Electrostatic Resonance Oscillations of a Nonuniform Hot Plasma in an External Field

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The frequency spectrum of a hydrodynamic model of a finite, warm, nonuniform plasma in an arbitrary external electric or magnetic field is considered. We find that the spectrum is real and the system stable, for an arbitrary configuration. A variational principle is given for estimating the eigenfrequencies. First-order perturbation theory is applied to a cylindrical plasma, and formulas are obtained for the first-order correction to the eigenfrequencies (resonances) for the case of an applied magnetic field or transverse electric field, arbitrary electron density $n_{00}(r)$, and arbitrary angular dependence $e^{i\mu\theta}$ ($\mu=0, \pm1, \pm2, \cdots$), the effect of the applied fields on the zero-order electron density being included. We find that for $\mu\neq0$, the modes have a twofold degeneracy, and that a uniform axial magnetic field splits the resonances in two. The first-order correction to the resonances is found to vanish for a uniform transverse electric or magnetic field. These results are discussed relative to other models and to experiment, and appear to be in agreement with the available experimental data for the behavior of the main dipole resonance in both transverse and axial magnetic fields.

I. INTRODUCTION

THE dipole resonances of a cylindrical plasma column have been studied both experimentally and theoretically for many years.[1] Cold-plasma theory for a uniform column predicts precisely one resonance,[2] whereas several are observed.[3] Gould[4] proposed that these resonances might be consequences of the nonzero electron temperature. The application of linearized hydrodynamic equations including a scalar electron pressure to a uniform column led to a series of dipole resonances and achieved a qualitative explanation of the experimental observations.[4] This same hydrodynamic description gave excellent quantitative agreement with experiment, when the nonuniformity of the electron density was considered.[5]

Attention has recently been focused on the behavior of the resonances in the presence of an applied magnetic field.[1,3,6–9] Measurements have been made for cylindrical plasmas in uniform axial and transverse magnetic fields.[1,3,7] The theoretical work has, with the exception of Ref. 9, been confined to cold-plasma models which are limited to a study of the main resonance.[1,6,8] Hoh,[9] while considering a warm-plasma model, confines his discussion to a qualitative one-dimensional analysis, which does not include the main resonance. We shall attempt to extend the theory further by considering a model which, in principle, appears capable of predicting quantitatively the behavior of both the main and higher order resonances in an applied weak magnetic or electric field.

The model which immediately suggests itself is a simple modification of the warm hydrodynamic model used by Nickel, Parker, and Gould,[5] which so successfully accounted for the zero-magnetic-field observations. We merely include the magnetic component $q(\mathbf{U}\times\mathbf{B})$ of the Lorentz force in the equation of motion. The numerical calculations of Nickel, Parker, and Gould[5] were based on a value of γ (the ratio of specific heats) $=3$; the theoretical analysis presented here, however, is limited to the case $\gamma=1$, simply because we have not found the problem for $\gamma\neq1$ to be tractable.[10] While the main resonance should not be sensitive to the particular value of γ chosen (since cold-plasma theory and the warm hydrodynamic theory agree reasonably well here), the situation is not as clear concerning the additional resonances. However, present theory on the subject of Dattner resonances in applied magnetic fields has been limited to cold-plasma models,[1,6,8] with the exception of Hoh's[9] qualitative one-dimensional analysis of the $\gamma=1$ model. Under these circumstances, further development of the $\gamma=1$ theory is certainly appropriate.

The equations for our model are then those used by Hoh,[9] with the addition of a real-valued scalar dielectric constant $\kappa(\mathbf{x})$ in Poisson's equation [κ represents the dielectric properties of the enclosure, e.g., a glass bulb or tube, confining the hot plasma—$\kappa(\mathbf{x})=1$ within the plasma volume V_p], viz

$$\nabla\cdot(N\mathbf{U})+\partial N/\partial t=0\,, \tag{1}$$

$$m\{\partial\mathbf{U}/\partial t+(\mathbf{U}\cdot\nabla)\mathbf{U}\} =e(\nabla\Phi-\mathbf{U}\times\mathbf{B}_0)-k_BTN^{-1}\nabla N\,, \tag{2}$$

$$\nabla\cdot\{\kappa\nabla\Phi\}=-(e/\epsilon_0)(N_i-N+N_{\rm ext})\,, \tag{3}$$

[1] An extensive list of references can be found in the paper by F. W. Crawford, G. S. Kino, and A. B. Cannara, J. Appl. Phys. 34, 3168 (1963).
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[3] A. Dattner, Ericcson Technics 2, 309 (1957).
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[6] E. Åström, Arkiv Fysik 19, 163 (1961).
[7] A. M. Messiaen and P. E. Vandenplas, Physica 28, 537 (1962).
[8] A. M. Messiaen and P. E. Vandenplas, Phys. Letters 2, 193 (1962).
[9] F. C. Hoh, Phys. Rev. 133, A1016 (1964).

[10] For $\gamma=1$, the operators involved are Hermitian. This does not appear to be the case for $\gamma\neq1$.

where $N_i(\mathbf{x})$ is the ion density, $N(\mathbf{x},t)$ the electron density, $\mathbf{U}(\mathbf{x},t)$ the macroscopic electron velocity, $\Phi(\mathbf{x},t)$ the electrostatic potential, N_{ext} the source of any externally applied electric field $\mathbf{E}$, $-e$ the electronic charge, $\mathbf{B}_0(\mathbf{x})$ the applied magnetic field, m the electron mass, k_B the Boltzmann constant, and T the electron temperature (assumed constant). The motion of the ions is neglected, as is the magnetic field produced by the plasma.

