where $T^{\lambda\mu}$ is the usual Klein-Gordon energy-momentum tensor,

$$T_{\lambda\mu}=\Phi_\lambda{}^*\partial_\mu\Phi+\partial_\mu\Phi^*\Phi_\lambda-g_{\lambda\mu}(\Phi_\nu{}^*\Phi^\nu-m^2\Phi^*\Phi)\,.$$

As $x_0\to-\infty$, $\langle p^\mu\rangle$ tends to the expectation value in the free-particle state specified by φ. If φ is sharply peaked around some value of p, so that Φ closely approximates $\Phi_p{}^{\text{in}}$, then $\langle p^\mu\rangle$ must tend to p^μ in the remote past. However, $\langle p^\mu\rangle$ is not time-independent. Its time dependence may be found from the relation

$$\partial_\lambda T^{\lambda\mu}=j_\lambda\partial^\mu A^\lambda\,,$$

where

$$j_\lambda=ie(\Phi^*\Phi_\lambda-\Phi_\lambda{}^*\Phi)\,.$$

If we define expectation values in the natural way

$$\langle eA_\mu\rangle=\int d^3x\; j^0(x)A_\mu(x)\,,$$

$$\left\langle\frac{e}{v^0}v_\lambda F^{\lambda\mu}\right\rangle=\int d^3x\; j_\lambda(x)F^{\lambda\mu}(x)\,,$$

then we find that $\langle p^\mu\rangle$ satisfies the classical equation of motion

$$\frac{d}{dt}(\langle p^\mu\rangle-e\langle A^\mu\rangle)=\left\langle\frac{e}{v^0}v_\lambda F^{\lambda\mu}\right\rangle\,.$$

In particular, the expectation value of $\langle p^\mu-eA^\mu\rangle$ during the presence of the beam is on the average equal to $\bar{p}^\mu$.

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Phenomenological Theory of Laser Beam Fluctuations and Beam Mixing*

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The paper contains a quantum theoretical analysis of laser beam fluctuations and of the light beat experiments with two lasers. With the help of experimental results on photon counting fluctuations in a single-mode laser field, some correlation properties of the field are derived. It is shown that the correlation equations are satisfied by states of the field which are much more general than "coherent" states. The equations lead directly to the spectral density of the intensity operator in the light beat experiments, which can be obtained from photoelectric measurements. The resulting expression is practically identical to that found by Forrester for light having thermal statistical properties. The reasons for this are discussed by a comparison of the corresponding probability distributions of photon counts and of the classical wave amplitude.

1. INTRODUCTION

THE problem of determining the optical spectrum of a laser beam from beat experiments with two or more lasers is of interest, not only because of its practical importance, but because it involves the fluctuation properties of the optical field. Immediately after the development of the first continuously operating laser it was recognized that the spectral profile of one mode of the extremely narrow band light beam could not be determined by conventional interferometry. The first order of magnitude determination of the spectral linewidth was based on a photoelectric analysis of the "beats" resulting from the superposition of two similar but independent laser beams,[1] and variations of this method have become standard practice.[2]

If we picture each Fourier component of one light beam as "beating" against each Fourier component of the other, we are naturally led to regard the spectral excursion of the beat notes, reflected in the spectral range of the photoelectric signals, as a measure of the spectral width of the light itself. To an order of magnitude this measure will undoubtedly be valuable. However, in order to arrive at a quantitative relation between the spectral densities of the light beams and the spectral density of the measured photoelectric signal, we need to have information on the statistical properties of the optical fields. This information was not available to the first experimenters, and indeed the proper description of a laser field is still the subject of debate.[3–8]

By treating the classical wave amplitude of the optical field as a Gaussian random process, Forrester[9] obtained

* This research was supported in part by the U. S. Army Research Office (Durham) and by the U. S. Air Force Cambridge Research Laboratories.

[1] A. Javan, E. A. Ballik, and W. L. Bond, J. Opt. Soc. Am. **52**, 96 (1962).

[2] See, for example, D. R. Herriot, J. Opt. Soc. Am. **52**, 31 (1962); B. J. McMurtry and A. E. Siegman, Appl. Opt. **2**, 767 (1963); M. S. Lipsett and L. Mandel, Nature (London) **199**, 553 (1963).

[3] W. E. Lamb, Jr., Phys. Rev. **134**, A1429 (1964).

[4] H. Paul, W. Brunner, and G. Richter, Ann. Physik **12**, 325 (1963).

[5] L. Mandel, Phys. Rev. **134**, A10 (1964).

[6] T. F. Jordan and F. Ghielmetti, Phys. Rev. Letters **12**, 607 (1964).

[7] L. Mandel, Phys. Letters **10**, 166 (1964).

[8] H. Haken, Phys. Rev. Letters **13**, 329 (1964).

[9] A. T. Forrester, J. Opt. Soc. Am. **51**, 253 (1961), and *Advances in Quantum Electronics* (Columbia University Press, New York, 1961), p. 233. Actually Forrester did not make the Gaussian random assumption explicitly, but implicitly, in treating the Fourier components of the classical wave amplitude as statistically independent variates.

a simple relation between the spectral densities, but the assumption was very properly criticized as inapplicable to a laser field.[10] Forrester's formula has consequently tended to be discounted.

In the following we shall approach the problem phenomenologically in quantum-mechanical terms. We first examine the implications of some recent experimental results on photon counting fluctuations in a single-mode laser field for the correlation properties[11,12] of the field. The equations obeyed by the correlations are satisfied by states of the field which are much more general than the coherent states often taken to characterize a laser field. Moreover it is pointed out that "coherent" states do not satisfy the requirements of stationarity and ergodicity that one might reasonably expect for this physical process. However, an explicit form for the field density operator turns out to be unnecessary. It is shown that the phenomenological correlation equations, together with the assumption of stationarity, lead directly to the spectral density of the light-intensity fluctuations in the "beat" experiment. The resulting expression is practically identical to that given by Forrester,[9] despite the fact that the properties of the field are here very different from those implicitly assumed by him.

The result is partly a reflection of the situation that the field resulting from the superposition of two independent laser modes has quite different statistical properties from the field of one mode. This is illustrated by a comparison of the photon counting distributions and of the distribution of the classical wave amplitude for the two cases.

2. MOMENTS AND CORRELATIONS OF PHOTON NUMBERS

It has been emphasized by Glauber[11–13] that the operator of the field which most nearly corresponds to the "observable" in a photoelectric measurement is the complex-field or configuration-space annihilation operator $\mathbf{A}(\mathbf{x},t)$ corresponding to the detection of a photon at the space-time point $\mathbf{x},t$. We can expand $A_j(\mathbf{x},t)$ in the form

$$A_j(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{\mathbf{k},s} a_{\mathbf{k},s}(\boldsymbol{\varepsilon}_{\mathbf{k},s})_j \exp i(\mathbf{k}\cdot\mathbf{x}-ckt)\,, \quad (1)$$

where L^3 is the normalization volume, $a_{\mathbf{k},s}$ is the annihilation operator for a photon of momentum $\hbar\mathbf{k}$ and polarization s, and the $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ form a set of complex orthogonal unit vectors, defined up to a unitary transformation. The operators $A_i(\mathbf{x},t)$ and $A_j^\dagger(\mathbf{x},t)$ obey the equal time commutation rules[14]

$$\begin{aligned}&[A_i(\mathbf{x},t),A_j(\mathbf{x}',t)]=0=[A_i^\dagger(\mathbf{x},t),A_j^\dagger(\mathbf{x}',t)]\\ &[A_i(\mathbf{x},t),A_j^\dagger(\mathbf{x}',t)]=\delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}')\\ &\qquad\qquad -\frac{1}{(2\pi)^3}\int\frac{k_ik_j}{k^2}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]d^3k\,,\end{aligned} \quad (2)$$

and can be used to construct the number operator for the number of photons in a volume δV of linear dimensions large compared to the wavelength at a given time,[14]

$$N_{\delta V,t}=\sum_j\int_{\delta V}A_j^\dagger(\mathbf{x},t)A_j(\mathbf{x},t)d^3x\,. \quad (3)$$

