum_{i=1}^{N+1}\omega_i n)\prod_{i=1}^{r}[2p\cdot n(\omega_i-\omega_i'+i\epsilon)]^{-1} \times\prod_{i=r+1}^{r+s}[2p'\cdot n(\omega_i'-\omega_i+i\epsilon)]^{-1}\prod_{i=r+s+1}^{N}2\pi 2\omega_i\delta(\omega_i'-\omega_i). \quad (47)$$

This is just a symmetrized version of the Feynman diagram of Fig. 2.

Now that we have an explicit form for the scattering amplitude, we are in a position to discuss the changes in energy and momentum of the beam. In the form (47), conservation of energy and momentum is quite manifest. Clearly, the momentum transfer at each vertex must be small, for otherwise the energy denominators become large. However, since the number of photons is very great, there can nevertheless be a significant net transfer of momentum if there is any tendency for these small momentum transfers to be in the same sense for each photon.

The amplitude will certainly be small unless there is a substantial overlap between the wave functions $\tilde{\alpha}(\omega)$ and $\tilde{\alpha}'(\omega)$, for if not the last term in the square brackets in (46) becomes very small. (This is normally the largest term, provided that e^2 is small in an appropriate sense—to be made precise below.) Thus the average momentum transfer to each photon $\Delta\omega$ must be small compared with the momentum spread γ of the wave functions, and must therefore tend to zero with γ. However, if the photon density is held fixed, then the number of photons N must tend to infinity like $1/\gamma$, so that the total momentum transfer can still be finite.

The crucial question is whether there is in fact a general tendency for the small momentum transfers to be all in the same direction. If there is, then the integration over τ in (46) should yield a distribution of values of q which is peaked not at ω, but at a somewhat smaller value, $\omega-N\Delta\omega$. (Here $N\Delta\omega$ is the quantity we have previously denoted by $\zeta\omega$.) This is the effect we found in our earlier calculation. There is, however, a second way in which this tendency should manifest itself. Suppose we examine the transition probability as a function of the mean momentum ω_0' of the photons in the final state [the center of the distribution $\tilde{\alpha}'(\omega)$]. Then we ought to find that the maximum occurs at a value rather higher than the mean momentum ω_0 of the initial state photons. The maximum should occur when $\delta\omega=\omega_0'-\omega_0$ is somewhere near $\Delta\omega$, though the precise position will depend on the shape we choose for the functions $\tilde{\alpha}(\omega)$ and $\tilde{\alpha}'(\omega)$. It will occur when $\tilde{\alpha}'(\omega)$ most closely approximates the actual distribution of momenta in the final state. If the shape is changed, as is likely, this need not happen when the mean values coincide. In principle, it would be possible to determine the distribution of momenta in the final state by evaluating the transition probabilities to all of a complete set of final states, and taking the appropriate phase-space average. However, the determination of the position of maximum probability for a fixed shape is sufficient to demonstrate the existence of a shift, though not to determine its precise magnitude. (For that, we must rely on finding the distribution of values of q.)

One point which might perhaps be a source of confusion should be mentioned here. The transition probability is in fact very close to its maximum value even when the wave functions coincide, and $\delta\omega=0$. Thus one might think that there should be a large probability that the total momentum transfer is zero, or even of the wrong sign. There is indeed a significant probability that a measurement of the momenta of photons in the beam would not reveal such a shift, but this in no way contradicts conservation of energy and momentum. The point is that the momentum transfer to each photon is much less than the uncertainty in its momentum. The determination of the average final-state momentum requires a large number of measurements, which will produce results of varying sign, but with positive values slightly predominating.