We assume small oscillations about static equilibrium and write

$$\begin{aligned} N&=n_0(\mathbf{x})+n(\mathbf{x})e^{i\omega t},\\ \mathbf{U}&=\mathbf{O}+\mathbf{V}(\mathbf{x})e^{i\omega t}, \qquad (4)\\ \Phi&=\phi_0(\mathbf{x})+\phi(\mathbf{x})e^{i\omega t}. \end{aligned}$$

The linearization of Eqs. (1)–(3) leads to the following expressions:

$$n(\mathbf{x})=-\nabla\cdot(n_0\boldsymbol{\xi}), \qquad (5)$$

$$-\omega^2 n_0\boldsymbol{\xi}=i\omega(e/m)n_0(\mathbf{B}_0\times\boldsymbol{\xi})+(k_BT/m)\{\nabla[\nabla\cdot(n_0\boldsymbol{\xi})] -[\nabla\cdot(n_0\boldsymbol{\xi})]n_0^{-1}\nabla n_0\}+(e/m)n_0\nabla\phi, \qquad (6)$$

$$\nabla\cdot(\kappa\nabla\phi)=-(e/\epsilon_0)\nabla\cdot(n_0\boldsymbol{\xi}), \qquad (7)$$

where we have defined

$$\boldsymbol{\xi}(\mathbf{x})\equiv\mathbf{V}(\mathbf{x})/i\omega. \qquad (8)$$

The unperturbed quantities, N_i, ϕ_0, and n_0 are related by

$$\phi_0(\mathbf{x})=(k_BT/e)\ln n_0(\mathbf{x})+\text{const}, \quad \mathbf{x}\in V_p, \qquad (9\text{a})$$

$$\nabla\cdot\{\kappa\nabla\phi_0\}=-(e/\epsilon_0)[N_i(\mathbf{x})-n_0(\mathbf{x})+N_{\text{ext}}(\mathbf{x})]. \qquad (9\text{b})$$

The unperturbed electron density n_0 will in general be a function of the applied fields and the various system parameters (e.g., T, collision frequencies, etc.), viz.

$$n_0=n_0(\mathbf{x};\lambda_1,\cdots,\lambda_n,\mathbf{F}_0), \qquad (9\text{c})$$

where $\lambda_1,\cdots,\lambda_n$ denote the pertinent system parameters and $\mathbf{F}_0$ denotes the zero-order applied field ($\mathbf{E}_0$ or $\mathbf{B}_0$).[11] An externally applied zero-order field $\mathbf{E}_0$ will manifest itself in (6) through n_0.

It remains to specify the boundary conditions. Let S_p denote the boundary surface of the plasma, i.e., S_p is the surface enclosing V_p. Then we require that

$$0=\mathbf{n}\cdot\mathbf{V}=\mathbf{n}\cdot\boldsymbol{\xi} \text{ on } S_p,$$

where $\mathbf{n}$ denotes the unit normal to S_p. Equation (7) then implies that the normal component of $\kappa\nabla\phi$ is continuous across S_p, as well as at the boundary surface of the dielectric enclosure. The tangential component of $\nabla\phi$ is to be continuous over these boundaries, $\phi(\mathbf{x})$ must vanish as $|\mathbf{x}|\to\infty$ for a bounded plasma, and ϕ must be everywhere continuous if the spectrum is discrete.

In the following section, the gross features of the spectrum of this model of a bounded plasma are determined, for an arbitrary configuration. We find that the spectrum is real and that the system is stable. A variational principle is given for computing the eigenfrequencies. Perturbation theory is applied to determine the first-order correction to the eigenfrequencies for a plasma subjected to an external electric or magnetic field, the perturbation n_{01} of the zero-order electron density due to the external field being included. We find that the first-order effect for a nondegenerate mode is due entirely to n_{01}.

Our first-order perturbation theory is then applied to a cylindrical plasma with an unperturbed electron density $n_{00}=n_{00}(r)$, and expressions for the first-order correction to the eigenfrequencies for a uniform axial magnetic field, a uniform transverse electric or magnetic field, arbitrary density $n_{00}(r)$, and arbitrary angular dependence $e^{i\mu\theta}$ ($\mu=0,\pm1,\pm2,\cdots$) are determined. We find that for $\mu\neq0$, the modes have a twofold degeneracy and that an axial component of the magnetic field splits the corresponding resonance in two. A transverse electric or magnetic field is found to have no first-order effect.

Finally, the agreement between theory and experiment is discussed. We note that for the main dipole resonance, the available experimental evidence suggests no first-order effect for a transverse magnetic field,[7] while a first-order splitting for an axial field is observed,[1] in agreement with our theory.

II. THE SPECTRUM OF A BOUNDED PLASMA IN AN EXTERNAL FIELD

We proceed to determine the general characteristics of the spectrum of our $\gamma=1$ model of a bounded plasma of density $n_0(\mathbf{x})$ in an external field. The analysis is based upon Eq. (6), which we rewrite as

$$\omega^2 n_0\boldsymbol{\xi}=\omega\mathbf{H}_1(\boldsymbol{\xi})+\mathbf{H}_2(\boldsymbol{\xi}), \qquad (10)$$

where

$$\begin{aligned} \mathbf{H}_1(\boldsymbol{\xi})&\equiv-i(e/m)n_0\mathbf{B}_0\times\boldsymbol{\xi},\\ \mathbf{H}_2(\boldsymbol{\xi})&\equiv-(k_BT/m)\mathbf{H}_3(\boldsymbol{\xi})-(e/m)\mathbf{H}_4(\boldsymbol{\xi}),\\ \mathbf{H}_3(\boldsymbol{\xi})&\equiv\nabla[\nabla\cdot(n_0\boldsymbol{\xi})]-[\nabla\cdot(n_0\boldsymbol{\xi})]n_0^{-1}\nabla n_0, \qquad (11)\\ \mathbf{H}_4(\boldsymbol{\xi})&\equiv n_0\nabla\phi_{\xi}, \end{aligned}$$

and ϕ_ξ is defined to be the (unique) solution of Eq. (7) in E_3 (Euclidean three-dimensional space) which vanishes at least as fast as $|\mathbf{x}|^{-1}$ as $|\mathbf{x}|\to\infty$ [n_0 is assumed bounded in space, i.e., there exists $R<\infty$ such that $|\mathbf{x}|>R$ implies $n_0(\mathbf{x})=0$].

