New Approach to the Theory of Fluctuations in a Plasma

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A rigorous but simple method is presented for calculating autocorrelation functions (fluctuations) of non-equilibrium plasmas—including inhomogeneous and nonstationary plasmas—in external fields. This method is based upon the derivation of an exact and remarkably simple formal relation between autocorrelation functions and the usual one-particle distribution function $f_1(\mathbf{R},\mathbf{v},t)$ for an explicitly defined initial value. This relation explicitly reduces the problem of calculating fluctuation spectra to the problem of solving the usual kinetic equations for $f_1(\mathbf{R},\mathbf{v},t)$. Consequently, the central quantity of fluctuation theory is one and the same with the central quantity of kinetic theory, and the two theories are completely and explicitly united. To first order in the plasma parameter it is shown that one need only solve the linearized Vlasov equation for $f_1(\mathbf{R},\mathbf{v},t)$, and when this solution is substituted into our formal relation we obtain a general formula for autocorrelation functions which is valid for nonstationary systems and includes the effects of the transverse motion of the plasma in addition to the longitudinal motion. For stationary systems this formula approaches the numerous calculations of previous authors in the limits where the transverse terms vanish. As a result of the transverse terms, the spectrum of scattered light can have resonant peaks at frequencies which correspond to transverse modes of oscillation as well as to the well-known longitudinal modes.

I. INTRODUCTION

It is generally recognized that space-time correlation functions (autocorrelation functions) provide a unified and powerful means for studying the properties of plasmas. The autocorrelations of particle density, current density, electric field density, and particle velocity are directly related to such quantities as light-scattering cross sections, conductivity tensors, diffusion coefficients, dielectric tensors, and radiation absorption and emission.[1] Several calculations of autocorrelation functions for plasmas have been made of varying degrees of rigor and complexity,[2–13] particularly for electron densities (scattering). An important and general theory, based on a test-particle problem, is developed in Ref. 6. This theory, like the others (except Ref. 13), is restricted to homogeneous and stationary systems, and—as discussed in Sec. IV—is not entirely valid when the velocity-distribution function is not Maxwellian. Due to the influence of Ref. 6 the test-particle and field-particle distribution functions have emerged as *fundamental quantities* in the theory of fluctuations. That is, autocorrelation functions are expressed in terms of certain functions (denoted by W_{11} and W_{12} in Ref. 6) related to test- and field-particle distribution functions. Equations are then derived, and solved, for W_{11} and W_{12} to a given order in the plasma parameter.

It is the purpose of this communication to present an alternative approach to the theory of fluctuations in which the usual[14] one-particle distribution function $f_1(\mathbf{R},\mathbf{v},t)$ appears as the fundamental quantity. This approach is based on the derivation of an exact and surprisingly simple relation between autocorrelation functions and $f_1(\mathbf{R},\mathbf{v},t)$ (this relation is a formal identity and is exact to all orders for all systems). As a consequence of this relation, the fundamental quantity of fluctuation theory is one and the same with the fundamental quantity of kinetic theory, and the two theories are completely and explicitly united. The present approach is rigorous, yet simple, and provides a basis for dealing with nonstationary as well as inhomogeneous plasmas.

The procedure we follow can be divided into two parts. The first part is to derive the formal relation between autocorrelation functions and $f_1(\mathbf{R},\mathbf{v},t)$. This will be found in Sec. II. The second part is to solve the usual kinetic equations for $f_1(\mathbf{R},\mathbf{v},t)$ in terms of its initial value.

It is shown that, to first order in the plasma parameter, we need only solve the linearized Vlasov equation, when the system is homogeneous. Since, however, the linearized Vlasov equation has been solved by many authors, we find that, to first order, the second part has already been done for us. [To higher orders in the plasma parameter we must ultimately solve the usual nonlinear kinetic equation for $f_1(\mathbf{R},\mathbf{v},t)$, with collision

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[14] The usual distribution function $f_1(\mathbf{R},\mathbf{v},t)$ is the distribution function for a system in which no particles are singled out. It satisfies the usual kinetic equation

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{R}}+\frac{q}{m}\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial f_1}{\partial \mathbf{v}}=\left(\frac{\delta f_1}{\delta t}\right)_c$$

[see Eq. (14) for nomenclature].

terms intact.] When the well-known solution of the linearized Vlasov equation for $f_1(\mathbf{R},\mathbf{v},t)$ is substituted into the formal relation of part one, we obtain a general formula for autocorrelation functions which is exact to first order, and which applies to a large class of correlation functions, including density, current, electric field and kinetic pressure tensor. This expression is an improvement of previous results for correlation functions in three respects: (1) it is valid for *nonstationary* as well as stationary systems; (2) it includes the effects of the transverse motion of the plasma (see also Sec. 5 of Ref. 13), and (3) the effects of drift motion and velocity anisotropy are calculated rigorously.