In many problems encountered in practice, one is interested in the number of photons $N(S,\,t,\,t+T)$ falling on a given photoelectric surface S, of linear dimensions large compared to the wavelength, in a time interval t to $t+T$, when a plane beam of light strikes the surface normally. Under these conditions we evidently have for the expectation value of any function of the N operators

$$\langle f[N(S,\,t,\,t+T)]\rangle=\langle f[N_{\delta V,t}]\rangle\,, \quad (4)$$

when the volume δV is taken to be the cylinder of base S and length cT. We may look on

$$\sum_j A_j^\dagger(\mathbf{x},t)A_j(\mathbf{x},t)\equiv\sum_j I_j(\mathbf{x},t)$$

as the light-intensity operator of the beam. In the following we will for simplicity restrict ourselves entirely to plane and polarized beams falling normally on detectors.

In the statistical description of the field we have to distinguish between moments and correlations of the number operator N.[15,16] The rth moment $\langle N^r\rangle$ is expressed by

$$\langle N^r\rangle=\sum_{j_1}\cdots\sum_{j_r}\int\cdots\int_{\delta V}\langle A_{j_1}^\dagger(\mathbf{x}_1,t)A_{j_1}(\mathbf{x}_1,t)\cdots A_{j_r}^\dagger(\mathbf{x}_r,t)A_{j_r}(\mathbf{x}_r,t)\rangle d^3x_1\cdots d^3x_r\,,\quad r=1,\,2,\,3,\text{ etc.,} \quad (5)$$

[10] A. W. Smith and G. W. Williams, J. Opt. Soc. Am. **52**, 337 (1962).
[11] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[12] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[13] R. J. Glauber, *Quantum Electronics III*, edited by P. Grivet and N. Bloembergen (Columbia University Press, New York, 1964), p. 111.
[14] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row Publishers, Inc., New York, 1961), p. 172.
[15] L. Mandel, Phys. Rev. **136**, B1221 (1964). My attention has been drawn by Professor M. L. Goldberger to the fact that the second term in the commutator given in Eq. (2) was dropped prematurely in this paper, although it makes no contribution ultimately. The more correct derivation of the equation connecting moments and correlations is given in the Appendix.
[16] T. F. Jordan, Phys. Letters **11**, 289 (1964).

whereas the rth order correlation of N is given by the normally ordered integral

$$\langle :N^r: \rangle = \sum_{j_1} \cdots \sum_{j_r} \int_{\delta V} \cdots \int \langle A_{j_1}^\dagger(\mathbf{x}_1,t) \cdots A_{j_r}^\dagger(\mathbf{x}_r,t) A_{j_1}(\mathbf{x}_1,t) \cdots A_{j_r}(\mathbf{x}_r,t) \rangle d^3x_1 \cdots d^3x_r, \quad r=1,2,3, \text{ etc.,} \tag{6}$$

where $:O:$ denotes normal ordering of the operator O. The relation between these two quantities (5) and (6) can be obtained by repeated application of the commutation rules for the A and $A^\dagger$ operators.[15] Alternatively it can be shown that the characteristic generating functions for $\langle N^r \rangle$ and $\langle :N^r: \rangle$ are very simply connected by[17]

$$\langle \exp iyN \rangle = \langle :\exp(e^{iy}-1)N: \rangle, \tag{7}$$

or

$$1+\sum_{r=1}^{\infty} \frac{(iy)^r}{r!} \langle N^r \rangle = 1+\sum_{r=1}^{\infty} \frac{(e^{iy}-1)^r}{r!} \langle :N^r: \rangle, \tag{8}$$

from which the relation between the $\langle N^r \rangle$ and $\langle :N^r: \rangle$ follows by direct comparison of the coefficients of y^r.[18]

The Eqs. (5) and (6), which apply to a single volume δV can be generalized for r separate volumes. In practice, the moments are usually measured for a single region, whereas the correlations are most readily measured for different regions of space time. Corresponding to (6) we may write

$$\langle :N_{\delta V_1,t} \cdots N_{\delta V_r,t}: \rangle = \sum_{j_1} \cdots \sum_{j_r} \int_{\delta V_1} \cdots \times \int_{\delta V_r} \langle A_{j_1}^\dagger(\mathbf{x}_1,t) \cdots A_{j_r}^\dagger(\mathbf{x}_r,t) A_{j_1}(\mathbf{x}_1,t) \cdots \times A_{j_r}(\mathbf{x}_r,t) \rangle d^3x_1 \cdots d^3x_r. \tag{9}$$

Following (4), with a special choice of δV_1, δV_2, etc., we may also look on this expression as an autocorrelation of the number operator at the same surface S at different time intervals T (with $\delta V = ScT$), when a plane beam is falling normally on the surface. For short T, (9) then represents the rth order autocorrelation of the intensity operator.

3. DEDUCTIONS FROM EXPERIMENTS ON LASER BEAMS

The field of a gas laser which is oscillating continuously in a single mode above threshold has been studied photoelectrically by several workers.[19,20] In these investigations the laser beam was allowed to fall normally on a photoelectric detector, whose photocurrent fluctuations were analyzed. Similar experiments have also recently been carried out with a continuously oscillating gallium arsenide laser.[21] It has been found that the mean-squared fluctuation of the photoelectric current agrees with the expected shot-noise fluctuation of a current of the given average magnitude. Moreover it appears that no intensity correlation of the type observed by Hanbury Brown and Twiss[22] is detectable when a single-mode laser beam is split into two beams by a half-silvered mirror and each beam falls on a detector.[21] As shot noise is well known to be a Poisson process,[23] it follows that the photoelectric counts have the variance of a Poisson distribution.

If we look on a photoelectric detector as a photon counter, which allows us to determine the expectation values of the projection operators $|N\rangle\langle N|$ from a histogram of counts, these results have an immediate interpretation in terms of the properties of the field. That a photodetector may be consistently viewed in this way, even when the quantum efficiency is below unity, has been confirmed in several recent analyses.[24–26] Accordingly we interpret the experimental results as implying that the number operator N for a laser beam has the variance of a Poisson distribution.

Now, from a comparison of coefficients of y^2 in the expansion of (8) it follows that (as shown directly in the Appendix)

$$\langle N^2 \rangle = \langle :N^2: \rangle + \langle N \rangle,$$

and, as $\langle N^2 \rangle = \langle N \rangle + \langle N \rangle^2$ for a Poisson process,

$$\langle :N^2: \rangle = \langle N \rangle^2,$$

which we may write in the form

$$\langle :N^r: \rangle = \langle N \rangle^r, \quad r=1,2, \tag{10a}$$

since the relation holds identically for $r=1$. If this result holds for a light beam polarized in the j direction and for any volume δV whose linear dimensions are large compared with the wavelength (or alternatively for any counting interval T much longer than a period when plane waves are falling normally on the detector),

[17] Various derivations of this result may be found in J. Schwinger, J. Math. Phys. **2**, 407 (1961); W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill Book Company, Inc., New York, 1964); F. Ghielmetti, Phys. Letters **12**, 210 (1964), and in Ref. 15.