Let Q denote the set of all complex vectors $\mathbf{f}(\mathbf{x})$ defined on V_p which satisfy some differentiability criterion so that $\mathbf{H}_i(\mathbf{f})$ ($i=1$, 2, 3, 4) is well defined within V_p [e.g., one might require $\mathbf{f}$ as well as $n_0(\mathbf{x})$ to be twice continuously differentiable in V_p] and

[11] Note that $\mathbf{B}_0$ and $\mathbf{E}_0$ are *defined* to be the zero-order externally applied magnetic and electric fields. ϕ_0 is by definition the total zero-order potential, due to the plasma and $\mathbf{E}_0$. Therefore, in general, $-\nabla\phi_0\neq\mathbf{E}_0$.

satisfy the condition $\mathbf{f}\cdot\mathbf{n}=0$ on S_p, where $\mathbf{n}$ denotes the unit normal to S_p. The operators $\mathbf{H}_i$ ($i=1, 2, 3, 4$) are clearly linear operators; furthermore, they are all Hermitian on the space Q. It then follows that all the eigenvalues ω of (10) are real. We form the inner product of (10) with $\boldsymbol{\xi}$ to obtain

$$\omega^2(\boldsymbol{\xi},n_0\boldsymbol{\xi})=\omega(\boldsymbol{\xi},\mathbf{H}_1(\boldsymbol{\xi}))+(\boldsymbol{\xi},\mathbf{H}_2(\boldsymbol{\xi})), \tag{12}$$

where

$$(\boldsymbol{\xi},\boldsymbol{\eta})\equiv\int_{V_p}\boldsymbol{\xi}^*\cdot\boldsymbol{\eta}d^3x.$$

Solving for ω, there results

$$\omega=\frac{(\boldsymbol{\xi},\mathbf{H}_1(\boldsymbol{\xi}))\pm[(\boldsymbol{\xi},\mathbf{H}_1(\boldsymbol{\xi}))^2+4(\boldsymbol{\xi},n_0\boldsymbol{\xi})(\boldsymbol{\xi},\mathbf{H}_2(\boldsymbol{\xi}))]^{1/2}}{2(\boldsymbol{\xi},n_0\boldsymbol{\xi})}. \tag{13}$$

The inner products are all real, as $\boldsymbol{\xi}$ is in Q and the operators are Hermitian. Furthermore, for any nontrivial $\boldsymbol{\xi}$ in Q, we have $(\boldsymbol{\xi},n_0\boldsymbol{\xi})>0$. Therefore ω will be real if $(\boldsymbol{\xi},\mathbf{H}_2(\boldsymbol{\xi}))\geq 0$. Now

$$\boldsymbol{\xi}^*\cdot\mathbf{H}_3(\boldsymbol{\xi})=\nabla\cdot[\boldsymbol{\xi}^*\nabla\cdot(n_0\boldsymbol{\xi})]-n_0^{-1}|\nabla\cdot(n_0\boldsymbol{\xi})|^2 \tag{14}$$

and

$$\boldsymbol{\xi}^*\cdot\mathbf{H}_4(\boldsymbol{\xi})=\nabla\cdot\left\{n_0\phi_\xi\boldsymbol{\xi}^*+\frac{\epsilon_0}{e}\phi_\xi\kappa\boldsymbol{\nabla}\phi_\xi^*\right\}-\frac{\epsilon_0}{e}\kappa|\boldsymbol{\nabla}\phi_\xi|^2 \tag{15}$$

so that

$$(\boldsymbol{\xi},\mathbf{H}_2(\boldsymbol{\xi}))=(k_BT/m)\int_{V_p}\frac{|\nabla\cdot(n_0\boldsymbol{\xi})|^2}{n_0}d^3x+\frac{\epsilon_0}{m}\int_{E_3}\kappa|\boldsymbol{\nabla}\phi_\xi|^2d^3x \tag{16}$$

which is non-negative. Thus, the spectrum is real and the system stable.

One may readily deduce from the fact that the spectrum is real and $\mathbf{H}_1$ and $\mathbf{H}_2$ are Hermitian on Q that the eigenfunctions $\boldsymbol{\xi}$ of (10) ($\boldsymbol{\xi}$ in Q) corresponding to nonzero eigenfrequencies ω make the expression

$$\frac{(\boldsymbol{\zeta},\mathbf{H}_1(\boldsymbol{\zeta}))\pm[(\boldsymbol{\zeta},\mathbf{H}_1(\boldsymbol{\zeta}))^2+4(\boldsymbol{\zeta},n_0\boldsymbol{\zeta})(\boldsymbol{\zeta},\mathbf{H}_2(\boldsymbol{\zeta}))]^{1/2}}{2(\boldsymbol{\zeta},n_0\boldsymbol{\zeta})} \tag{17}$$

an extremum on Q. Hence, Eq. (13) furnishes a variational principle $\delta[\omega]=0$ for estimating the eigenfrequencies ω.

We now turn to the situation where the magnitude of the applied zero-order field $\mathbf{E}_0(\mathbf{x})$ or $\mathbf{B}_0(\mathbf{x})$ is sufficiently small so that its effect may be treated as a small perturbation of the field-free case. Suppose $\mathbf{E}_0$ and $\mathbf{B}_0$ vanish, and let $\boldsymbol{\zeta}_j(\mathbf{x})$ denote the eigenfunction of Eq. (10) satisfying the boundary condition $\boldsymbol{\zeta}_j\cdot\mathbf{n}=0$ on S_p (i.e., $\boldsymbol{\zeta}_j$ is in Q) and possessing the eigenvalue ω_j. Then $\boldsymbol{\zeta}_j$ satisfies the equation

$$\omega_j^2n_0\boldsymbol{\zeta}_j=\mathbf{H}_2(\boldsymbol{\zeta}_j), \tag{18}$$

where n_0 is now given by Eq. (9c) with $\mathbf{F}_0\equiv 0$, $\mathbf{H}_2$ is a linear Hermitian operator on Q, and the ω_j's are all real. The initial velocity field $\mathbf{V}(\mathbf{x},0)$ is an arbitrarily prescribed initial condition, and an arbitrary member of Q; it is therefore physically reasonable to assume that the eigenfunctions $\boldsymbol{\zeta}_j$ form a complete orthonormal basis for Q. In the one-dimensional case, the entire problem can be cast into the form of a Sturm-Liouville system with a discrete spectrum and a complete orthonormal set of eigenfunctions. We therefore anticipate, and shall assume, that Eq. (18) leads to a discrete spectrum and that the eigenfunctions $\boldsymbol{\zeta}_j$ form a complete orthonormal basis for Q. This provides the foundation for the ensuing perturbation calculations.