In Sec. IV we find that the terms arising from the transverse motion lead to an interesting, and quite understandable, consequence for the scattering of light from plasmas (density correlation function): Namely, that the frequency spectrum of scattered light can have resonant peaks at those frequencies which correspond to the transverse modes of a plasma (these are in addition to the well-known peaks which correspond to longitudinal modes). That is, previous results have the expression for the spectrum inversely proportional to the longitudinal dielectric constant ϵ_L. Resonant peaks thus occur at the zeros of ϵ_L and, hence, correspond to longitudinal modes. Our calculation, however, shows that the spectrum is inversely proportional to the plasma dispersion function $|\mathbf{R}|$, the zeros of which correspond to all the oscillatory modes, transverse as well as longitudinal, of a collisionless plasma. It follows that the spectrum can have resonant peaks associated with transverse (and mixed) modes as well as with longitudinal modes. The transverse motion is important for light scattering whenever it is coupled to the longitudinal motion—for example, by an external magnetic field.

II. FORMAL IDENTITY FOR AUTOCORRELATION FUNCTIONS

We consider fluctuations, and autocorrelation functions, of a nonequilibrium, partly ionized plasma in an external field. Let $\mathbf{R}_i(t)$ denote the position of particle i at time t, $\mathbf{v}_i(t)$ denote the velocity of particle i at time t, $\mathbf{x}_i(t)\equiv[\mathbf{R}_i(t),\mathbf{v}_i(t)]$ denote both the position and velocity of particle i at time t and $\mathbf{X}(t)\equiv[\mathbf{R}_1(t)\cdots\mathbf{R}_N(t), \mathbf{v}_1(t)\cdots\mathbf{v}_N(t)]$ denote the positions and velocities of all N particles of the plasma at time t. In addition, let $M(\mathbf{R},t)$ denote a macroscopic quantity (such as the particle density and current density) at point $\mathbf{R}$ and time t. We restrict our discussion to fluctuations of those macroscopic quantities which can be written in the form

$$M(\mathbf{R},t)\equiv\sum_{j=1}^{N} M_j(\mathbf{R},x_j(t))\equiv\sum_{j=1}^{N} M_j(\mathbf{R},t). \qquad (1)$$

[For example, if $M_j(\mathbf{R},t)\equiv\delta(\mathbf{R}-\mathbf{R}_j(t))$, $\mathbf{v}_j(t)\delta(\mathbf{R}-\mathbf{R}_j(t))$, or $\mathbf{v}_j(t)\mathbf{v}_j(t)\delta(\mathbf{R}-\mathbf{R}_j(t))$, then $M(\mathbf{R},t)$ is the particle density, current density, or kinetic pressure tensor, respectively.] If the macroscopic state of the system is defined by an ensemble of systems $\langle\ \rangle$ with distribution function $F_N(\{\mathbf{X}\},t)$ then the autocorrelation function of $M(\mathbf{R},t)$, denoted by $C(M)$, is defined by

$$\begin{aligned}C(M)&\equiv\langle M(\mathbf{R}+\mathbf{R}', t+\tau)M(\mathbf{R},t)\rangle,\\ &\equiv\int d\{\mathbf{X}\}M(\mathbf{R}+\mathbf{R}', t+\tau)M(\mathbf{R},t)F_N(\{\mathbf{X}\},0),\\ &\equiv\int d\{\mathbf{X}\}M(\mathbf{R}+\mathbf{R}', 0)M(\mathbf{R}, -\tau)\\ &\qquad\times F_N(\{\mathbf{X}\}, t+\tau), \qquad (2)\end{aligned}$$

where $\{\mathbf{X}\}\equiv(\mathbf{R}_1\cdots\mathbf{R}_N,\mathbf{v}_1\cdots\mathbf{v}_N)$ denotes the *initial* ($t=0$) phase (initial positions and velocities) of all N particles, and $F_N(\{\mathbf{X}\}, t+\tau)$ is the N-particle distribution function of the system at time $t+\tau$ normalized to unity. It is seen from (2) that the autocorrelation function $C(M)$ depends upon the state of the system (depends upon F_N). We shall find, however, that $C(M)$ depends only upon F_1. The problem here, as in Ref. 6, is to calculate $C(M)$ in terms of F_1. [If the system is in equilibrium, then F_N is simply the thermal distribution function $Ae^{-H/KT}$, where H is the Hamiltonian of the system, T is the temperature, and A is a normalization factor. In this case F_1 is just the Maxwell velocity distribution,

$$F_1=m_1(2\pi KT)^{-3/2}\exp[-m_1v_1^2/2KT].$$

If the system is in a stationary state, then F_N, F_1, and $C(M)$ are independent of the time t. For a nonstationary system, however, F_N and F_1 as well as $C(M)$ will depend upon the time t as well as the time interval τ. In Ref. 6 F_N (denoted there by D_1), F_1 and $C(M)$ are eventually assumed to be independent of t (stationary assumption).]

We now note that if $G(\{\mathbf{X}(t)\})$ denotes any function of the positions and velocities of the particles at time t then G necessarily satisfies the equation of motion[15]:

$$dG(\{\mathbf{X}(t)\})/dt=iLG(\{\mathbf{X}(t)\}), \qquad (3)$$

where L is the Liouville operator of our plasma defined by the Poisson bracket with the Hamilton H (that is, $LG\equiv-i[H,G]$). The formal solution of Eq. (3) in terms of the initial value of G is given by

$$G(\{\mathbf{X}(t)\})=e^{itL}G(\{\mathbf{X}\}). \qquad (4)$$

Substituting (4) and (1) into (2), and making use of

$$F_N(\{\mathbf{X}\}, t+\tau)\equiv F_N(\{\mathbf{X}(-\tau)\}, t)\equiv e^{-i\tau L}F_N(\{\mathbf{X}\},t)$$

we obtain

$$\begin{aligned}C(M)&=\int d\{\mathbf{X}\}M(\mathbf{R}+\mathbf{R}', 0)e^{-i\tau L}[M(\mathbf{R},0)F_N(\{\mathbf{X}\},t)]\\ &\equiv\int d\{\mathbf{X}\}M(\mathbf{R}+\mathbf{R}', 0)f_N(\{\mathbf{X}\},\tau), \qquad (5)\end{aligned}$$