We now wish to show that both these manifestations of the frequency shift do indeed occur. We shall again choose an exponential cutoff function. We set

$$\tilde{\alpha}(\omega)=\frac{2\alpha_0\gamma\omega}{(\omega-\omega_0)^2+\frac{1}{4}\gamma^2},\quad \alpha_0=\frac{1}{2}\left(\frac{\gamma}{\omega_0}\right)^{1/2},$$

$$\tilde{\alpha}'(\omega)=\frac{2\alpha_0'\gamma\omega}{(\omega-\omega_0')^2+\frac{1}{4}\gamma^2},\quad \alpha_0'=\frac{1}{2}\left(\frac{\gamma}{\omega_0'}\right)^{1/2}.$$

Provided that $\gamma\ll\omega$, we may extend the lower limit of integration in (41) to $-\infty$, and obtain

$$\alpha(\tau)=\alpha_0 e^{-i\omega_0\tau-\frac{1}{2}\gamma|\tau|},\quad \alpha'(\tau)=\alpha_0' e^{-i\omega_0'\tau-\frac{1}{2}\gamma|\tau|}. \quad (48)$$

Substituting in (42), (43), and (46), we find

$$f(q)=2e^2(N+1)^{1/2}\alpha_0\int d\tau\, e^{i\tau(q-\omega_0)-\frac{1}{2}\gamma|\tau|} \times\left[C+i\Delta\omega\frac{e^{i\delta\omega\tau-\gamma|\tau|}-1}{i\delta\omega-\gamma\epsilon(\tau)}\right]^N, \quad (49)$$

where

$$\Delta\omega=e^2\alpha_0'\alpha_0\left(\frac{1}{p'\cdot n}-\frac{1}{p\cdot n}\right), \quad (50)$$

and

$$C=\frac{1}{(\delta\omega)^2+\gamma^2}\left[\gamma^2\frac{\omega_0'+\omega_0}{2(\omega_0'\omega_0)^{1/2}}+\delta\omega\Delta\omega - \frac{ie^2\gamma^2}{2(\omega_0'\omega_0)^{1/2}}\left(\frac{1}{p'\cdot n}+\frac{1}{p\cdot n}\right)\right]. \quad (51)$$

Note that the quantity which here corresponds to $-a^*\cdot a$ in our earlier work is $N\alpha_0'\alpha_0$. Hence we should expect the total momentum transfer to the beam to be $N\Delta\omega$, with $\Delta\omega$ given by (50). We shall show below that this is indeed the case. Note that $\Delta\omega$ is proportional to γ, so that $N\Delta\omega$ remains constant as γ tends to zero.

We are interested in the expression (49) for small values of γ and large values of N. We have already remarked that $\delta\omega$ must be small compared to γ, so that both $\delta\omega$ and $1/N$ must go to zero with γ. Thus the square bracket in (49) may be approximated by

$$C^N e^{iN\Delta\omega\tau/C}[1+O(\gamma)].$$

For $\delta\omega \ll \gamma$ and e^2 small in the sense that $e^2/m\omega_0 \ll 1$, C is approximately equal to unity, so the leading term in (49) is

$$f(q)\approx\frac{2e^2(N+1)^{1/2}\alpha_0 C^N\gamma}{(q-\omega_0+N\Delta\omega)^2+\frac{1}{4}\gamma^2}. \quad (52)$$

It is clear from this expression that $f(q)$ is peaked, as expected, around $q=\omega_0-N\Delta\omega$, and not around $q=\omega_0$. Just as before, in the limit $\gamma\to 0$, $f(q)$ becomes proportional to a δ function, $\delta(q-\omega_0+N\Delta\omega)$.

It is not hard to verify that the second effect we discussed above also occurs. Clearly, $\delta\omega$ appears in $f(q)$ only through the factor C^N, and so $f(q)$ will have its maximum value as a function of $\delta\omega$ when $|C|$ is a maximum. Now, the second and third terms of (51) are small compared to the first term, so that

$$|C|\approx \mathrm{Re}(C)\approx\frac{\gamma^2+\delta\omega\Delta\omega}{\gamma^2+(\delta\omega)^2}.$$

Hence the position of the maximum is given approximately by

$$\delta\omega\approx\tfrac{1}{2}\Delta\omega.$$

The fact that this value differs from zero shows that the distribution of frequencies in the final state is shifted relative to that in the initial state. The factor of $\frac{1}{2}$ suggests that the average value $\Delta\omega$ is in fact produced by relatively few photons acquiring substantially larger momenta (but with momentum transfers still less than γ), and a larger number for which the momentum transfer is smaller than $\Delta\omega$.

It is not hard to verify that the inclusion of higher order terms does not affect these conclusions, and that similar results would be obtained for other choices of the shape of the functions $\tilde{\alpha}(\omega)$ and $\tilde{\alpha}'(\omega)$. However, since no new points of principle emerge from such a discussion, we shall omit it.