We substitute $\epsilon\mathbf{F}_0$ for $\mathbf{F}_0$ in Eq. (9c), and expand n_0, the eigenfunction $\boldsymbol{\xi}$, and its corresponding eigenvalue ω in a power series in the perturbation parameter ϵ, viz.

$$\begin{aligned} n_0(\mathbf{x};\lambda_1,\cdots,\lambda_n,\epsilon\mathbf{F}_0)&=n_{00}(\mathbf{x};\lambda_1,\cdots,\lambda_n)+\epsilon n_{01}+\epsilon^2n_{02}+\cdots,\\ \boldsymbol{\xi}&=\boldsymbol{\zeta}_j+\epsilon\boldsymbol{\xi}_1+\epsilon^2\boldsymbol{\xi}_2+\cdots,\\ \omega&=\omega_j+\epsilon\Omega_1+\epsilon^2\Omega_2+\cdots, \end{aligned} \tag{19}$$

where $\boldsymbol{\zeta}_j$ and ω_j satisfy (18) with n_0 replaced by n_{00}. The substitution of (19) into Eqs. (10) and (11) together with the replacement of $\mathbf{B}_0$ by $\epsilon\mathbf{B}_0$ in the first of Eqs. (11) gives Eq. (18) (with n_0 replaced by n_{00}) for the zero-order term, while the first-order term is

$$\begin{aligned} 2\omega_j\Omega_1n_{00}\boldsymbol{\zeta}_j+\omega_j^2n_{01}\boldsymbol{\zeta}_j+\omega_j^2n_{00}\boldsymbol{\xi}_1&=\omega_j\mathbf{H}_{01}(\boldsymbol{\zeta}_j)+\mathbf{H}_{02}\left(\boldsymbol{\xi}_1+\frac{n_{01}}{n_{00}}\boldsymbol{\zeta}_j\right)+(k_BT/m)\nabla\\ &\quad\cdot(n_{00}\boldsymbol{\zeta}_j)\boldsymbol{\nabla}\left(\frac{n_{01}}{n_{00}}\right)-(e/m)n_{01}\boldsymbol{\nabla}\Phi_{\zeta_j}, \end{aligned} \tag{20}$$

where $\mathbf{H}_{0i}(\boldsymbol{\xi})$ is defined to be $\mathbf{H}_i(\boldsymbol{\xi})$ as given by (11) with n_0 replaced by n_{00} ($i=1, 2, 3, 4$), and $\boldsymbol{\nabla}\Phi_{\zeta_j}$ satisfies

$$\nabla\cdot(\kappa\boldsymbol{\nabla}\Phi_{\zeta_j})=-(e/\epsilon_0)\nabla\cdot(n_{00}\boldsymbol{\zeta}_j). \tag{21}$$

We expand $\boldsymbol{\xi}_1$ in terms of the $\boldsymbol{\zeta}_n$,

$$\boldsymbol{\xi}_1=\sum_n a_n\boldsymbol{\zeta}_n, \tag{22}$$

substitute this expression into (20), and take the inner product of the result with $\boldsymbol{\zeta}_k$. Noting that

$$\omega_j^2n_{00}\boldsymbol{\zeta}_j=\mathbf{H}_{02}(\boldsymbol{\zeta}_j) \tag{23}$$

and

$$(\boldsymbol{\zeta}_k,n_{00}\boldsymbol{\zeta}_j)=\delta_{jk}, \tag{24}$$

we obtain, for $k\neq j$,

$$\begin{aligned} a_k(\omega_j^2-\omega_k^2)&=\omega_j(\boldsymbol{\zeta}_k,\mathbf{H}_{01}[\boldsymbol{\zeta}_j])-\omega_j^2(\boldsymbol{\zeta}_k,n_{01}\boldsymbol{\zeta}_j)+\left(\boldsymbol{\zeta}_k,\mathbf{H}_{02}\left[\frac{n_{01}}{n_{00}}\boldsymbol{\zeta}_j\right]\right)\\ &\quad+(k_BT/m)\left(\boldsymbol{\zeta}_k,\nabla\cdot[n_{00}\boldsymbol{\zeta}_j]\boldsymbol{\nabla}\left[\frac{n_{01}}{n_{00}}\right]\right)-(e/m)(\boldsymbol{\zeta}_k,n_{01}\boldsymbol{\nabla}\Phi_{\zeta_j}), \end{aligned} \tag{25}$$

while $k=j$ gives (for $\omega_j \neq 0$)

$$\Omega_1 = \tfrac{1}{2}(\boldsymbol{\zeta}_j, \mathbf{H}_{01}[\boldsymbol{\zeta}_j]) - \tfrac{1}{2}\omega_j(\boldsymbol{\zeta}_j, n_{01}\boldsymbol{\zeta}_j) + (1/2\omega_j)\left(\boldsymbol{\zeta}_j, \mathbf{H}_{02}\left[\frac{n_{01}}{n_{00}}\boldsymbol{\zeta}_j\right]\right) + (k_B T/2m\omega_j)\left(\boldsymbol{\zeta}_j, \nabla\cdot[n_{00}\boldsymbol{\zeta}_j]\nabla\left[\frac{n_{01}}{n_{00}}\right]\right) - (e/2m\omega_j)(\boldsymbol{\zeta}_j, n_{01}\nabla\Phi_{\zeta_j}). \quad (26)$$

We began with the unperturbed system in the jth eigenstate; Eq. (25) gives the first-order admixture of states due to the perturbation, while Eq. (26) specifies the first-order shift in the eigenfrequency. Suppose that ω_j is nondegenerate and nonzero. Then the first term on the right-hand side of (26) vanishes. Indeed, since ω_j and $\mathbf{H}_{02}$ are real, without loss of generality we may take $\boldsymbol{\zeta}_j$ to be real, so that

$$(\boldsymbol{\zeta}_j, \mathbf{H}_{01}[\boldsymbol{\zeta}_j]) = -(ie/m)\int_{V_p} n_{00}\boldsymbol{\zeta}_j\cdot\mathbf{B}_0\times\boldsymbol{\zeta}_j\, d^3x = 0$$

for arbitrary $\mathbf{B}_0(\mathbf{x})$. Thus, the first-order correction to the eigenfrequency corresponding to a nondegenerate mode is due entirely to the perturbed electron density n_{01}. This is not necessarily the case if the mode is degenerate, as will be demonstrated in the following section, where the specific case of cylindrical geometry is considered.