[15] Herbert Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), Chap. 8, Paragraph 5, Eq. (8–58); J. Weinstock, Phys. Rev. **132**, 454 (1963).

where we have defined $f_N(\{\mathbf{X}\},\tau)$ by

$$f_N(\{\mathbf{X}\},\tau)\equiv e^{-i\tau L}[M(\mathbf{R},0)F_N(\{\mathbf{X}\},t)]$$
$$(\equiv e^{-i\tau L}[\sum_{j=1}^{N} M_j(\mathbf{R},x_j)F_N(\{\mathbf{X}\},t)]). \quad (6)$$

Clearly, $f_N(\{\mathbf{X}\},\tau)$ satisfies Liouville's equation

$$\partial f_N/\partial\tau=-iLf_N$$

and has the initial value (value at $\tau=0$)

$$f_N(\{\mathbf{X}\},0)\equiv M(\mathbf{R},0)F_N(\{\mathbf{X}\},t). \quad (7)$$

This means that $f_N(\{\mathbf{X}\},\tau)$ is the distribution function that our system would have at time τ if its initial value were $M(\mathbf{R},0)F_N(\{\mathbf{X}\},t)$.[16] [The distribution function f_N depends upon $\mathbf{R}$, and t through the initial value. These can be considered parameters in the same sense that the temperature and density are considered parameters, and for convenience of notation we shall not always explicitly state this parametric dependence. Thus, we shall use the notation

$$f_N(\tau)\equiv f_N(\{\mathbf{X}\},\tau)\equiv f_N(\{\mathbf{X}\},\tau|\mathbf{R},t).$$

It is also to be noted that $f_N(\{\mathbf{X}\},\tau)$ is not normalized to unity.]

If we let

$$\int d\{\mathbf{X}\}^j$$

denote the integral over the positions and velocities of all particles except particle j, so that $\int d\{\mathbf{X}\}\equiv\int dx_j \times\int d\{\mathbf{X}\}^j$, then, with (1), we find that (5) can be written as

$$C(M)=\sum_{j=1}^{N}\int d\mathbf{x}_j M_j(\mathbf{R}+\mathbf{R}',\mathbf{x}_j)\int d\{\mathbf{X}\}^j f_N(\{\mathbf{X}\},\tau). \quad (8)$$

But

$$\int d\{\mathbf{X}\}^j f_N(\{\mathbf{X}\},\tau)$$
$$\equiv V^{-1}f_\alpha(\mathbf{x}_j,\tau)$$
$$\left(\equiv\int d\{\mathbf{X}\}^j e^{-i\tau L}[M(\mathbf{R},0)F_N(\{\mathbf{X}\},t)]\right), \quad (9)$$

where $f_\alpha(\mathbf{x}_j,\tau)$ is simply the one-particle distribution function (the probability density in μ space), and α refers to the species of particle j (electron, ion, or neutral). The important point here is that $f_N(\{\mathbf{X}\},\tau)$ is the exact solution of Liouville's equation and, consequently, $f_\alpha(\mathbf{x},\tau)$ must be the exact solution of the kinetic equation, to all orders in the plasma parameter. (There have been many derivations of approximate kinetic equations, usually in the form of an expansion in powers of the plasma parameter, which are familiar to all. To lowest order the kinetic equation is simply the well-known Vlasov equation. If, however, the kinetic equation could be derived exactly to all orders then f_α would be the solution.)

Substituting (9) into (8), and denoting the total number of particles of species α in the system by N_α, we finally obtain the desired formal identity

$$C(M)=\sum_\alpha\frac{N_\alpha}{V}\int d\mathbf{x}_1 M_1(\mathbf{R}+\mathbf{R}',\mathbf{x}_1)f_\alpha(\mathbf{x}_1,\tau). \quad (10)$$

Equation (10) is an exact expression for the autocorrelation function of M in terms of the one-particle distribution function $f_\alpha(\mathbf{x}_1,\tau)$. This equation thus *exactly* reduces the problem of calculating autocorrelation functions to the very familiar problem of solving the kinetic equations for f_α. It holds for arbitrary states of the system, including inhomogeneous and nonstationary states, with or without magnetic fields. The distribution function f_α is determined by the kinetic equations in terms of its initial value $f_\alpha(\mathbf{x}_1,0)$, and $f_\alpha(\mathbf{x}_1,0)$ is immediately given, from (9) and (6) or (7), by

$$f_\alpha(\mathbf{x}_1,0)\equiv V\sum_{j=1}^{N}\int d\{\mathbf{X}\}^1 M_j(\mathbf{R},\mathbf{x}_j)F_N(\{\mathbf{X}\},t)$$
$$=M_1(\mathbf{R},\mathbf{x}_1)F_\alpha(\mathbf{x}_1,t)$$
$$+\sum_\beta\left(\frac{N_\beta}{V}\right)\int d\mathbf{x}_2 M_2(\mathbf{R},\mathbf{x}_2)F_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2,t);$$
$$F_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2,t)\equiv F_\alpha(\mathbf{x}_1,t)F_\beta(\mathbf{x}_2,t)+P_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2,t), \quad (11)$$