[18] For some explicit relations between the coefficients see, for example, L. Mandel, *Progress in Optics II*, edited by E. Wolf (John Wiley & Sons, Inc., New York, 1963), p. 181.

[19] J. A. Bellisio, C. Freed, and H. A. Haus, Appl. Phys. Letters **4**, 5 (1964).

[20] R. L. Bailey and J. H. Sanders, Phys. Letters **10**, 295 (1964).

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[25] F. Ghielmetti, Phys. Letters **12**, 210 (1964).

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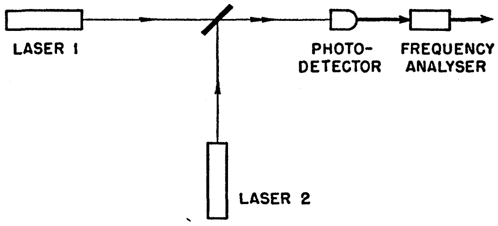

FIG. 1. The principle of the beat experiment.

we see from (3) that (10a) implies

$$\langle[A_j^\dagger(\mathbf{x},t)]^r[A_j(\mathbf{x},t)]^r\rangle=\langle A_j^\dagger(\mathbf{x},t)A_j(\mathbf{x},t)\rangle^r,$$

or

$$\langle{:}I_j(\mathbf{x},t)^r{:}\rangle=\langle I_j(\mathbf{x},t)\rangle^r,\quad r=1,2, \tag{10b}$$

where $I_j(\mathbf{x},t)$ is the intensity operator. More generally we interpret the absence of Hanbury Brown-Twiss type of intensity correlation to mean that

$$\langle{:}I_j(\mathbf{x},t_1)I_j(\mathbf{x},t_2){:}\rangle=\langle I_j(\mathbf{x},t_1)\rangle\langle I_j(\mathbf{x},t_2)\rangle. \tag{11}$$

Although the higher order moments of the photoelectric fluctuations have not so far received any attention experimentally, it is interesting to note the consequences of making the plausible assumption that they also correspond to a Poisson process. If we take N to obey a Poisson distribution, its characteristic function will have the well-known form[27]

$$\langle\exp iyN\rangle=\exp[(e^{iy}-1)\langle N\rangle], \tag{12}$$

and comparison with Eq. (7) and expansion of the exponential then shows immediately that Eqs. (10) will hold for all positive integral values of r. It is possible to make a similar plausible generalization of (11). Fortunately, however, Eqs. (10) and (11) with $r=1$, 2 are sufficient for determining the spectrum of the intensity fluctuations in the light beat experiments.

The state of the single-mode laser field can be described by the most general density operator satisfying Eqs. (10) and (11). Unfortunately this general solution does not appear to have a simple explicit form. We can see at once that any "coherent" state[12] of the field

$$|\{v_{\mathbf{k},s}\}\rangle\equiv\prod_{\mathbf{k},s}|v_{\mathbf{k},s}\rangle,$$

where $|v_{\mathbf{k},s}\rangle$ is any eigenstate of the $a_{\mathbf{k},s}$ operator, satisfies (10) and (11), but that much more complicated solutions are possible. In any case a coherent state does not describe a stationary field,[7] and the device of averaging over the total phase to ensure stationarity does not ensure the ergodicity that one might reasonably expect for this physical process. For a stationary quasi-monochromatic light beam we require the expectation values of all operators to be independent of the origin of time, and that, at least for small integral values of M, N,

$$\begin{gathered}\langle A_j^\dagger(\mathbf{x},t_1)\cdots A_j^\dagger(\mathbf{x},t_N)\rangle=0,\\ \langle A_j(\mathbf{x},t_1)\cdots A_j(\mathbf{x},t_M)\rangle=0,\\ \langle A_j^\dagger(\mathbf{x},t_1)\cdots A_j^\dagger(\mathbf{x},t_N)\\ \times A_j(\mathbf{x},t_{N+1})\cdots A_j(\mathbf{x},t_{N+M})\rangle=0,\quad\text{if}\quad N\neq M.\end{gathered} \tag{13}$$

As an example of the more general type of solution of Eq. (10), valid at the point $\mathbf{x}$, consider the following density operator ρ, which we write in the "diagonal" Sudarshan representation[28–30] in which the $|\{v_{\mathbf{k},s}\}\rangle$ states are the basis states:

$$\rho=\int f(\{v_{\mathbf{k},s}\})\delta\Big[B-\frac{1}{L^{3/2}}\Big|\sum_{\mathbf{k},s}v_{\mathbf{k},s}(\boldsymbol{\varepsilon}_{\mathbf{k},s})_j \times\exp i(\mathbf{k}\cdot\mathbf{x}-ckt)\Big|\Big]|\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}|\,d^2\{v_{\mathbf{k},s}\}. \tag{14a}$$

Here, $\{v_{\mathbf{k},s}\}$ stands for the set of all $v_{\mathbf{k},s}$, and the integral extends over the entire complex $\{v_{\mathbf{k},s}\}$ plane. B is a real positive number, and $f(\{v_{\mathbf{k},s}\})$ is any function of the $\{v_{\mathbf{k},s}\}$ which ensures the required normalization and stationarity of ρ. Stationarity requires $f(\{v_{\mathbf{k},s}\})$ to be chosen so that the expectation value of the complex-field operator $A_j(\mathbf{x},t)$ is zero. In view of (1), the eigenvalue of $A_j(\mathbf{x},t)$ is evidently

$$V_j(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{\mathbf{k},s}v_{\mathbf{k},s}(\boldsymbol{\varepsilon}_{\mathbf{k},s})_j\exp i(\mathbf{k}\cdot\mathbf{x}-ckt), \tag{14b}$$

and the $v_{\mathbf{k},s}$ should have random phase. It is easy to show that (14a) satisfies (10) for all positive integral r, with $\langle I_j(\mathbf{x},t)\rangle=B^2$.

Fortunately, for the problem of calculating the spectral density of the intensity operator it is not necessary to know the general form of ρ explicitly, and the defining relations (10) and (11) can be used directly.

4. ANALYSIS OF THE LIGHT BEAT EXPERIMENT

Consider the experiment outlined in Fig. 1, in which the beams from two independent lasers 1 and 2 are brought together and superposed with the aid of a 45° half-silvered mirror. The beams strike a photoelectric detector normal to the two wave fronts (assumed to be plane) whose output is to be analyzed. We suppose that both beams are linearly polarized in the same direction j as viewed from the detector.

Insofar as a spectral analysis of the photoelectric current corresponds to a spectral analysis of the total field-intensity operator,[11,24] we begin by considering the

[27] J. F. Kenney and E. S. Keeping, *Mathematics of Statistics, Part 2* (D. Van Nostrand Company, New York, 1951), p. 74.

[28] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

[29] E. C. G. Sudarshan, in *Proceedings of the Symposium on Optical Masers* (Polytechnic Institute of Brooklyn, New York, 1963), p. 45.