III. PERTURBATION TREATMENT OF A CYLINDRICAL PLASMA IN AN APPLIED FIELD

The perturbation theory of the previous section will now be applied to a cylindrical plasma. The system considered can be characterized as follows (see Fig. 1): $\mathbf{E}_0(\mathbf{x}) = \mathbf{E}_\perp(r,\theta)$ (no z dependence, no z component),

$$n_{00}(\mathbf{x}) = n_{00}(r) \quad (\text{no } \theta \text{ or } z \text{ dependence}),$$

$$\mathbf{B}_0(\mathbf{x}) = \mathbf{B}_0(r,\theta) \quad (\text{no } z \text{ dependence}),$$

$$\begin{aligned} \kappa(\mathbf{x}) = \kappa(r) &= 1 \quad 0\leq r<a, \quad b<r \\ &= \kappa_0 \quad a\leq r\leq b, \end{aligned} \quad (27)$$

$$V_p \begin{cases} = 0\leq r<a \\ = 0\leq\theta<2\pi \\ = -\infty<z<\infty\,. \end{cases}$$

As n_{00}, $\mathbf{B}_0$, and $\mathbf{E}_0$ are independent of z and $\mathbf{E}_0$ has no z component, we expect (and shall assume) that n_0 (and therefore n_{0i} for all i) is independent of z. We may then restrict our attention to solutions of (6) which are themselves independent of z, and we write

$$\begin{aligned} \phi(\mathbf{x}) &= \phi(r,\theta)\,, \\ \boldsymbol{\xi}(\mathbf{x}) &= \boldsymbol{\xi}(r,\theta)\,. \end{aligned} \quad (28)$$

In this circumstance, the operator $\mathbf{H}_2$ [see Eq. (11)]

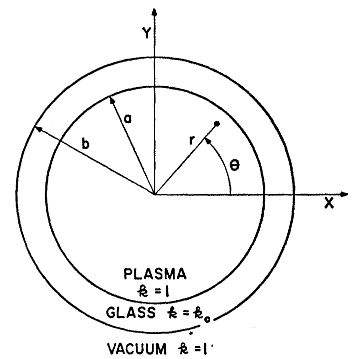

FIG. 1. Geometry of cylindrical plasma column.

has no z component. This will not necessarily be the case for $\boldsymbol{\xi}$, if $\mathbf{B}_0$ has an r or θ component, for the z component of Eq. (10) gives

$$\omega^2 n_0 \xi_z = -i\omega(e/m)n_0(B_{0\theta}\xi_r - B_{0r}\xi_\theta)\,. \quad (29)$$

The use of the perturbation theory developed in the previous section then requires that the eigenfunctions of the unperturbed problem, given, of course, by Eq. (23) together with the boundary condition $\xi_r(a,\theta)=0$, form a complete orthonormal basis for Q, where Q now denotes the set of all suitably well-behaved three-vectors $\boldsymbol{\xi}(r,\theta)$ satisfying $\xi_r(a,\theta)=0$. This presents no difficulty, however, for $\boldsymbol{\zeta}=f(r,\theta)\mathbf{e}_z$ is an eigenfunction satisfying (23) with $\omega=0$, for arbitrary $f(r,\theta)$; $\mathbf{H}_{02}$ is Hermitian on Q, where the appropriate representation of the inner product is

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_0^a \int_0^{2\pi} \boldsymbol{\xi}^* \cdot \boldsymbol{\eta}\, r dr d\theta\,; \quad (30)$$

and one certainly expects the two-dimensional restriction (to the r, θ plane) of Eq. (23) to admit of a complete set of eigenfunctions for the r, θ subplane of Q. The perturbation theory of Eqs. (19)–(26) can then be carried over directly to our present problem—we need only redefine the inner product by Eq. (30).

The matter of degenerate modes remains to be investigated. It is illuminating to case the unperturbed problem into a differential equation for ϕ. Equations (6) and (7) combine to yield, for $\mathbf{B}_0\equiv 0$, $\mathbf{E}_0\equiv 0$,

$$\nabla\cdot\left\{\nabla(\nabla^2\phi) - \nabla^2\phi\frac{\nabla n_{00}}{n_{00}} + \frac{m}{k_B T}[\omega^2 - \omega_p{}^2(r)]\nabla\phi\right\} = 0\,, \quad 0\leq r<a \quad (31)$$

where $\omega_p{}^2(r) = e^2 n_{00}(r)/\epsilon_0 m$, and Eq. (7) gives

$$\nabla\cdot(\kappa\nabla\phi) = 0 \quad a<r\,. \quad (32)$$

Since $n_{00}=n_{00}(r)$ we may separate variables, and write

$$\phi = \phi_{n,\mu}(r)e^{i\mu\theta} \quad n=1, 2, \cdots; \ \mu=0, \pm1, \pm2, \cdots, \quad (33)$$

where the double subscript n, μ denotes the nth mode with angular dependence $e^{i\mu\theta}$. The insertion of (33) into (31) leads to

$$(d^4\phi_{n,\mu}/dr^4)+f_1(d^3\phi_{n,\mu}/dr^3)+f_2(d^2\phi_{n,\mu}/dr^2)+f_3(d\phi_{n,\mu}/dr)+f_4\phi_{n,\mu}(r)=0 \quad 0\leq r<a, \tag{34}$$

where

$$\begin{aligned} f_1&=\frac{2}{r}-\frac{d}{dr}(\ln n_{00}),\\ f_2&=-\frac{(2\mu^2+1)}{r^2}-\frac{2}{r}\frac{d}{dr}(\ln n_{00})-\frac{d^2}{dr^2}(\ln n_{00})+\frac{m}{k_BT}(\omega_{n,\mu}{}^2-\omega_p{}^2),\\ f_3&=\frac{(2\mu^2+1)}{r^3}+\frac{\mu^2}{r^2}\frac{d}{dr}(\ln n_{00})-\frac{1}{r}\frac{d^2}{dr^2}(\ln n_{00})+\frac{m}{k_BT}\left[\frac{(\omega_{n,\mu}{}^2-\omega_p{}^2)}{r}-\frac{d\omega_p{}^2}{dr}\right],\\ f_4&=\frac{\mu^2(\mu^2-4)}{r^4}-\frac{\mu^2}{r^3}\frac{d}{dr}(\ln n_{00})+\frac{\mu^2}{r^2}\frac{d^2}{dr^2}(\ln n_{00})-\frac{m\mu^2}{k_BTr^2}(\omega_{n,\mu}{}^2-\omega_p{}^2).\end{aligned} \tag{35}$$