where $F_\alpha(\mathbf{x}_1,t)$ is the *actual* one-particle distribution function of species α in our system at time t, $F_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2,t)$ is the two-particle distribution function of species α and β in our system at time t, and $P_{\alpha\beta}$ is the pair-*correlation* function of our system at time t. The distribution functions F_α describes the state, at time t, of the system whose fluctuations we are considering and is assumed to be known (the pair function $P_{\alpha\beta}$ can be calculated in terms of F_α and F_β). If, for example, the system is in equilibrium then F_α and $F_{\alpha\beta}$ are simply given by[17]

$$F_\alpha(\mathbf{x}_1,t)=F_\alpha{}^0(\mathbf{v}_1)\equiv\left(\frac{m_\alpha}{2\pi KT}\right)^{3/2}\exp\left[-\frac{m_\alpha v_1{}^2}{2KT}\right];$$
$$F_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2,t)=F_\alpha{}^0(\mathbf{v}_1)F_\beta{}^0(\mathbf{v}_1)g_{\alpha\beta}(|\mathbf{R}_1-\mathbf{R}_2|)$$
$$\equiv F_\alpha{}^0(\mathbf{v}_1)F_\beta{}^0(\mathbf{v}_1)+P_{\alpha\beta}(\mathbf{v}_1,\mathbf{v}_2,|\mathbf{R}_1-\mathbf{R}_2|), \quad (12)$$

[16] The distribution function $f_N(\{\mathbf{X}\},\tau)$ must not be confused with the distribution function $F_N(\{\mathbf{X}\},t)$. They are distribution functions for two different states (different initial values) of the same system. (It is necessary to specify both the Hamiltonian of a system and the initial state of the system in order to uniquely determine the distribution function.) Thus, $f_N(\{\mathbf{X}\},\tau)$ is the distribution function that our system *would* have at time τ if its initial value (initial state) had been $f_N(\{\mathbf{X}\},0)=M(R,0)F_N(\{\mathbf{X}\},t)$; whereas $F_N(\{\mathbf{X}\},t)$ is the distribution function that our system actually has at time t, and is assumed to be known—that is, $F_1(\mathbf{x},t)$ is assumed to be known. (It is assumed we know F_1 in the sense that the fluctuations and autocorrelation function of a system obviously depend upon the state F_1 of the system and, hence, must be calculated in terms of the state of the system. We shall calculate autocorrelation functions for arbitrary states of the system—arbitrary F_1.)

[17] As another example, we note that if the system is oscillating about equilibrium (in a normal mode) then $F_\alpha(\mathbf{x}_1,t)$ might be given by

$$F_\alpha{}^0(\mathbf{v}_1)[1+Ae^{i(\mathbf{k}\cdot\mathbf{R}+\omega t)}],$$

where $\mathbf{k}$, ω, and A are wave vector, frequency and amplitude, respectively, of the normal mode.

where $g_{\alpha\beta}$ is the equilibrium pair-distribution function, and $P_{\alpha\beta}$ is the familiar equilibrium pair-*correlation* function. Hence, the initial value of f_α in the equilibrium case is

$$f_\alpha(\mathbf{x}_1,0)=M_1(\mathbf{R},\mathbf{x}_1)F_\alpha{}^0(\mathbf{v}_1)+\sum_\beta \frac{N_\beta}{V}\int d\mathbf{x}_2 M_2(\mathbf{R},\mathbf{x}_2)[F_\alpha{}^0F_\beta{}^0+P_{\alpha\beta}]. \quad (13)$$

III. FORMULA FOR $C(M)$ IN TERMS OF F_α

The remaining step in the calculation of $C(M)$ from (10) is to substitute into (10) the solution of the kinetic equation for $f_\alpha(\mathbf{x}_1,\tau)$, in terms of the initial value in (11), which we shall now do to lowest order. The kinetic equation is, of course, given by

$$\frac{\partial f_\alpha}{\partial\tau}+\mathbf{v}_1\cdot\frac{\partial f_\alpha}{\partial\mathbf{R}_1}+\frac{q_\alpha}{m_\alpha}\left(\mathbf{E}+\frac{\mathbf{v}_1}{c}\times\mathbf{B}\right)\cdot\frac{\partial f_\alpha}{\partial\mathbf{v}_1}=\left(\frac{\delta f_\alpha}{\delta\tau}\right)_c, \quad (14)$$

where $\mathbf{E}$ is the self-consistent electric field, $\mathbf{B}$ is the total magnetic field (external magnetic field $\mathbf{B}_0$ plus the self-consistent magnetic field), q_α is the charge (including sign) of species α, m_α is the mass of species α, and $(\delta f_\alpha/\delta\tau)_c$ is the change in f_α due to collisions (the collision term). To lowest order, however, we can neglect the collision term. Furthermore, it is shown in the Appendix that, to lowest order, Eq. (14) can be linearized when the system is homogeneous. That is, if we write

$$f_\alpha(\mathbf{x}_1,\tau)=f_\alpha{}^0(\mathbf{v}_1)+{}^Pf_\alpha(\mathbf{x}_1,\tau), \quad (15)$$