8. DISCUSSION

The principal conclusions of this paper are two. Firstly, we have shown that the discrepancy between the results of BK and of Goldman,[3] and those of Fried and Eberly[4] is due to the use by the latter of an infinitely long wave train rather than a finite one, and not to any difference between the results of quantum-mechanical and semiclassical calculations. (Any such difference, apart from radiative corrections, would in fact be a violation of the correspondence principle.)

Secondly, we showed that the apparent failure of energy-momentum conservation is not a real failure. It is possible to see precisely where this energy goes. As we showed in detail in the case of the one-dimensional model, the beam itself takes up this extra energy and momentum in the form of a shift of the average momentum of the photons remaining in the beam. It may be well to recall at this point that the magnitude of the effect is very small indeed. It will be hard enough to see the frequency shift of the scattered photon. It would be quite impossible to detect the same amount of energy distributed among 10^{20} or more photons.

We showed in Sec. 2 that the frequency shift has a very simple classical interpretation as a Doppler shift arising from an average velocity of the electron in the direction of the beam. It is also clear from this discussion that the variation of the amplitude as the beam is switched on and off is absolutely crucial to the existence of the effect. No calculation involving an infinitely long wave train of constant amplitude could ever reveal such an effect.

Our calculations were made possible by the essentially one-dimensional nature of the problem, when the beam is taken to have infinite plane-wave fronts. It would certainly be desirable to repeat the calculation for a more realistic shape of beam, and in particular for a focused beam. In that case, the Klein-Gordon equation cannot be solved in closed form, so that the calculation would be much more difficult. In principle, however, it could be solved numerically, and for some special choices of the beam shape this might not be too hard. A perturbation calculation like the one outlined in Sec. 3 would of course be possible. However, as we have seen, the frequency shift is compounded of large numbers of very small terms, so that it would almost certainly not appear until a very high order. From the one-dimensional model calculation, it is reasonable to conjecture that one must go to an order comparable with the total number of photons.

An alternative approach would be to use classical (relativistic) electrodynamics. At least in the case of a unidirectional beam, the effect is completely classical, and the classical calculation gives exactly the same answer as the semiclassical or fully quantum-mechanical ones.[21] In principle, one can solve the equation of motion for an electron in any prescribed field, and it is then straightforward, though possibly tedious, to compute the radiation from it.

Another problem which deserves further attention concerns the radiative corrections to this process. Though these corrections are individually small, the frequency shift itself is an example of a significant effect arising from the sum of many small terms, and it is far from obvious that the corrections are negligible in sum. It would be desirable to show explicitly that they are so.

It may be useful to conclude this discussion with some remarks concerning the possibility of observing this effect experimentally. We shall need a more general expression for the frequency shift than the one which was given in Sec. 2. Let us denote the velocity of the incoming electron by v (in units of c), the angles between its direction and those of the incoming and scattered photons by α and β, and the angle between the incoming and scattered photons by θ. (See Fig. 3, in which the outgoing electron is not shown.) Then from (11) we find that the complete expression for the shift in wavelength is

$$\frac{\lambda'-\lambda}{\lambda}=v\frac{\cos\alpha-\cos\beta}{1-v\cos\alpha}+\frac{\lambda_C}{\lambda}\frac{2(1-v^2)^{1/2}\sin^2(\frac{1}{2}\theta)}{1-v\cos\alpha}+\mu^2\frac{(1-v^2)\sin^2(\frac{1}{2}\theta)}{(1-v\cos\alpha)^2}, \quad (53)$$

where λ_C is the electron Compton wavelength. The first term of this expression represents the Doppler effect, and the second the Compton effect. The third is the intensity-dependent shift we are interested in. Its magnitude is determined by the parameter μ^2, which may be expressed in terms of the beam intensity (energy flux) I by the relation

$$\mu^2=r_0\lambda^2 I/\pi mc^3, \quad (54)$$

where $r_0=e^2/4\pi mc^2$ is the classical electron radius.