These equations show that $r=0$ is a regular singularity of (34), if $n_{00}(0)\neq 0$. We require that ϕ, $\nabla\phi$, and $n=(\epsilon_0/e)\nabla^2\phi$ be well behaved at $r=0$, and an examination of the behavior of the solutions of (34) near $r=0$ shows that there are precisely two linearly independent solutions of (34), denoted by $P_{n,\mu}(r)$ and $R_{n,\mu}(r)$, which satisfy these conditions. Hence, we must have

$$\phi_{n,\mu}(r)=A_{n,\mu}P_{n,\mu}(r)+B_{n,\mu}R_{n,\mu}(r) \quad 0\leq r\leq a, \tag{36}$$

where $A_{n,\mu}$ and $B_{n,\mu}$ are constants. The solution of (32) gives

$$\begin{aligned}\phi_{n,\mu}(r)&=C_{n,\mu}(b/r)^{|\mu|}, \quad r\geq b\\ &=(C_{n,\mu}/2\kappa_0)[(\kappa_0-1)(r/b)^{|\mu|}+(\kappa_0+1)(b/r)^{|\mu|}], \quad a\leq r\leq b,\end{aligned} \tag{37}$$

where $C_{n,\mu}$ is a constant, and $C_{n,0}=0$. The potential $\phi_{n,\mu}(r)$ must be matched at $r=a$ so that $\phi_{n,\mu}$ and $\kappa(r)(d\phi_{n,\mu}/dr)$ are continuous there (note that the boundary condition $\xi_r(a,\theta)=0$ implies that there is no plasma surface charge at $r=a$). These two conditions then serve to determine $A_{n,\mu}$ and $B_{n,\mu}$ in terms of $C_{n,\mu}$, if $\mu\neq 0$; if $\mu=0$, so that $C_{n,\mu}=0$, they determine the ratio $A_{n,\mu}/B_{n,\mu}$. Therefore, we conclude that, for given values of μ and $\omega_{n,\mu}$, the radial function $\phi_{n,\mu}(r)$ is uniquely determined (up to a constant multiplier) and that for fixed μ, all the $\omega_{n,\mu}$ $(n=1, 2, \cdots)$ are distinct. The eigenfunction $\boldsymbol{\zeta}$ of Eq. (23) corresponding to the potential $\phi_{n,\mu}(r)e^{i\mu\theta}$ is

$$\boldsymbol{\zeta}_{n,\mu}(r,\theta)=\boldsymbol{\zeta}_{n,\mu}(r)e^{i\mu\theta}, \tag{38}$$

where

$$\omega_{n,\mu}{}^2\boldsymbol{\zeta}_{n,\mu}(r)=\frac{dW_{n,\mu}(r)}{dr}\mathbf{e}_r+\frac{i\mu}{r}W_{n,\mu}(r)\mathbf{e}_\theta, \tag{39}$$

$$W_{n,\mu}(r)\equiv\frac{e}{m}\lambda_D{}^2\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{d\phi_{n,\mu}}{dr}\right]-\left(\frac{\mu^2}{r^2}+\frac{1}{\lambda_D{}^2}\right)\phi_{n,\mu}(r)\right\}, \tag{40}$$

and

$$\lambda_D{}^2(r)\equiv k_BT/m\omega_p{}^2(r)=\epsilon_0k_BT/e^2n_{00}(r). \tag{41}$$

It follows that, for $\omega_{n,\mu}\neq 0$, $\boldsymbol{\zeta}_{n,\mu}$ is uniquely determined by $\phi_{n,\mu}$. Now Eqs. (35) and (37) depend on μ only through $|\mu|$, so that the sets of eigenfrequencies $\omega_{n,\mu}$ and radial potential functions $\phi_{n,\mu}(r)$ for μ and $-\mu$ coincide, and we may write

$$\begin{aligned}\omega_{n,\mu}=\omega_{n,|\mu|}, \quad \phi_{n,\mu}(r)=\phi_{n,|\mu|}(r),\\ W_{n,\mu}(r)=W_{n,|\mu|}(r).\end{aligned} \tag{42}$$

Without loss of generality, we may take $\phi_{n,|\mu|}(r)$ and $W_{n,|\mu|}(r)$ to be real. Therefore any nonzero eigenfrequency $\omega_{n,\mu}$, $|\mu|>0$, has a twofold degeneracy, represented by the two linearly independent eigenfunctions[12]

$$\begin{aligned}\boldsymbol{\zeta}_1(r,\theta)&\equiv\omega_{n,\mu}{}^{-2}\left\{\frac{dW_{n,|\mu|}}{dr}\mathbf{e}_r+\frac{i|\mu|}{r}W_{n,|\mu|}\mathbf{e}_\theta\right\}e^{i|\mu|\theta},\\ \boldsymbol{\zeta}_2(r,\theta)&\equiv\boldsymbol{\zeta}_1{}^*(r,\theta),\end{aligned} \tag{43}$$

$\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ satisfy the orthonormality condition

$$(\boldsymbol{\zeta}_i,n_{00}\boldsymbol{\zeta}_j)=\delta_{ij} \quad i, j=1, 2 \tag{44}$$

provided we normalize $W_{n,|\mu|}(r)$ so that

$$2\pi\omega_{n,\mu}{}^{-4}\int_0^a rn_{00}(r)\left\{\left(\frac{dW_{n,|\mu|}}{dr}\right)^2+\frac{\mu^2}{r^2}W_{n,|\mu|}{}^2\right\}dr=1. \tag{45}$$

We do not anticipate any further degeneracies for such an $\omega_{n,\mu}$; accordingly, the following discussion will assume a mode with precisely the twofold degeneracy of Eq. (43).