where $f_\alpha{}^0(\mathbf{v}_1)$ is the velocity distribution [see Eq. (11)] given by

$$f_\alpha{}^0(\mathbf{v}_1)\equiv F_\alpha(\mathbf{v}_1,t)\sum_\beta\frac{N_\beta}{V}\int d\mathbf{x}_2M_2(\mathbf{R},\mathbf{x}_2)F_\beta(\mathbf{v}_2,t), \quad (16)$$

then ${}^Pf_\alpha(x_1,\tau)$ can be considered a *small perturbation* whose initial value, determined from Eqs. (11), (15), and (16), is

$${}^Pf_\alpha(\mathbf{x}_1,0)=F_\alpha(\mathbf{v}_1,t)M_1(\mathbf{R},\mathbf{x}_1)+\sum_\beta\frac{N_\beta}{V}\int d\mathbf{x}_2M_2(\mathbf{R},\mathbf{x}_2)P_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2). \quad (17)$$

In other words $f_\alpha(\mathbf{x}_1,\tau)$ can, to lowest order, be linearized about $f_\alpha{}^0(\mathbf{v}_1)$. (It is interesting to note that similar linearizations have been used in previous theories with little or no attenuation drawn to the place where they are used or to their justification. To higher orders in the plasma parameter we must solve Eq. (14) with collision term, and nonlinearity, intact.) The Fourier transform of ${}^Pf_\alpha(\mathbf{x}_1,0)$, which will be required soon, is given by

$$\begin{aligned}{}^*f_\alpha(\mathbf{k},\mathbf{v}_1,0)&\equiv\int d\mathbf{R}_1e^{-i\mathbf{k}\cdot\mathbf{R}_1}\,{}^Pf_\alpha(\mathbf{x}_1,0)\\&=F_\alpha(\mathbf{v}_1,t){}^*M_1(\mathbf{R},\mathbf{k})+\sum_\beta\frac{N_\beta}{V}\int d\mathbf{v}_2{}^*M_2(\mathbf{R},\mathbf{k}){}^*P_{\alpha\beta}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{k}), \quad (18)\end{aligned}$$

where ${}^*M$ and ${}^*P_{\alpha\beta}$ denote the Fourier transforms of M and $P_{\alpha\beta}$.

The solution of the linearized Eq. (14) for ${}^Pf_\alpha(\mathbf{x}_1,\tau)$ has been obtained by many investigators. We shall use Bernstein's[18] Fourier-Laplace solution to obtain (in Bernstein's notation)

$${}^Pf_\alpha(\mathbf{x}_1,\tau)\equiv\int\frac{d\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{R}_1}\oint\frac{ds}{2\pi i}e^{s\tau}\,{}^Pf_\alpha(\mathbf{k},\mathbf{v}_1,s),$$

$${}^Pf_\alpha(\mathbf{k},\mathbf{v}_1,s)=-\int_{\pm\infty}^{\phi}d\phi'\frac{G_\alpha}{\Omega_\alpha}\left[{}^*f_\alpha(\mathbf{k},\mathbf{v}_1{}',0)-\frac{q_\alpha}{m_\alpha}\frac{\mathbf{R}^A\mathbf{a}}{|\mathbf{R}|}\cdot\left\{\frac{\partial f_\alpha{}^0(\mathbf{v}_1{}')}{\partial\mathbf{v}_1{}'}+\frac{i}{s}\left(\frac{\partial f_\alpha{}^0(\mathbf{v}_1{}')}{\partial\mathbf{v}_1{}'}\times\mathbf{v}_1{}'\right)\times\mathbf{k}\right\}\right], \quad (19)$$

where

$$\mathbf{R}\equiv(s^2+c^2k^2)\mathbf{I}-c^2\mathbf{k}\mathbf{k}-s\sum_\alpha\frac{4\pi q_\alpha{}^2N_\alpha}{m_\alpha V}\int d\mathbf{v}\int_{\pm\infty}^{\phi}d\phi'\frac{G_\alpha}{\Omega_\alpha}\mathbf{v}\frac{\partial F_\alpha(\mathbf{v}')}{\partial\mathbf{v}'},$$

$$\mathbf{a}\equiv\sum_\alpha\frac{4\pi q_\alpha N_\alpha}{\sum_\beta N_\beta\int d\mathbf{x}_2M_2F_\beta(\mathbf{v}_2)}\int d\mathbf{v}\left[\left(\mathbf{v}-\frac{is\mathbf{k}}{k^2}\right){}^*f_\alpha(\mathbf{v})+s\mathbf{v}\int_{\pm\infty}^{\phi}d\phi'\frac{G_\alpha}{\Omega_\alpha}{}^*f_\alpha(\mathbf{v}')\right],$$

$\mathbf{R}^A\equiv$adjoint of $\mathbf{R}\equiv\mathbf{R}^{-1}|\mathbf{R}|$,

$$G_\alpha\equiv\exp\left[\frac{s+iku\cos\theta}{\Omega_\alpha}(\varphi-\varphi')+\frac{ik\omega\sin\theta}{\Omega_\alpha}(\sin\varphi-\sin\varphi')\right],$$

$\Omega_\alpha\equiv q_\alpha B_0/(m_\alpha c)$ (gyrofrequency),

$\mathbf{v}\equiv\mathbf{e}_1\omega\cos\varphi+\mathbf{e}_2\omega\sin\varphi+\mathbf{e}_3u$,

$\mathbf{v}'\equiv\mathbf{e}_1\omega\cos\varphi'+\mathbf{e}_2\omega\sin\varphi'+\mathbf{e}_3u$,

$\mathbf{k}\equiv\mathbf{e}_1k\sin\varphi+\mathbf{e}_3k\cos\varphi$.