It is not hard to obtain intensities sufficient to give a shift of measurable size. The real difficulty lies in distinguishing this from the Doppler shifts given by the first term of (53). In order to achieve sufficient intensity, the beam must be sharply focused, and the range of values of α will be considerable. Because of this uncertainty in α, the Doppler shift will be uncertain by an amount of order v/c (quite apart from any uncertainty in v). Hence the Doppler shifts will dominate unless $\mu^2 \gtrsim v/c$. To first order, the Doppler shifts will be distributed symmetrically about zero. The magnitude of the intensity-dependent shift may vary to some extent

[21] See BK, Appendix C.

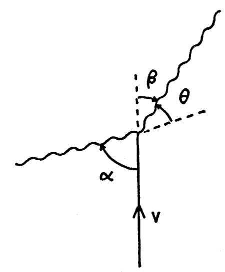

FIG. 3. Definitions of angles in the Compton scattering process. The outgoing electron is not shown.

(mainly because of uncertainty in the angle θ), but its sign is always positive. Thus, to verify the existence of the effect, it is sufficient to observe an asymmetry in the distribution (besides that due to the Compton effect, which will be small in practice). For this, it is enough if μ^2 is of the same order as v/c.

It is clearly desirable to keep the electron velocities as small as possible, both to satisfy this criterion, and because the intensity-dependent shift becomes small for very large velocities, as may be seen from (53). Moreover, as we pointed out in Sec. 5, the edge effects could conceivably become important for large velocities and sharply focused beams. However, since the relevant parameter in that case is $\mu(v/c)^2$, this is not an important restriction.

If we assume optimistically that the electron kinetic energies can be kept down to a fraction of an electron volt, then we require at least $\mu^2\approx 10^{-3}$. Now in cgs units (54) becomes

$$\mu^2\approx 4\times 10^{-18}\lambda^2 I.$$

Thus for $\lambda\approx 5000$ Å we need an intensity $I\approx 10^{16}$ W cm^{-2}. This is beyond the limit of currently available intensities, but not by an absurdly large factor. Intensities of this order might be obtained if the size of the focal spot could be reduced, approaching the diffraction limit more closely. For example, to achieve this intensity with a 10-nsec pulse of 10^3J, we should need a focal spot of radius about 2×10^{-3} cm.

It is easy to compute the intensity of the scattered light. Since the cross section is essentially the Thomson scattering cross section, the number of photons scattered out of a single pulse per unit solid angle is

$$N=\frac{n_e I\lambda V T r_0^2(1+\cos^2\theta)}{4\pi\hbar c},$$

where V is the volume of the interaction region, and T is the duration of the pulse. With the figures chosen above, this yields $N\approx 10^{-6}\,n_e$, where n_e is in cm^{-3}. (It should be noted that the probability of a single electron suffering more than one collision is quite large. However, this is not serious, because the change in velocity of the electron at each collision is only of the order of 10^{-5}.) The acceptable value of N depends on the level of background radiation, but it does not seem that it would be impossible to obtain adequate electron densities.

It is important to note that the experiment must be done with genuinely free electrons in vacuum. Bound electrons would probably not exhibit the effect, since they are not free to be accelerated by the field. In any case, at the intensities considered here, any gas present would be ionized, and the effects of the resulting plasma would be likely to obscure the effect we are interested in. A frequency shift of light scattered from such a plasma has in fact been observed.[22] This is apparently also a Doppler shift arising from motion of the plasma, but it is in the opposite direction to that predicted for free electrons, and probably of quite different origin.

ACKNOWLEDGMENTS

I am indebted to many colleagues for helpful discussions and in particular to Dr. Lowell S. Brown for pointing out some errors and possible improvements in an earlier version of this paper, to Dr. J. M. Charap for a number of valuable suggestions, and to Dr. G. Feldman for healthy skepticism. I should like to thank Dr. Z. Fried for sending me a preliminary version of his work. I wish to acknowledge a correspondence with Professor Philip Stehle.

APPENDIX

A further argument has recently been advanced by Stehle and de Baryshe[23] in support of their contention that the frequency shift does not appear in a fully quantum-mechanical calculation. They have evaluated the expectation value of the operator $-i\partial/\partial z$ in the state represented by the wave function $\Phi_p{}^{\text{in}}(x)$, using box normalization, and shown that it is time-independent, and, in the limit of an infinitely large box, equal to p_z.