The first-order correction to the unperturbed eigenfrequency $\omega_{n,\mu}$ is obtained from Eq. (26). Due to the twofold degeneracy, we must choose for the $\boldsymbol{\zeta}_j$ and $\boldsymbol{\zeta}_k$ of (25) and (26) linear combinations of $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ of

[12] In the work to follow it will be convenient to return to the single-subscript notation of Sec. II. To this end, we suppose a particular mode is selected for study (i.e., a particular pair of values of n and μ are prescribed, say n_0 and μ_0); we define $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ by Eq. (43) for this selected pair (n_0,μ_0). $\boldsymbol{\zeta}_j$ for $j\neq 1, 2$ would then denote an eigenfunction corresponding to a pair $(n,\mu)\neq(n_0, \pm\mu_0)$.

Eq. (43) which make the right-hand side of (25) vanish. We shall now demonstrate that if $\rho_s(r)$ vanishes for $s=\pm 2\mu$, where

$$n_{01}(r,\theta)=\sum_{s=-\infty}^{\infty}\rho_s(r)e^{is\theta},\quad \rho_s=\rho_{-s}^*,\tag{46}$$

then $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ are themselves appropriate eigenfunctions for our perturbation theory.

Lemma I: Let $n_{01}(r,\theta)$ be expanded in the Fourier series of Eq. (46), and let $\boldsymbol{\zeta}_j=\mathbf{F}_1(r)e^{i\mu\theta}$, $\boldsymbol{\zeta}_k=\mathbf{F}_2(r)^{-i\mu\theta}$ ($\mu=0,\pm1,\pm2,\cdots$) in Eqs. (25) and (26). Then, if $\rho_s(r)\equiv 0$ for $s=\pm 2\mu$, all the inner products of Eq. (25) containing n_{01} vanish. If $\rho_0(r)\equiv 0$, then all the inner products of Eq. (26) containing n_{01} vanish.

Proof: The inner products of (25) and (26) containing n_{01} are linear functionals of n_{01}, and so it suffices to consider each term of the series (46) separately. Setting $\boldsymbol{\zeta}_j=\mathbf{F}_1(r)e^{i\mu\theta}$, $\boldsymbol{\zeta}_k=\mathbf{F}_2(r)e^{-i\mu\theta}$, and replacing n_{01} by $\rho_s(r)e^{is\theta}$, it is easily seen that each of the inner products of (25) that contains n_{01} is proportional to

$$\int_0^{2\pi}e^{i(s+2\mu)\theta}d\theta=0,\quad s\neq-2\mu$$
$$=2\pi,\quad s=-2\mu$$

while each of the inner products of (26) that contains n_{01} is proportional to

$$\int_0^{2\pi}e^{is\theta}d\theta=0,\quad s\neq 0$$
$$=2\pi,\quad s=0$$

and the lemma follows immediately.

If we now choose $\boldsymbol{\zeta}_j=\boldsymbol{\zeta}_1$, $\boldsymbol{\zeta}_k=\boldsymbol{\zeta}_2$, then the right-hand side of (25) will vanish, provided $\rho_s\equiv 0$ for $s=\pm 2\mu$ and $(\boldsymbol{\zeta}_1,\mathbf{H}_{01}[\boldsymbol{\zeta}_2])=0$. But since $\mathbf{F}\cdot\mathbf{B}_0\times\mathbf{F}\equiv 0$ for any $\mathbf{F}$, we have

$$(\boldsymbol{\zeta}_1,\mathbf{H}_{01}[\boldsymbol{\zeta}_2])=-i(e/m)\int_0^a\int_0^{2\pi}[\boldsymbol{\zeta}_1^*\cdot\mathbf{B}_0\times\boldsymbol{\zeta}_1^*]\times n_{00}(r)rdrd\theta=0.$$

To summarize, $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ will be appropriate eigenfunctions for our degenerate perturbation theory if $\rho_s\equiv 0$ for $s=\pm 2\mu$. If, in addition, $\rho_0\equiv 0$, then (26) reduces to

$$\Omega_{1,i}=\tfrac{1}{2}(\boldsymbol{\zeta}_i,\mathbf{H}_{01}[\boldsymbol{\zeta}_i])\quad i=1,2.\tag{47}$$

Let us assume for the moment that (47) holds for some μ. Writing $\mathbf{B}_0(r,\theta)=B_{0r}(r,\theta)\mathbf{e}_r+B_{0\theta}(r,\theta)\mathbf{e}_\theta+B_{0z}(r,\theta)\mathbf{e}_z$, Eqs. (45) and (47) imply that the first-order correction to the eigenfrequency $\omega_{n,\mu}$ is

$$\Delta\omega_{n,\mu}=\pm|\mu|(e/4\pi m)\times\frac{\displaystyle\int_0^a\int_0^{2\pi}n_{00}(r)B_{0z}(r,\theta)\frac{dW_{n,|\mu|}{}^2}{dr}d\theta dr}{\displaystyle\int_0^a rn_{00}(r)\left[\left(\frac{dW_{n,|\mu|}}{dr}\right)^2+\frac{\mu^2}{r^2}W_{n,|\mu|}{}^2\right]dr},\tag{48}$$

where the positive sign is associated with $\boldsymbol{\zeta}_1$ and the negative sign with $\boldsymbol{\zeta}_2$. Thus only the axial component of the magnetic field has a first-order effect, providing (47) holds.

Lemma II: Let $n_{01}(r,\theta+\pi)=-n_{01}(r,\theta)$. Then for every μ ($\mu=0,\pm1,\pm2,\cdots$), the first-order correction to the eigenfrequency $\omega_{n,\mu}$ is given by Eq. (48).

Proof: It suffices to show that $\rho_{2\mu}(r)\equiv 0$ for $\mu=0,\pm1,\pm2,\cdots$. But this follows immediately from the hypothesis on n_{01}.

Theorem: The first-order correction to the eigenfrequency $\omega_{n,\mu}$ due to a transverse applied field $\mathbf{F}_0(r,\theta)$ (electric or magnetic) vanishes for every μ ($\mu=0,\pm1,\pm2,\cdots$), if $\mathbf{F}_0(r,\theta+\pi)=\mathbf{F}_0(r,\theta)$. (In particular, $\Delta\omega_{n,\mu}$ vanishes for a uniform transverse applied field.)