Here $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a rectangular basis of unit vectors, $\mathbf{e}_3$ is in the direction of $\mathbf{B}_0$, and θ is the angle between $\mathbf{B}_0$ and $\mathbf{k}$. [The determinant of $\mathbf{R}$, denoted by $|\mathbf{R}|$, is the *dispersion function* for a collisionless plasma. The zeroes of $|\mathbf{R}|$ correspond to the normal modes of a plasma, and have been discussed in detail in Ref. 18. The terms $(\partial f_\alpha{}^0/\partial\mathbf{v})\times\mathbf{v}\times\mathbf{k}$ and $\int d\mathbf{v}\,\mathbf{v}{}^*f_\alpha(\mathbf{v})$ are important when the "temperature" of species α is different in different directions or when there is a drift velocity.]

A general formula for autocorrelation functions is obtained, to first order in the plasma parameter, by combining (15), (19), and (10):

$$C(M)=\sum_\alpha\frac{N_\alpha}{V}\int d\mathbf{x}_1M_1(\mathbf{R},\mathbf{x}_1)[f_\alpha{}^0(\mathbf{v}_1)+{}^Pf_\alpha(\mathbf{x}_1,\tau)], \quad (20)$$

[18] I. B. Bernstein, Phys. Rev. 109, 10 (1958). It will be noted that we have generalized Bernstein's work to include non-Maxwellian velocity-distribution functions.

where ${}^{P}f_\alpha(\mathbf{x}_1,\tau)$ is explicitly given by (19). Equations (20) and (19) for autocorrelation functions are valid for nonequilibrium homogeneous plasmas in the presence of a uniform field $\mathbf{B}_0$. They remain valid even when the plasma is not in a stationary state. In such an event $F_\alpha(\mathbf{v}_1)$ changes with the initial time t and, hence, $C(M)$ will change with t, as well as with the time difference τ. To numerically evaluate the autocorrelation function of any quantity (density, electric field, current, etc.) one need only substitute the corresponding M into (20) and (19) and carry through the indicated integrations—by computer, if necessary. The autocorrelation function is considerably simpler when there is no external magnetic field and $\mathbf{v}\mathbf{x}(\partial f_\alpha{}^0/\partial\mathbf{v})=0$, since then ${}^{P}f_\alpha(\mathbf{k},\mathbf{v}_1,s)$ reduces to

$$
\begin{aligned}
{}^{P}f_\alpha &= \frac{{}^{*}f_\alpha(\mathbf{k},\mathbf{v}_1,0)}{s+i\mathbf{k}\cdot\mathbf{v}_1}+\frac{q_\alpha}{m_\alpha}\sum_\beta \frac{U_\beta}{\epsilon_L}\mathbf{k}\cdot\frac{\partial F_\alpha{}^0/\partial\mathbf{v}_1}{s+i\mathbf{k}\cdot\mathbf{v}_1}\,,\\
\epsilon_L &\equiv 1-\frac{4\pi i}{k^2}\sum_\alpha \frac{q_\alpha{}^2N_\alpha}{m_\alpha V}\int d\mathbf{v}\frac{k\cdot\partial F_\alpha{}^0/\partial\mathbf{v}}{s+i\mathbf{k}\cdot\mathbf{v}}\,,\qquad (21)\\
U_\beta &\equiv \frac{i4\pi q_\beta N_\beta}{k^2 V}\int d\mathbf{v}\frac{{}^{*}f_\beta(\mathbf{k},\mathbf{v},0)}{s+i\mathbf{k}\cdot\mathbf{v}}\,.
\end{aligned}
$$

IV. EXAMPLE: LIGHT SCATTERING

As an example, let us use (20) to write the spectral distribution of electron-density fluctuations. This quantity, denoted by $S(\mathbf{k},\omega)$, is proportional to the cross section of light scattered from a plasma (in the first Born approximation) and is defined by

$$
S(\mathbf{k},\omega)\equiv\int d\mathbf{R}'e^{-i\mathbf{k}\cdot\mathbf{R}'}\times\int d\tau e^{-i\omega\tau}\langle M(\mathbf{R}+\mathbf{R}',\,t+\tau)M(\mathbf{R},t)\rangle. \qquad (22)
$$

Here, $M(\mathbf{R},t)$ denotes the electron density

$$
M(\mathbf{R},t)\equiv\sum_{j=1}^{Ne}\delta(\mathbf{R}-\mathbf{R}_j(t))\,,\quad M_j(\mathbf{R},\mathbf{x}_j)\equiv\delta(\mathbf{R}-\mathbf{R}_j) \qquad (23)
$$