[30] See J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. (to be published), for a rigorous proof of the representation. An alternative proof has recently been given by C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, 27 (1965).

second-order autocorrelation of this operator. In view of the statistical independence of the two superposed fields, the density operator ρ of the combined field will factorize into the product $\rho_1\rho_2$ of the density operators ρ_1 and ρ_2 for the two separate fields. We represent each in the "diagonal" Sudarshan form, so that we may write

$$\rho=\int\int p_1(\{v_{\mathbf{k},s}'\})p_2(\{v_{\mathbf{k},s}''\})|\{v_{\mathbf{k},s}''\}\rangle|\{v_{\mathbf{k},s}'\}\rangle \times\langle\{v_{\mathbf{k},s}'\}|\langle\{v_{\mathbf{k},s}''\}|d^2\{v_{\mathbf{k},s}'\}d^2\{v_{\mathbf{k},s}''\}. \quad (15)$$

The second-order autocorrelation of the intensity operator at the point $\mathbf{x}$ at times t and $t+\tau$ will depend only on $\mathbf{x}$ and τ for a stationary field, and, from the discussion of Sec. 2, will be given by

$$R(\mathbf{x},\tau)=\mathrm{Tr}[\rho A_j^\dagger(\mathbf{x},t) \times A_j^\dagger(\mathbf{x},t+\tau)A_j(\mathbf{x},t)A_j(\mathbf{x},t+\tau)]. \quad (16)$$

The operators A here act on the combined field, so that

$$A_j(\mathbf{x},t)|\{v_{\mathbf{k},s}'\}\rangle|\{v_{\mathbf{k},s}''\}\rangle=[A_j(\mathbf{x},t)|\{v_{\mathbf{k},s}'\}\rangle]|\{v_{\mathbf{k},s}''\}\rangle +|\{v_{\mathbf{k},s}'\}\rangle[A_j(\mathbf{x},t)|\{v_{\mathbf{k},s}''\}\rangle]. \quad (17)$$

By introducing (15) into (16) and making use of (17), together with the fact that the trace remains invariant under cyclic permutation of operators, we can express $R(\mathbf{x},\tau)$ as the sum of 16 terms[31] (we suppress the index j and parameter $\mathbf{x}$ for brevity):

$$\begin{aligned}R(\mathbf{x},\tau)=&\mathrm{Tr}[\rho_1A^\dagger(t)]\,\mathrm{Tr}[\rho_2A^\dagger(t+\tau)A(t)A(t+\tau)]+\mathrm{Tr}[\rho_2A^\dagger(t)A^\dagger(t+\tau)A(t)A(t+\tau)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t)A(t)A(t+\tau)]+\mathrm{Tr}[\rho_1A^\dagger(t)A^\dagger(t+\tau)]\,\mathrm{Tr}[\rho_2A(t)A(t+\tau)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t)A(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t+\tau)A(t)]+\mathrm{Tr}[\rho_1A(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t)A^\dagger(t+\tau)A(t)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t+\tau)A(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t)A(t)]+\mathrm{Tr}[\rho_1A^\dagger(t)A^\dagger(t+\tau)A(t+\tau)]\,\mathrm{Tr}[\rho_2A(t)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t)A(t)]\,\mathrm{Tr}[\rho_2A^\dagger(t+\tau)A(t+\tau)]+\mathrm{Tr}[\rho_1A(t)]\,\mathrm{Tr}[\rho_2A^\dagger(t)A^\dagger(t+\tau)A(t+\tau)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t+\tau)A(t)]\,\mathrm{Tr}[\rho_2A^\dagger(t)A(t+\tau)]+\mathrm{Tr}[\rho_1A^\dagger(t)A^\dagger(t+\tau)A(t)]\,\mathrm{Tr}[\rho_2A(t+\tau)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t)A(t)A(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t+\tau)]+\mathrm{Tr}[\rho_1A(t)A(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t)A^\dagger(t+\tau)]\\&+\mathrm{Tr}[\rho_1A^\dagger(t+\tau)A(t)A(t+\tau)]\,\mathrm{Tr}[\rho_2A^\dagger(t)]+\mathrm{Tr}[\rho_1A^\dagger(t)A^\dagger(t+\tau)A(t)A(t+\tau)].\end{aligned} \quad (18)$$

Of these the 1st, 3rd, 4th, 6th, 8th, 10th, 12th, 13th, 14th, and 15th terms vanish by virtue of the stationarity conditions (13). The 2nd and 16th terms have the form of the left-hand side of Eq. (11) and, because of stationarity, reduce to $\langle I_2(\mathbf{x})\rangle^2$ and $\langle I_1(\mathbf{x})\rangle^2$, respectively, where

$$\begin{aligned}\langle I_1(\mathbf{x})\rangle&=\mathrm{Tr}[\rho_1I_j(\mathbf{x},t)]\\\langle I_2(\mathbf{x})\rangle&=\mathrm{Tr}[\rho_2I_j(\mathbf{x},t)]\end{aligned} \quad (19)$$

are the expected intensities of the separate fields before superposition. The remaining terms are all of the same general form. If we write

$$\begin{aligned}\Gamma_{11}(\mathbf{x},\tau)&=\mathrm{Tr}[\rho_1A_j^\dagger(\mathbf{x},t)A_j(\mathbf{x},t+\tau)]\\\Gamma_{22}(\mathbf{x},\tau)&=\mathrm{Tr}[\rho_2A_j^\dagger(\mathbf{x},t)A_j(\mathbf{x},t+\tau)],\end{aligned} \quad (20)$$

where $\Gamma_{11}(\mathbf{x},\tau)$ and $\Gamma_{22}(\mathbf{x},\tau)$ are autocorrelations of the complex-field operators for the two separate fields,[11] and correspond to the self-coherence functions introduced into the classical coherence theory by Wolf[32,33] then

$$R(\mathbf{x},\tau)=[\langle I_1(\mathbf{x})\rangle+\langle I_2(\mathbf{x})\rangle]^2 +2\,\mathrm{Re}[\Gamma_{11}^*(x,\tau)\Gamma_{22}(\mathbf{x},\tau)]. \quad (21)$$

It is convenient to make use of normalized correlation functions defined by

$$\gamma_{11}(\mathbf{x},\tau)=\frac{\Gamma_{11}(\mathbf{x},\tau)}{\langle I_1(\mathbf{x})\rangle},$$

$$\gamma_{22}(\mathbf{x},\tau)=\frac{\Gamma_{22}(\mathbf{x},\tau)}{\langle I_2(\mathbf{x})\rangle},$$

$$r(\mathbf{x},\tau)=\frac{R(\mathbf{x},\tau)-[\langle I_1(\mathbf{x})\rangle+\langle I_2(\mathbf{x})\rangle]^2}{R(\mathbf{x},0)-[\langle I_1(\mathbf{x})\rangle+\langle I_2(\mathbf{x})\rangle]^2}.$$

Then (21) becomes

$$r(\mathbf{x},\tau)=\mathrm{Re}[\gamma_{11}^*(\mathbf{x},\tau)\gamma_{22}(\mathbf{x},\tau)]. \quad (22)$$

We now introduce the normalized spectral densities of both the $A_j(\mathbf{x},t)$ and $I_j(\mathbf{x},t)$ operators, which are Fourier transforms of the corresponding autocorrelations. Let

$$\begin{aligned}\phi_{11}(\mathbf{x},\nu)&=\int_{-\infty}^{\infty}\gamma_{11}(\mathbf{x},\tau)\exp(2\pi i\nu\tau)d\tau,\\\phi_{22}(\mathbf{x},\nu)&=\int_{-\infty}^{\infty}\gamma_{22}(\mathbf{x},\tau)\exp(2\pi i\nu\tau)d\tau,\\\psi(\mathbf{x},\nu)&=\int_{-\infty}^{\infty}r(\mathbf{x},\tau)\exp(2\pi i\nu\tau)d\tau,\end{aligned} \quad (23)$$

where $\nu=kc/2\pi$. We observe that, from the definition

[31] See also Ref. 5 for a method of simplifying the expression for intensity correlations.