This is, however, precisely the result one should expect on the basis of the argument presented in this paper. As these authors remark, the wave function $\Phi_p{}^{\text{in}}(x)$ at no time represents an electron spatially separated from the beam. In fact, it is a non-normalizable wave function representing an infinite beam of electrons, spread over the whole of space. But this does not mean that the expectation value of momentum in this state is the average value of the electron momentum in the beam. For a finite wave train, each electron spends only a finite time inside the beam. Thus at any time all but a finite number of the electrons are outside the beam. The expectation value of momentum in this state is therefore determined by the momenta of the infinite number of electrons outside the beam, and is not sensitive to changes of momentum occurring in the region of the beam.

To arrive at a correct description of the physical scattering process, it is necessary to use a wave-packet description of the electron, as well as of the photon beam. In other words, we should use a normalizable solution $\Phi(x)$ of the Klein-Gordon equation. Such a solution may be represented as a linear superposition of the solutions $\Phi_p{}^{\text{in}}(x)$, of the form

$$\Phi(x)=\int \frac{d^3p}{(2\pi)^3 2p^0}\Phi_p{}^{\text{in}}(x)\varphi(\mathbf{p}). \tag{A1}$$

From the asymptotic condition (21), which is valid for any fixed values of the spatial coordinates $\mathbf{x}$, we may conclude that as $x^0\to-\infty$, Φ approaches a free-particle wave function,

$$\Phi(x)\sim\int \frac{d^3p}{(2\pi)^3 2p^0}e^{-ip\cdot x}\varphi(\mathbf{p}). \tag{A2}$$

It is worth noting that while (21) holds only as a weak limit (in particular, it is nonuniform in $\mathbf{x}$), this Eq. (A2) holds as a strong limit in the sense of the norm topology induced by the scalar product

$$\langle\Psi|\Phi\rangle=i\int d^3x(\Psi^*\Phi_0-\Psi_0{}^*\Phi), \tag{A3}$$

where

$$i\Phi_\mu=(i\partial_\mu-eA_\mu)\Phi.$$

It has been suggested by Stehle and de Baryshe that the wave functions $\Phi_p{}^{\text{in}}$ for different values of p may fail to be orthogonal. In fact, however, they satisfy the same orthogonality relations as the free-particle states, namely

$$\langle\Phi_{p'}{}^{\text{in}}|\Phi_p{}^{\text{in}}\rangle=(2\pi)^3 2p^0\delta_3(\mathbf{p}'-\mathbf{p}). \tag{A4}$$

First, let us note that the scalar product (A3) of two normalizable wave functions is necessarily independent of time. Therefore it may be evaluated in the remote past to yield

$$\langle\Psi|\Phi\rangle=\int \frac{d^3p}{(2\pi)^3 2p^0}\psi^*(\mathbf{p})\varphi(\mathbf{p}).$$

The validity of this equation is in fact all that one means by the assertion (A4). However, (A4) can also be verified by direct evaluation of the integral.

Now let us examine the expectation value of momentum in the state defined by (A1). It may be defined by

$$\langle p^\mu\rangle=\int d^3x\, T^{0\mu}(x),$$

[22] S. A. Ramsden and W. E. R. Davies, Phys. Rev. Letters **13**, 227 (1964).
[23] P. Stehle and P. G. de Baryshe (unpublished).

where $T^{\lambda\mu}$ is the usual Klein-Gordon energy-momentum tensor,

$$T_{\lambda\mu}=\Phi_\lambda{}^*\partial_\mu\Phi+\partial_\mu\Phi^*\Phi_\lambda-g_{\lambda\mu}(\Phi_\nu{}^*\Phi^\nu-m^2\Phi^*\Phi).$$