(the index j refers to electrons only). It is more convenient to consider the related Laplace transform spectrum $S^+(\mathbf{k},s)$ defined by

$$
S^+(\mathbf{k},s)\equiv\int d\mathbf{R}'e^{-i\mathbf{k}\cdot\mathbf{R}'}\times\int d\tau e^{-s\tau}\langle M(\mathbf{R}+\mathbf{R}',\,t+\tau)M(\mathbf{R},t)\rangle\,, \qquad (24)
$$

from which $S(\mathbf{k},\omega)$ can be obtained by the simple relation

$$
S(\mathbf{k},\omega)=2\,\mathrm{Re}S^+(\mathbf{k},i\omega)\,. \qquad (25)
$$

Substituting (20), (19), and (23) into (24), and taking advantage of the spatial homogeneity to set $\mathbf{R}$ equal to zero, we obtain

$$
\begin{aligned}
S^+(\mathbf{k},s) &= \int d\mathbf{R}'e^{-i\mathbf{k}\cdot\mathbf{R}'}\int d\tau e^{-s\tau}\int d\mathbf{x}_1\\
&\quad\times\delta(\mathbf{R}'-\mathbf{R}_1)[f_e{}^0(\mathbf{v}_1)+{}^{P}f_e(\mathbf{x}_1,\tau)]\\
&= \int d\mathbf{v}_1\int d\mathbf{R}_1e^{-i\mathbf{k}\cdot\mathbf{R}_1}\int d\tau e^{-s\tau}[f_e{}^0(\mathbf{v}_1)+{}^{P}f_e(\mathbf{x}_1,\tau)]\\
&= \frac{(2\pi)^3\delta(\mathbf{k})(N_e/V)}{s}+\int d\mathbf{v}_1\,{}^{P}f_e(\mathbf{k},\mathbf{v}_1,s)\\
&= \frac{(2\pi)^3\delta(\mathbf{k})(N_e/V)}{s}\\
&\quad-\int d\mathbf{v}_1\int_{-\infty}^{\phi}d\phi'\frac{G_e}{\Omega_e}\Bigg[{}^{*}f_e(\mathbf{k},\mathbf{v}_1,0)+\frac{e}{m_e}\frac{\mathbf{R}^A\mathbf{a}}{|\mathbf{R}|}\\
&\quad\cdot\left\{\frac{\partial f_e{}^0}{\partial\mathbf{v}_1'}+\frac{i}{s}\left(\frac{\partial f_e{}^0}{\partial\mathbf{v}_1'}\mathbf{x}\mathbf{v}_1'\right)\mathbf{x}\mathbf{k}\right\}\Bigg]\,, \qquad (26)
\end{aligned}
$$

where, from (16), (18), and (23),

$$
\begin{aligned}
f_e{}^0(\mathbf{v}) &\equiv \frac{N_e}{V}F_e(\mathbf{v},t)\,,\\
{}^{*}f_e(\mathbf{k},\mathbf{v}_1,0) &\equiv F_e(\mathbf{v}_1,t)+\frac{N_e}{V}\int d\mathbf{v}_2\,{}^{*}P_{ee}\,,\qquad (27)\\
{}^{*}f_\alpha(\mathbf{k},\mathbf{v}_1,0) &\equiv \frac{N_\alpha}{V}\int d\mathbf{v}_2\,{}^{*}P_{\alpha e}\,,\quad (\alpha\neq e)\,,
\end{aligned}
$$

and $\mathbf{R}$ and $\mathbf{a}$ are defined by (19).

Equation (26) provides an expression for the light scattered from a homogeneous nonequilibrium plasma in a uniform magnetic field which is exact to first order in the plasma parameter. Furthermore, it accounts for the effects of nonstationary [the change of $F_e(\mathbf{v}_1,t)$ and (N_e/V) with time t] upon the spectrum of scattered light. Hence, when the system is changing with time Eq. (26) simply states that the spectrum of scattered light depends on F_e and (N_e/V) at the time at which the scattering occurs. If the scattering occurs over a time interval Δt, then Eq. (26) must be integrated over Δt. Previous results can be seen to be special cases of (26). That is, aside from notation, Eq. (26) reduces to previous results (for stationary plasmas) in the asymptotic limits where

$$
s/k\ll c\ \text{(speed of light)} \qquad (28)
$$

and

$$
\int d\mathbf{v}\frac{G_e}{\Omega_e}\left(\frac{\partial f_e{}^0}{\partial\mathbf{v}}\times\mathbf{v}\right)\times\frac{\mathbf{k}}{s}\ll\int d\mathbf{v}\frac{G_e}{\Omega_e}\frac{\partial f_e{}^0}{\partial\mathbf{v}}\,. \qquad (29)
$$

Under the limiting conditions of (28) and (29) $\mathbf{R}^A\mathbf{a}/|\mathbf{R}|$ reduces to

$$\frac{\mathbf{R}^A\mathbf{a}}{|\mathbf{R}|}=i\sum_\alpha \frac{4\pi q_\alpha \mathbf{k}}{\epsilon_L k^2}\int d\mathbf{v}\int_{\pm\infty}^{\phi} d\phi' \frac{G_\alpha}{\Omega_\alpha}{}^* f_\alpha, \qquad (30)$$

where ϵ_L is the *longitudinal dielectric constant:*

$$\epsilon_L \equiv 1+i\sum_\alpha \frac{4\pi q_\alpha^2 N_\alpha}{m_\alpha k^2 V}\int d\mathbf{v}\int_{\pm\infty}^{\phi} d\phi' \frac{G_\alpha}{\Omega_\alpha}\mathbf{k}\cdot\frac{\partial F_\alpha}{\partial \mathbf{v}_1'}.$$