[32] See M. Born and E. Wolf, *Principles of Optics* (The Macmillan Company, New York, 1964), 2nd ed., p. 500.

[33] See, for example, E. Wolf, *Quantum Electronics, III* (Columbia University Press, New York, 1964), p. 13.

(1), $A_j(\mathbf{x},t)$ is analytic in the lower half complex t plane. It follows that $\Gamma_{11}(\mathbf{x},\tau)$ and $\Gamma_{22}(\mathbf{x},\tau)$ defined by (20) are also analytic in the lower half complex τ plane (and similarly for $\gamma_{11}(\mathbf{x},\tau)$ and $\gamma_{22}(\mathbf{x},\tau)$), and, by a well-known theorem,[34] that $\phi_{11}(\mathbf{x},\nu)$ and $\phi_{22}(\mathbf{x},\nu)$ vanish for negative values of ν.

Since $\phi_{11}(\mathbf{x},\nu)$ and $\phi_{22}(\mathbf{x},\nu)$ are spectral densities of two similar, but independent, laser beams of narrow spectral range, it seems reasonable to assume that the two functions are similar, except for a possible translation along the frequency axis. The center frequency of each beam depends very critically on the geometry of the cavity and cannot easily be predetermined. Let ν_1 and ν_2 be the center frequencies of the two beams, with $\nu_2 \geqslant \nu_1$. We therefore assume that

$$\phi_{11}(\mathbf{x},\, \nu_1+\nu')=\phi_{22}(\mathbf{x},\, \nu_2+\nu'). \tag{24}$$

Because both $\phi_{11}(\mathbf{x},\nu)$ and $\phi_{22}(\mathbf{x},\nu)$ vanish for $\nu<0$ it follows from (23) and (24) that

$$\begin{aligned}\gamma_{22}(\mathbf{x},\tau)&\\ &=\int_0^\infty \phi_{22}(\mathbf{x},\nu)\exp(-2\pi i\nu\tau)d\nu\\ &=\exp(-2\pi i\nu_2\tau)\int_{-\nu_2}^\infty \phi_{22}(\mathbf{x},\,\nu_2+\nu')\exp(-2\pi i\nu'\tau)d\nu'\\ &=\exp(-2\pi i\nu_2\tau)\int_{-\nu_1}^\infty \phi_{11}(\mathbf{x},\,\nu_1+\nu')\exp(-2\pi i\nu'\tau)d\nu'\\ &=\exp[-2\pi i(\nu_2-\nu_1)\tau]\gamma_{11}(\mathbf{x},\tau),\end{aligned} \tag{25}$$

and, by substituting (25) into (22), we find

$$\begin{aligned}r(\mathbf{x},\tau)&=|\gamma_{11}(\mathbf{x},\tau)|^2\cos 2\pi(\nu_2-\nu_1)\tau\\ &=|\gamma_{22}(\mathbf{x},\tau)|^2\cos 2\pi(\nu_2-\nu_1)\tau.\end{aligned} \tag{26}$$

To find the spectral density of the intensity fluctuations $\psi(\mathbf{x},\nu)$ we merely Fourier transform (26) according to the relations (23). Thus

$$\begin{aligned}\psi(\mathbf{x},\nu)&=\int_{-\infty}^\infty r(\mathbf{x},\tau)\exp(2\pi i\nu\tau)d\tau\\ &=\int_{-\infty}^\infty\int_0^\infty\int_0^\infty \phi_{11}(\nu')\phi_{11}(\nu'')\exp[2\pi i(\nu+\nu''-\nu')\tau]\cos[2\pi(\nu_2-\nu_1)\tau]d\nu'd\nu''d\tau\\ &=\tfrac{1}{2}\int_0^\infty\int_0^\infty \phi_{11}(\nu')\phi_{11}(\nu'')[\delta(\nu+\nu''-\nu'+\nu_2-\nu_1)+\delta(\nu+\nu''-\nu'-\nu_2+\nu_1)]d\nu'd\nu''\\ &=\tfrac{1}{2}\int_0^\infty \phi_{11}(\nu')[\phi_{11}(\nu'+\nu+\nu_2-\nu_1)+\phi_{11}(\nu'+\nu-\nu_2+\nu_1)]d\nu',\end{aligned} \tag{27}$$

when we make explicit use of the fact that $\phi_{11}(\nu)$ vanishes for $\nu<0$. If the photoelectric detector is regarded as a photon counter, the same quantity $\psi(\mathbf{x},\nu)$ also describes the normalized spectral density of the photoelectric current, which can be determined by passing the photoelectric signal to a frequency analyzer. It is easy to see from (27) that $\psi(\mathbf{x},-\nu)=\psi(\mathbf{x},\nu)$, which also follows directly from the fact that $r(\mathbf{x},\tau)$ is an autocorrelation function and real. By applying the Schwarz inequality to both terms on the right-hand side of Eq. (27), and using the fact that $\phi_{11}(\mathbf{x},\nu)$ is non-negative, we find that

$$\begin{aligned}\psi(\mathbf{x},\nu)&\leqslant\tfrac{1}{2}\left[\int_0^\infty \phi_{11}^2(\nu')d\nu'\int_0^\infty \phi_{11}^2(\nu'+\nu+\nu_2-\nu_1)d\nu'\right]^{1/2}+\tfrac{1}{2}\left[\int_0^\infty \phi_{11}^2(\nu')d\nu'\int_0^\infty \phi_{11}^2(\nu'+\nu-\nu_2+\nu_1)d\nu'\right]^{1/2}\\ &\leqslant\int_0^\infty \phi_{11}^2(\nu')d\nu'\equiv\xi.\end{aligned} \tag{28}$$

The quantity ξ, which is an upper bound for $\psi(\mathbf{x},\nu)$, can be shown to be a measure of the coherence time of each incident laser beam.[35,36] We can also see from (27) that $\psi(\mathbf{x},\nu)$ is the sum of two identical, symmetric spectral distributions centered on $\nu_2-\nu_1$ and $\nu_1-\nu_2$, respectively, so that we may justifiably regard $\psi(\mathbf{x},\nu)$ as the spectral density of "beat notes." Since each of the two separate spectral distribution functions in (27) has a peak value of $\frac{1}{2}\xi$, and since the integral of $\psi(\mathbf{x},\nu)$ over all frequencies is unity by definition, we can see that the spectral width of $\psi(\mathbf{x},\nu)$ will be an order of magnitude measure of the reciprocal coherence time or linewidth.

When $\nu_2=\nu_1$ the expression (27) is identical to that found by Forrester[9,35] for a light beam having the sta-

[34] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon Press, Oxford, England, 1948), 2nd ed.

[35] Cf. L. Mandel, *Progress in Optics, II*, edited by E. Wolf (North-Holland Publishing Company, Amsterdam; John Wiley & Sons, Inc., New York, 1963), p. 181.