As $x_0\to-\infty$, $\langle p^\mu\rangle$ tends to the expectation value in the free-particle state specified by φ. If φ is sharply peaked around some value of p, so that Φ closely approximates $\Phi_p{}^{\text{in}}$, then $\langle p^\mu\rangle$ must tend to p^μ in the remote past. However, $\langle p^\mu\rangle$ is not time-independent. Its time dependence may be found from the relation

$$\partial_\lambda T^{\lambda\mu}=j_\lambda\partial^\mu A^\lambda,$$

where

$$j_\lambda=ie(\Phi^*\Phi_\lambda-\Phi_\lambda{}^*\Phi).$$

If we define expectation values in the natural way

$$\langle eA_\mu\rangle=\int d^3x\; j^0(x)A_\mu(x),$$

$$\left\langle\frac{e}{v^0}v_\lambda F^{\lambda\mu}\right\rangle=\int d^3x\; j_\lambda(x)F^{\lambda\mu}(x),$$

then we find that $\langle p^\mu\rangle$ satisfies the classical equation of motion

$$\frac{d}{dt}(\langle p^\mu\rangle-e\langle A^\mu\rangle)=\left\langle\frac{e}{v^0}v_\lambda F^{\lambda\mu}\right\rangle.$$

In particular, the expectation value of $\langle p^\mu-eA^\mu\rangle$ during the presence of the beam is on the average equal to $\bar{p}^\mu$.

PHYSICAL REVIEW VOLUME 138, NUMBER 3B 10 MAY 1965

Phenomenological Theory of Laser Beam Fluctuations and Beam Mixing*

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(Received 16 December 1964)

The paper contains a quantum theoretical analysis of laser beam fluctuations and of the light beat experiments with two lasers. With the help of experimental results on photon counting fluctuations in a single-mode laser field, some correlation properties of the field are derived. It is shown that the correlation equations are satisfied by states of the field which are much more general than "coherent" states. The equations lead directly to the spectral density of the intensity operator in the light beat experiments, which can be obtained from photoelectric measurements. The resulting expression is practically identical to that found by Forrester for light having thermal statistical properties. The reasons for this are discussed by a comparison of the corresponding probability distributions of photon counts and of the classical wave amplitude.

1. INTRODUCTION

THE problem of determining the optical spectrum of a laser beam from beat experiments with two or more lasers is of interest, not only because of its practical importance, but because it involves the fluctuation properties of the optical field. Immediately after the development of the first continuously operating laser it was recognized that the spectral profile of one mode of the extremely narrow band light beam could not be determined by conventional interferometry. The first order of magnitude determination of the spectral linewidth was based on a photoelectric analysis of the "beats" resulting from the superposition of two similar but independent laser beams,[1] and variations of this method have become standard practice.[2]

If we picture each Fourier component of one light beam as "beating" against each Fourier component of the other, we are naturally led to regard the spectral excursion of the beat notes, reflected in the spectral range of the photoelectric signals, as a measure of the spectral width of the light itself. To an order of magnitude this measure will undoubtedly be valuable. However, in order to arrive at a quantitative relation between the spectral densities of the light beams and the spectral density of the measured photoelectric signal, we need to have information on the statistical properties of the optical fields. This information was not available to the first experimenters, and indeed the proper description of a laser field is still the subject of debate.[3–8]

By treating the classical wave amplitude of the optical field as a Gaussian random process, Forrester[9] obtained

* This research was supported in part by the U. S. Army Research Office (Durham) and by the U. S. Air Force Cambridge Research Laboratories.

[1] A. Javan, E. A. Ballik, and W. L. Bond, J. Opt. Soc. Am. **52**, 96 (1962).

[2] See, for example, D. R. Herriot, J. Opt. Soc. Am. **52**, 31 (1962); B. J. McMurtry and A. E. Siegman, Appl. Opt. **2**, 767 (1963); M. S. Lipsett and L. Mandel, Nature (London) **199**, 553 (1963).

[3] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).

[4] H. Paul, W. Brunner, and G. Richter, Ann. Physik **12**, 325 (1963).

[5] L. Mandel, Phys. Rev. **134**, A10 (1964).

[6] T. F. Jordan and F. Ghielmetti, Phys. Rev. Letters **12**, 607 (1964).

[7] L. Mandel, Phys. Letters **10**, 166 (1964).

[8] H. Haken, Phys. Rev. Letters **13**, 329 (1964).

[9] A. T. Forrester, J. Opt. Soc. Am. **51**, 253 (1961), and *Advances in Quantum Electronics* (Columbia University Press, New York, 1961), p. 233. Actually Forrester did not make the Gaussian random assumption explicitly, but implicitly, in treating the Fourier components of the classical wave amplitude as statistically independent variates.