The expression which results from substituting (30) into (26) and neglecting $[(\partial f_e^0/\partial \mathbf{v}_1')\times \mathbf{v}_1']\times \mathbf{k}$ is, aside from notation, the same as the combined Eq. (13) and (42) of Ref. 7 (this can be proven by expanding G_α as an infinite sum of Bessel functions and then carrying through the φ' integration for each term of the sum.) {The quantity $[(\partial f_e^0/\partial \mathbf{v}_1')\times \mathbf{v}_1']\times \mathbf{k}$ in (26) is important when the plasma is not in equilibrium. Equation (13) of Ref. 7 is missing this quantity and is not strictly valid for nonequilibrium systems unless s/k is large enough. The source of this oversight can be traced back to the solution of the test-particle problem in Sec. 2 [Eq. (12)] of Ref. 6 in which the Lorentz force term $\mathbf{v}\times\mathbf{B}\cdot(\partial f_e^0/\partial \mathbf{v})$ is neglected.}

An interesting result that can be immediately deduced from Eq. (26) is that the spectrum of scattered light $S(\mathbf{k},\omega)$ can have resonant peaks at all frequencies which correspond to the normal modes of a plasma (transverse modes as well as longitudinal modes). This is a consequence of the fact that $S(\mathbf{k},\omega)$ is proportional to $|\mathbf{R}|^{-1}$ and $|\mathbf{R}|$ vanishes at all frequencies which correspond to a normal mode of a collisionless plasma. That is, $|\mathbf{R}|=0$ is the dispersion relation of a collisionless plasma. The spectral distribution calculated in Ref. 7, on the other hand, is proportional to ϵ_L^{-1} (rather than to $|\mathbf{R}|^{-1}$) and, hence, predicts resonant peaks at only the purely longitudinal modes ($\epsilon_L=0$). The results of Ref. 7 are limited by the so-called electrostatic or longitudinal approximation in which it is assumed that the longitudinal motion of the plasma is not coupled to the transverse motion. This assumption, however, is not always valid in the presence of magnetic fields.[19,20,21,18] When such coupling does exist the correct spectrum is given by (26) and the resonant peaks are given by the zeros of $|\mathbf{R}|$, instead of the zeros of ϵ_L. The conditions under which transverse coupling affects the resonant peaks of scattered light will be discussed in a future communication together with a discussion of the effects of spatial inhomogeneities.

[19] J. F. Denisse and J. L. Delcroix, *Plasma Waves* (Interscience Publishers, Inc., New York, 1963).

[20] W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasma* (Technology Press, Cambridge, Massachusetts, 1963).

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APPENDIX

We wish to show that, to lowest order in the plasma parameter, we need only consider the linearized Vlasov equation for $f_\alpha(\mathbf{x}_1,\tau)$ when the system is homogeneous. Consider first the initial value of f_α which, from Eq. (11), can be written as

$$f_\alpha(\mathbf{x}_1,0)=f_\alpha^0(\mathbf{v}_1)+{}^Pf_\alpha(\mathbf{x}_1,0), \qquad (A1)$$

where we have defined

$$f_\alpha^0(\mathbf{v}_1)\equiv F_\alpha(\mathbf{v}_1,t)\sum_\beta \frac{N_\beta}{V}\int d\mathbf{x}_2 M_2(\mathbf{R},\mathbf{x}_2)F_\beta(\mathbf{v}_2,t) \qquad (A2)$$

$${}^Pf_\alpha(\mathbf{x}_1,0)\equiv F_\alpha(\mathbf{v}_1,t)M_1(\mathbf{R},t)+\sum_\beta \frac{N_\beta}{V}\int d\mathbf{x}_2 M_2(\mathbf{R},\mathbf{x}_2)P_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2),$$

and we have used the fact that for homogeneous systems

$$F_{\alpha\beta}=F_\alpha(\mathbf{v}_1,t)F_\beta(\mathbf{v}_1,t)+P_{\alpha\beta}(\mathbf{x}_1,\mathbf{x}_2). \qquad (A3)$$

If, however, the plasma parameter is very small then

$$P_{\alpha\beta}\ll F_\alpha F_\beta,$$

and, since the system is very large,

$$M_1(\mathbf{R}_1 t)\ll\sum_\beta \frac{N_\beta}{V}\int d\mathbf{x}_2 M_2(\mathbf{R},\mathbf{x}_2)F_\beta(\mathbf{v}_2,t)$$

so that we consequently have

$${}^Pf_\alpha(\mathbf{x}_1,0)\ll f_\alpha^0(\mathbf{v}_1). \qquad (A4)$$

Equation (A4) states that, initially, ${}^Pf_\alpha$ is much smaller than f_α^0. If, however, there are no instabilities then ${}^Pf_\alpha$ will remain smaller than f_α^0 for all time:

$${}^Pf_\alpha(\mathbf{x}_1,\tau)\ll f_\alpha^0(\mathbf{v}_1). \qquad (A5)$$

It follows from Eqs. (A5) and (A1) that we can linearize $f_\alpha(\mathbf{x}_1,\tau)$ about the "unperturbed" quantity f_α^0 and regard ${}^Pf_\alpha$ as a small perturbation. If there is an instability, (A5) still holds for small enough τ, and the smaller the plasma parameter, the larger τ can be permitted to become.