[36] See also L. Mandel and E. Wolf, Proc. Phys. Soc. **80**, 516 (1962).

tistical properties of thermal light. While it is remarkable that similar equations describe the behavior of optical fields as different as thermal and laser fields, it is important not to overlook certain very important differences. Thus, Eqs. (26) and (27) apply only to the superposition of two *independent* laser fields, and not to the separate fields. Indeed, in view of the properties (10) and (11), $r(\mathbf{x},\tau)=1$ for all $\mathbf{x}$, τ for a single-mode laser field, and $\psi(\mathbf{x},\nu)=\delta(\nu)$. On the other hand, when $\nu_2=\nu_1$, (26) and (27) correctly describe the field of one thermal source. Thus the superposition plays a much more significant role for laser fields than for thermal ones. We illustrate this feature by calculating the distributions of photon numbers and of the classical wave amplitude for a field produced by superposition of two fields described by density operators of the form (14). We shall refer to the resulting field as a two-mode laser field.

5. SOME PROPERTIES OF A LASER FIELD CONSISTING OF TWO INDEPENDENT MODES

For simplicity we again take the two modes to correspond to parallel plane waves with similar polarizations j, and assume that the density operators ρ_1 and ρ_2 for the two modes are both of the form (14a), with equal values of B (i.e., equal expectation values of intensity at $\mathbf{x}$), but not necessarily equal functions $f_1(\{v_{\mathbf{k},s}\})$ and $f_2(\{v_{\mathbf{k},s}\})$. The density operator ρ of the combined field will be given by (15), with

$$\begin{aligned} p_1(\{v_{\mathbf{k},s}'\}) &= f_1(\{v_{\mathbf{k},s}'\})\delta(B-|V_j'(\mathbf{x},t)|) \\ p_2(\{v_{\mathbf{k},s}''\}) &= f_2(\{v_{\mathbf{k},s}''\})\delta(B-|V_j''(\mathbf{x},t)|), \end{aligned} \tag{29}$$

where $V_j'(\mathbf{x},t)$ and $V_j''(\mathbf{x},t)$ are the eigenvalues of $A_j(\mathbf{x},t)$ corresponding to the states $|\{v_{\mathbf{k},s}'\}\rangle$ and $|\{v_{\mathbf{k},s}''\}\rangle$, and are defined by (14b) with an obvious extension of the notation. If we introduce these eigenvalues into (17), the equation becomes

$$\begin{aligned} A_j(\mathbf{x},t)|\{v_{\mathbf{k},s}'\}\rangle|\{v_{\mathbf{k},s}''\}\rangle \\ =[V_j'(\mathbf{x},t)+V_j''(\mathbf{x},t)]|\{v_{\mathbf{k},s}'\}\rangle|\{v_{\mathbf{k},s}''\}\rangle. \end{aligned} \tag{30}$$

However, by Sudarshan's theorem[28–30] ρ must also be expressible directly in the form

$$\rho=\int\Phi(\{v_{\mathbf{k},s}\})|\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}|d^2\{v_{\mathbf{k},s}\}, \tag{31}$$

where $|\{v_{\mathbf{k},s}\}\rangle$ is an eigenstate of the annihilation operator $A_j(\mathbf{x},t)$ for the combined field. Hence

$$A_j(\mathbf{x},t)|\{v_{\mathbf{k},s}\}\rangle=V_j(\mathbf{x},t)|\{v_{\mathbf{k},s}\}\rangle, \tag{32}$$

where $V_j(\mathbf{x},t)$ is given by the corresponding expansion (14b). Comparison of (30) and (32) shows that the eigenstates $|\{v_{\mathbf{k},s}\}\rangle$ and $|\{v_{\mathbf{k},s}'\}\rangle|\{v_{\mathbf{k},s}''\}\rangle$ will correspond to the same eigenvalues if

$$V_j(\mathbf{x},t)=V_j'(\mathbf{x},t)+V_j''(\mathbf{x},t). \tag{33}$$

We may use this equation to relate $\Phi(\{v_{\mathbf{k},s}\})$ and $p_1(\{v_{\mathbf{k},s}'\})$ and $p_2(\{v_{\mathbf{k},s}''\})$ by writing

$$\Phi(\{v_{\mathbf{k},s}\})=\int\int p_1(\{v_{\mathbf{k},s}'\})p_2(\{v_{\mathbf{k},s}''\})\delta[V_j(\mathbf{x},t)-V_j'(\mathbf{x},t)-V_j''(\mathbf{x},t)]d^2\{v_{\mathbf{k},s}'\}d^2\{v_{\mathbf{k},s}''\}. \tag{34}$$

Following the method of Ghielmetti,[25] we now express the probability distribution $p(\{n_{\mathbf{k},s}\})$ for a particular set of photon numbers $\{n_{\mathbf{k},s}\}$ in the form

$$\begin{aligned} p(\{n_{\mathbf{k},s}\}) &= \mathrm{Tr}[\rho|\{n_{\mathbf{k},s}\}\rangle\langle\{n_{\mathbf{k},s}\}|] \\ &= \mathrm{Tr}\int\Phi(\{v_{\mathbf{k},s}\})|\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}|\{n_{\mathbf{k},s}\}\rangle\langle\{n_{\mathbf{k},s}\}|d^2\{v_{\mathbf{k},s}\}, \end{aligned}$$

and when we introduce the known scalar product of $|\{v_{\mathbf{k},s}\}\rangle$ and $|\{n_{\mathbf{k},s}\}\rangle$,[12] this becomes

$$p(\{n_{\mathbf{k},s}\})=\int\Phi(\{v_{\mathbf{k},s}\})\prod_{\mathbf{k},s}\left\{\frac{(v_{\mathbf{k},s}^*v_{\mathbf{k},s})^{n_{\mathbf{k},s}}}{n_{\mathbf{k},s}!}\exp(-v_{\mathbf{k},s}^*v_{\mathbf{k},s})\right\}d^2\{v_{\mathbf{k},s}\}. \tag{35}$$

Now the probability distribution $p(n)$ of the total photon number n is related to $p(\{n_{\mathbf{k},s}\})$ by

$$p(n)=\sum_{\{n_{\mathbf{k},s}\}}p(\{n_{\mathbf{k},s}\})\delta_{n,m},$$

with

$$m=\sum_{\mathbf{k},s}n_{\mathbf{k},s}. \tag{36}$$

Hence from (35) and (36)

$$p(n)=\int\Phi(\{v_{\mathbf{k},s}\})\sum_{\{n_{\mathbf{k},s}\}}\prod_{\mathbf{k},s}\left\{\frac{(v_{\mathbf{k},s}^*v_{\mathbf{k},s})^{n_{\mathbf{k},s}}}{n_{\mathbf{k},s}!}\exp(-v_{\mathbf{k},s}^*v_{\mathbf{k},s})\right\}\delta_{n,m}\,d^2\{v_{\mathbf{k},s}\},$$

and, with the help of the multinomial theorem, this can be written

$$p(n)=\int\Phi(\{v_{\mathbf{k},s}\})\frac{[\sum_{\mathbf{k},s}|v_{\mathbf{k},s}|^2]^n}{n!}\exp(-\sum_{\mathbf{k},s}|v_{\mathbf{k},s}|^2)d^2\{v_{\mathbf{k},s}\}. \tag{37}$$

We can further rewrite this relation in a more convenient form, in which it applies to the numbers in a given volume δV, by identifying the normalization volume L^3 in Eq. (14b) with the volume δV, provided the linear dimensions of δV are very large compared with the wavelength of the light. For, by applying Parseval's theorem to the expansion (14b), we find

$$\sum_{\mathbf{k},s}|v_{\mathbf{k},s}|^2=\int_{\delta V} V_j^*(\mathbf{x},t)V_j(\mathbf{x},t)d^3\mathbf{x}\equiv U\,, \tag{38}$$

and this allows us to write (37) in the form

$$\begin{aligned} p(n)&=\int \Phi(\{v_{\mathbf{k},s}\})\frac{U^n}{n!}e^{-U}d^2\{v_{\mathbf{k},s}\} \\ &=\int_0^\infty P(U)\frac{U^n}{n!}e^{-U}dU\,, \end{aligned} \tag{39}$$

where[37]

$$P(U')=\int \Phi(\{v_{\mathbf{k},s}\})\delta(U'-U)d^2\{v_{\mathbf{k},s}\}\,. \tag{40}$$