Proof: Let $n(r,\theta)$ denote $n_{01}(r,\theta)$ due to the field $\mathbf{F}_0(r,\theta)$, and let $n_\pi(r,\theta)$ denote $n_{01}(r,\theta)$ due to the field $\mathbf{G}_0(r,\theta)$, where $\mathbf{G}_0(r,\theta)$ is $\mathbf{F}_0$ rotated through 180 degrees about the plasma axis. Since $\mathbf{F}_0(r,\theta)=\mathbf{F}_0(r,\theta+\pi)$, we have $\mathbf{F}_0+\mathbf{G}_0=0$. By definition, n_{01} is the *linear* perturbation due to $\mathbf{F}_0$, the principle of superposition holds for n_{01}, and therefore

$$n(r,\theta)+n_\pi(r,\theta)=0.\tag{49}$$

If we rotate the plasma together with the field $\mathbf{F}_0$ through 180 degrees about the plasma axis, we obtain

$$n(r,\theta+\pi)=n_\pi(r,\theta).\tag{50}$$

Equations (49) and (50) imply $n(r,\theta+\pi)=-n(r,\theta)$, and therefore Lemma II holds. The theorem now follows immediately from Eq. (48).

We shall now argue that Eqs. (47) and (48) should also hold for an axial magnetic field $\mathbf{B}_0=B_{0z}(r)\mathbf{e}_z$. Cylindrical symmetry implies that the perturbation n_{01} due to such an axial field is a function of r only. Since $\mathbf{B}_0$ is axial and therefore does not interact with any axial ion or electron currents, the $n_{01}(r)$ induced by $\mathbf{B}_0(r)$ [denoted by $n_+(r)$] should be identical to that induced by $-\mathbf{B}_0(r)$ [denoted by $n_-(r)$], i.e., $n_+(r)=n_-(r)$. But by definition, n_{01} is the *linear* perturbation due to $\mathbf{B}_0$, and so the principle of superposition is applicable and gives $n_+(r)+n_-(r)=0$. Thus the $n_{01}(r)$ induced by $\mathbf{B}_0(r)$ vanishes, and therefore (47) and (48) hold for all μ. For a uniform axial field $B_0\mathbf{e}_z$, Eq. (48) takes the form

$$\Delta\omega_{n,\mu}=\pm\tfrac{1}{2}\omega_c I_{n,|\mu|}\quad \mu=0,\pm1,\pm2,\cdots,\tag{51}$$

where $\omega_c=eB_0/m$ and

$$I_{n,|\mu|}=|\mu|\frac{\displaystyle\int_0^a n_{00}(r)(dW_{n,|\mu|}{}^2/dr)dr}{\displaystyle\int_0^a rn_{00}(r)\{(dW_{n,|\mu|}/dr)^2+(\mu^2/r^2)W_{n,|\mu|}{}^2\}dr}.\tag{52}$$

It is a simple matter to show that $I_{n,|\mu|}>0$ for $\mu\neq 0$, if

$n_{00}(r)$ is monotone decreasing, i.e., if $dn_{00}(r)/dr \leq 0$ on $(0,a)$. Integrating by parts, we have

$$\int_0^a n_{00}(r)\frac{dW_{n,|\mu|}{}^2}{dr}dr = n_{00}(a)W_{n,|\mu|}{}^2(a) - n_{00}(0)W_{n,|\mu|}{}^2(0) - \int_0^a \frac{dn_{00}}{dr}W_{n,|\mu|}{}^2 dr. \quad (53)$$

One can deduce from the behavior of the functions $P_{n,\mu}(r)$ and $R_{n,\mu}(r)$ near $r=0$ that $W_{n,|\mu|}$ vanishes at $r=0$, for $\mu \neq 0$. Indeed, our small-amplitude theory and Eq. (39) require that $W_{n,|\mu|}/r$ be uniformly bounded on $(0,a)$ for $\mu \neq 0$. The right-hand side of Eq. (53) will then be positive if $n_{00}(r)$ is monotone decreasing on $(0,a)$. The electron-density profiles given in Ref. 5 all satisfy this condition, and therefore we expect all the dipole resonances as well as all the higher multipole resonances to be split linearly by a uniform axial magnetic field.

For the uniform cold-plasma case of $T=0$, n_{00} = constant, one easily finds that

$$\begin{aligned} \omega_{1,|\mu|} &= \tfrac{1}{2}\omega_p{}^2, & \mu &= \pm 1, \pm 2, \cdots, \\ \phi_{1,|\mu|} &= A r^{|\mu|}, & 0 &\leq r \leq a, \\ W_{1,|\mu|} &= (e/m)A r^{|\mu|}, & 0 &\leq r \leq a, \end{aligned} \quad (54)$$

so that

$$I_{1,|\mu|} = 1, \quad \mu = \pm 1, \pm 2, \cdots.$$

In this circumstance, Eqs. (51) and (52) reduce to the results of the cold-plasma-uniform-axial-field models of Åström,[6] Messiaen and Vandenplas,[7] and Crawford, Kino, and Cannara.[1]

IV. CONCLUSION

The theory presented here has led to the following results, for a cylindrical plasma with a radially decreasing electron density $n_{00}(r)$:

(1) The first-order correction to all the dipole resonances (as well as all the other n-pole resonances) vanishes for a uniform transverse electric or magnetic field.

(2) A uniform axial magnetic field causes a nonvanishing first-order splitting of all the dipole resonances and all the higher multipole resonances.

The first-order splitting of the main dipole resonance in a uniform axial magnetic field has been demonstrated by Crawford, Kino, and Cannara.[1] Surprisingly enough, since the laboratory plasma is warm and presumably decidedly nonuniform,[5] they find that $I_{1,1} \approx 1$, in agreement with the result obtained from the simple uniform cold-plasma model.

Messiaen and Vandenplas[7] investigated the behavior of the dipole resonances in both uniform axial and transverse magnetic fields. While their results for a transverse magnetic field indicate essentially no first-order effect for the main dipole resonance, in agreement with our theory, it is difficult to interpret the data for the secondary dipole resonances, as they disappear for small transverse magnetic fields and new peaks apparently emerge to take their place. The data for a uniform axial magnetic field does not appear to indicate any splitting of the secondary dipole resonances; however, as the secondary resonances disappear completely in their experiment for a field approximately five times the magnitude of that required to produce a noticeable splitting of the main resonance, all one can reasonably say is that their results suggest that $I_{2,1} \leq (1/5)I_{1,1}$. Quantitative agreement between theory and experiment awaits the evaluation of the $I_{n,1}$ for realistic T and $n_{00}(r)$, as well as further experimental results.