We note that $p(n)$ is now expressed in exactly the same form as in the usual semiclassical theory of light fluctuations.[35]

For the purpose of the present discussion we choose the volume δV to be a cylinder of base S parallel to the wave front (which we may identify with the surface of a photoelectric detector) and of height cT (where T is the counting interval). Moreover we choose T to be very short compared with the reciprocal frequency spread of the light, but much longer than a period. Under these conditions U can be replaced by $V_j^*(\mathbf{x},t)V_j(\mathbf{x},t)ScT$ when it occurs under the integral (39) or (40), and we find from (29), (34), and (40)

$$\begin{aligned} P(U)=&\int\int\int f_1(\{v_{\mathbf{k},s}'\})f_2(\{v_{\mathbf{k},s}''\})\delta[B-|V_j'(\mathbf{x},t)|]\delta[B-|V_j''(\mathbf{x},t)|] \\ &\qquad\times\delta[V_j(\mathbf{x},t)-V_j'(\mathbf{x},t)-V_j''(\mathbf{x},t)]\delta[U-V_j^*(\mathbf{x},t)V_j(\mathbf{x},t)ScT]d^2\{v_{\mathbf{k},s}\}d^2\{v_{\mathbf{k},s}'\}d^2\{v_{\mathbf{k},s}''\} \\ =&\int\int f_1(\{v_{\mathbf{k},s}'\})f_2(\{v_{\mathbf{k},s}''\})\delta[U-2B^2ScT(1+\cos(\theta'-\theta''))]d^2\{v_{\mathbf{k},s}'\}d^2\{v_{\mathbf{k},s}''\}\,, \end{aligned} \tag{41}$$

where

$$\arg V_j'(\mathbf{x},t)=\theta'$$
$$\arg V_j''(\mathbf{x},t)=\theta''.$$

We have already noted in connection with Eq. (14b) that stationarity is assured if the functions $f_1(\{v_{\mathbf{k},s}'\})$ and $f_2(\{v_{\mathbf{k},s}''\})$ are such that the phases θ' and θ'' are uniformly distributed over 0 to 2π. The difference $\theta'-\theta''$ is therefore also distributed at random over 0 to 2π, and since according to (41),

$$U=2B^2ScT[1+\cos(\theta'-\theta'')]\,, \tag{42}$$

$$\begin{aligned} P(U)&=\frac{1}{2\pi}\frac{1}{B^2ScT\sin(\theta'-\theta'')} \\ &=\frac{1}{\pi[(2B^2ScT)^2-(U-2B^2ScT)^2]^{1/2}} \\ &\qquad\text{for}\quad 0\leqslant U\leqslant 4B^2ScT\,, \end{aligned} \tag{43}$$

and

$$P(U)=0\quad\text{otherwise.}$$

The constant $2B^2ScT$, or $2\langle I_1(\mathbf{x},t)\rangle ScT$, is the expectation value $\langle U\rangle$ of U. Apart from the factor ScT, U itself now corresponds to the light intensity in the classical description of the beam. The distribution $P(U)$ is illustrated in Fig. 2, in which the corresponding distributions for a single-mode laser field and for the field of a thermal source are also shown for comparison. It is evident from the density operator (14) for the single-mode field that the corresponding $p(U)$ must be a δ function. For the polarized thermal field one obtains an exponential distribution.[35]

We may now use (43) to calculate the counting distribution $p(n)$ from (39). We then obtain

$$p(n)=\int_0^{2\langle U\rangle}\frac{U^ne^{-U}}{\pi n![2U\langle U\rangle-U^2]^{1/2}}dU\,. \tag{44}$$

For very large values of $\langle U\rangle$ this distribution has a minimum at $n\approx\langle U\rangle$, and peaks at $n=0$ and $n\approx 2\langle U\rangle$. It should be compared[24] with the corresponding one-

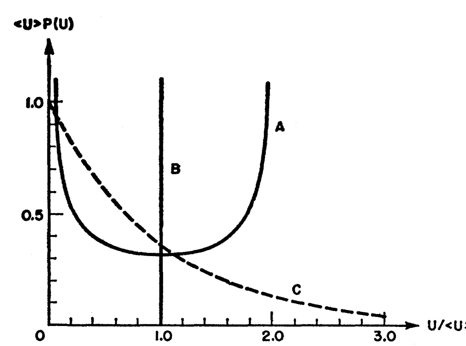

FIG. 2. The probability distributions of U for A, a two-mode laser field; B, a single-mode laser field; and C, a polarized thermal field.

[37] This simple representation was suggested by Dr. C. L. Mehta.

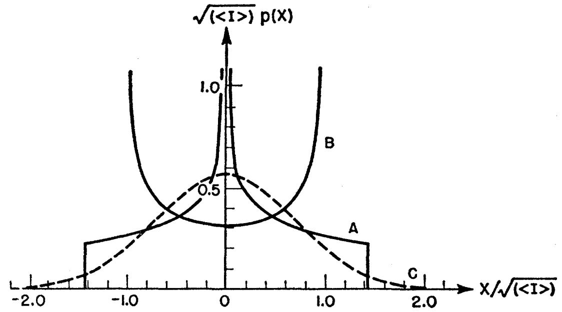

FIG. 3. Probability distributions of the classical wave amplitude for A, a two-mode laser field; B, a single-mode laser field; and C a polarized thermal field.

mode (Poisson) distribution having a single peak at $n\approx\langle n\rangle$, and the (Bose-Einstein) distribution for a polarized thermal field which decreases exponentially from $n=0$.

Finally let us make use of the distribution $P(U)$ given by (43) to calculate the distribution $p(X)$ of the real part X of the complex eigenvalue $V_j(\mathbf{x},t)$ of the field operator. $p(X)$ is essentially also the distribution of the instantaneous classical field amplitude. We have already noted that, for T very short compared with the reciprocal frequency spread of the light,

$$U=|V_j(\mathbf{x},t)|^2ScT, \qquad (45)$$

and, since stationarity is assured if the phase $\theta(\mathbf{x},t)$ of $V_j(\mathbf{x},t)$ is randomly distributed over 0 to 2π, we can immediately use (43) to write down the joint probability distribution $p'(|V|,\theta)$ of $|V|$ and θ. Thus

$$\begin{aligned} p'(|V|,\phi)&=\frac{2|V|ScT}{2\pi}P(|V|^2ScT)\\ &=\frac{|V|}{\pi^2[\langle I\rangle^2-(|V|^2-\langle I\rangle)^2]^{1/2}} \qquad \text{for} \quad |V|\leqslant\sqrt{(2\langle I\rangle)},\\ &=0 \quad \text{otherwise.} \end{aligned} \qquad (46)$$

In this expression the constant

$$2B^2=2\langle I_1(\mathbf{x},t)\rangle=2\langle I_2(\mathbf{x},t)\rangle$$

has been replaced by $\langle I\rangle$, the expectation value of the intensity of the composite beam. Hence, if X and Y are the real and imaginary parts of $V_j(\mathbf{x},t)$, the joint distribution $p''(X,Y)$ of X and Y is

$$\begin{aligned} p''(X,Y)&=\frac{1}{\pi^2[\langle I\rangle^2-(X^2+Y^2-\langle I\rangle)^2]^{1/2}}, \qquad \text{for} \quad X^2+Y^2\leqslant 2\langle I\rangle,\\ &=0 \quad \text{otherwise,} \end{aligned} \qquad (47)$$

and the probability distribution $\mathcal{P}(X)$ of X alone follows by integration over Y,

$$\mathcal{P}(X)=\int_{-\sqrt{(2\langle I\rangle-X^2)}}^{\sqrt{(2\langle I\rangle-X')}}\frac{1}{\pi^2[\langle I\rangle^2-(X^2+Y^2-\langle I\rangle)^2]^{1/2}}dY, \qquad X\leqslant\sqrt{(2\langle I\rangle)}.$$

With the help of the substitution $Y=[(2\langle I\rangle-X^2)\times(1-x^2)]^{1/2}$ the integral may be transformed to

$$\begin{aligned} \mathcal{P}(X)&=\frac{1}{\pi^2}\sqrt{\left(\frac{2}{\langle I\rangle}\right)}\int_0^1\frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-x^2[1-X^2/2\langle I\rangle])}}\\ &=\frac{1}{\pi^2}\sqrt{\left(\frac{2}{\langle I\rangle}\right)}K[\sqrt{(1-X^2/2\langle I\rangle)}] \qquad \text{for} \quad X^2\leqslant 2\langle I\rangle,\\ &=0 \quad \text{otherwise,} \end{aligned} \qquad (48)$$

where K is the complete elliptic integral of the first kind. This distribution is illustrated in Fig. 3, where the corresponding probability distributions for a single-mode laser beam and a beam from a polarized thermal source are shown for comparison.[38] The same distribution (48) was also found by Hodara[39] for the superposition of two strictly sinusoidal oscillations with random phases.

It is evident from inspection of Figs. 2 and 3 that the fluctuation properties of the two-mode laser field are in a sense intermediate between the properties of the other two fields. Moreover, in some significant respects its behavior is closer to that of the thermal field than of the single-mode laser field. Thus, we note that in Fig. 2 a minimum of the two-mode distribution coincides with a maximum (an infinity) of the one-mode distribution, and conversely in Fig. 3. It might well be difficult to construct two fields differing more than this in their fluctuation properties. It should not therefore be surprising that the photoelectric measurement of the superposition field carries information that measurements of the separate fields do not. The fact that Eq. (27) for the spectral density $\psi(\mathbf{x},\nu)$ coincides with the corresponding equation for a thermal field may be said to be a reflection of the dominance of phase fluctuations for this problem.

APPENDIX: THE RELATION BETWEEN SECOND-ORDER MOMENTS AND CORRELATIONS

The general relation of any order between moments and correlations of the number operator was recently obtained,[15] but the integral in the commutator given by Eqs. (2) was dropped prematurely. We show below that this term makes no contribution to the final relation, provided the linear dimensions of the volume δV of integration are large compared with the wavelength.

[38] L. Mandel, *Quantum Electronics, III* (Columbia University Press, New York, 1964), p. 101.
[39] H. Hodara (to be published).

Let N given by Eq. (3) be the number operator. Then, with the understanding that x stands for the space-time variables $\mathbf{x},t$ with t fixed, we may write

$$\langle N^2\rangle=\sum_i\sum_j\int\int_{\delta V}\langle A_i^\dagger(x)A_i(x)A_j^\dagger(x')A_j(x')\rangle d^3xd^3x'. \tag{A1}$$

We now apply the commutator given by Eqs. (2) to the inner operator product under the integral, and obtain

$$\begin{aligned}\langle N^2\rangle=&\sum_i\sum_j\int\int_{\delta V}\langle A_i^\dagger(x)A_j^\dagger(x')A_i(x)A_j(x')\rangle d^3xd^3x'+\sum_i\sum_j\int\int_{\delta V}\langle A_i^\dagger(x)A_j(x')\rangle\delta_{ij}\delta^3(x-x')d^3xd^3x'\\&-\sum_i\sum_j\int\int_{\delta V}\langle A_i^\dagger(x)A_j(x')\rangle\int\frac{1}{(2\pi)^3}\frac{k_ik_j}{\mathbf{k}^2}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]d^3kd^3xd^3x'\\=&\langle:N^2:\rangle+\langle N\rangle-\sum_i\sum_j\int\int_{\delta V}\langle A_i^\dagger(x)A_j(x')\rangle\int\frac{1}{(2\pi)^3}\frac{k_ik_j}{\mathbf{k}^2}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]d^3kd^3xd^3x',\end{aligned} \tag{A2}$$

where : : stands for normal ordering of operators. Except for the presence of the third term on the right-hand side, this is the relation used in Sec. 3 to derive Eqs. (10).

To show that the extra term vanishes, we make use of the Fourier series expansion (1) for $A_j(\mathbf{x},t)$ and substitute.[40] We then find

$$\begin{aligned}&\sum_i\sum_j\int\int_{\delta V}\langle A_i^\dagger(x)A_j(x')\rangle\int\frac{1}{(2\pi)^3}\frac{k_ik_j}{\mathbf{k}^2}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]d^3kd^3xd^3x'\\&=\frac{1}{(2\pi L)^3}\sum_i\sum_j\sum_{\mathbf{k}',s'}\sum_{\mathbf{k}'',s''}\int\int_{\delta V}\int\langle a_{\mathbf{k}',s'}^\dagger a_{\mathbf{k}'',s''}\rangle(\boldsymbol{\varepsilon}_{\mathbf{k}',s'}^*)_i(\boldsymbol{\varepsilon}_{\mathbf{k}'',s''})_j\frac{k_ik_j}{\mathbf{k}^2}\\&\qquad\times\exp i[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}+(\mathbf{k}''-\mathbf{k})\cdot\mathbf{x}']\exp ic[k't-k''t']d^3kd^3xd^3x'.\end{aligned}$$

If the linear dimensions of δV are large compared with any reciprocal wave number for which $\langle a_{\mathbf{k}',s'}^\dagger a_{\mathbf{k}'',s''}\rangle$ is nonzero, the integrals over $\mathbf{x}$ and $\mathbf{x}'$ lead to $\delta^3(\mathbf{k}-\mathbf{k}')\delta^3(\mathbf{k}''-\mathbf{k})$. We may therefore replace all $\mathbf{k}'$ and $\mathbf{k}''$ by $\mathbf{k}$ in the other terms under the integrals. On summing all terms that depend explicitly on i and j over i and j we obtain

$$\begin{aligned}\sum_i\sum_j(\boldsymbol{\varepsilon}_{\mathbf{k},s'}^*)_i(\boldsymbol{\varepsilon}_{\mathbf{k},s''})_jk_ik_j&=(\boldsymbol{\varepsilon}_{\mathbf{k},s'}^*\cdot\mathbf{k})(\boldsymbol{\varepsilon}_{\mathbf{k},s''}\cdot\mathbf{k})\\&=0,\end{aligned} \tag{A3}$$

since $\mathbf{k}$ and $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ are orthogonal. Equation (A2) is therefore correct even without the third term on the right-hand side. A similar argument may be used to eliminate extra terms in the expression for higher order moments.

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[40] I am indebted to Dr. C. L. Mehta for the suggestion of this step.