Diagonalization of the Antiferromagnetic Magnon-Phonon Interaction*

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A technique is developed for the diagonalization of quadratic forms consisting of operators whose commutators are c numbers. In particular, it is shown that the transformation matrix $\mathbf{S}$ which diagonalizes such quadratic forms, must satisfy $\mathbf{S}\,\mathbf{g}'\,\mathbf{S}^\dagger=\mathbf{g}$, where $\mathbf{g}$ is a matrix whose elements are c numbers depending upon the commutation relations of the original variables which constitute the quadratic form, and $\mathbf{g}'$ is similarly defined by the new variables. A perturbation expression is then derived for the elements of $\mathbf{S}$. These results are applied to the magnetoelastic interaction in antiferromagnets. It is found that a magnetic field oscillating at a frequency $\omega\ll\gamma(2H_EH_A)^{1/2}$ applied transverse to the z axis can parametrically excite phonons at half-frequency when the amplitude of the field exceeds a certain critical value.

I. INTRODUCTION

OWING to their success in high-energy phenomena, field-theoretical techniques are now being applied extensively to solid-state problems. Thus the fields associated with the various degrees of freedom in a crystal have been quantized. For example, phonons are quantized lattice vibrations, magnons are the elementary excitations of an exchange-coupled spin system, and plasmons are the collective Coulomb excitations of an electron gas. The introduction of such "particles" is particularly convenient because most of them behave like bosons. In determining the static properties of a system this boson nature greatly simplifies the evaluation of expectation values. Furthermore, in the calculation of dynamic quantities such as relaxation times and thermal or electrical conductivities, the concept of boson scattering is very convenient.

Briefly, the field quantization as applied to solid-state situations is achieved in the following manner. The field to be quantized is defined by its Hamiltonian. The canonical field variables associated with each normal mode are determined and expressed in terms of creation and annihilation operators $a_k^\dagger$ and a_k such that the quadratic terms in the Hamiltonian take the form $\sum\omega_k(a_k^\dagger a_k+\frac{1}{2})$. Here k is the normal-mode designation. In most cases the modes are taken to be plane waves, in which case $\mathbf{k}$ is the propagation vector.

If a perturbation is now introduced which couples this field to itself or to another field described, say, by $b_k^\dagger$ and b_k, then the total Hamiltonian will no longer be diagonal but will contain off-diagonal terms of the form $a_ka_{k'}$, $a_kb_{k'}$ $a_kb_{k'}^\dagger$, etc. In classical coupled-mode theory the procedure for the diagonalization of such terms is well known.[1] However, for noncommuting modes the inclusion of the subsidiary condition that the transformation is such that the new modes satisfy certain commutation relations, modifies the diagonalization procedure slightly. The second section of this paper considers this problem.

Actually, the technique developed in Sec. II is more general than that just described. It enables one to diagonalize any quadratic sum of operators whose commutators are c numbers, assuming that the diagonal form exists. Thus, one can directly diagonalize quadratic forms of coordinates q and momenta p and spin operators $S_\pm=S_x\pm iS_y$ (in the linearization approximation in which $S_z=$constant) without first introducing uncoupled boson operators.

It is found that the transformation is, in general, not unitary. For example, when boson creation operators couple to boson annihilation operators or vice versa, the diagonalizing transformation is not unitary. A perturbation theory is presented in Sec. III for finding the eigenfrequencies and the mode admixtures in the regions where the modes are nondegenerate. Finally, as an example of these procedures, we examine the modes resulting from the interaction between antiferromagnetic magnons and phonons.

II. GENERAL THEORY

A. Eigenvalue Condition

Any Hamiltonian which is quadratic in collective-mode amplitudes may be expressed in the form

$$\mathcal{H}=\mathbf{X}^\dagger\mathbf{H}\mathbf{X},\tag{1}$$

where $\mathbf{X}$ is a column matrix consisting of the n independent operators x_i, the row matrix $\mathbf{X}^\dagger$ is its transposed Hermitian adjoint, and $\mathbf{H}$ is the c-number Hermitian matrix which produces the original quadratic form. If $\mathbf{S}$ is a linear transformation of the form

$$\mathbf{X}=\mathbf{S}\mathbf{X}',\tag{2}$$

which diagonalizes $\mathcal{H}$, then it must satisfy

$$\mathbf{H}\mathbf{S}=(\mathbf{S}^\dagger)^{-1}\mathbf{\Omega}_H,\tag{3}$$

where $\mathbf{\Omega}_H$ is the diagonal eigenvalue matrix. If $\mathbf{S}$ were unitary, i.e., $\mathbf{S}^\dagger\mathbf{S}=\mathbf{I}$, then Eq. (3) would reduce to the

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[1] See, for example, W. H. Louisell, *Coupled Mode and Parametric Electronics* (John Wiley & Sons, Inc., New York, 1960).

familiar eigenvalue condition for c-number quadratic forms. However, as we shall see below, $\mathbf{S}$ need not in general be unitary.

The commutation relations may be written

$$[\mathbf{X},\mathbf{X}^\dagger]\equiv\mathbf{X}(\mathbf{X}^*)^T-(\mathbf{X}^*\mathbf{X}^T)^T=\mathbf{g}, \tag{4}$$

where $\mathbf{X}^*$ is the column matrix of the Hermitian adjoint operators, $\mathbf{X}^T$ is the transpose of $\mathbf{X}$, and $\mathbf{g}$ is a matrix whose elements g_{ij} are c-numbers. For boson operators $\mathbf{g}$ is diagonal. Notice that the transpose of the product of matrices whose elements do not commute is *not* equal to the product of the transposes in reverse order.

By using Eq. (2) in Eq. (4), we have

$$[\mathbf{S}\mathbf{X}',\mathbf{X}'^\dagger\mathbf{S}^\dagger]\equiv\mathbf{S}\mathbf{X}'(\mathbf{X}'^*)^T(\mathbf{S}^*)^T -(\mathbf{S}^*\mathbf{X}'^*\mathbf{X}'^T\mathbf{S}^T)^T=\mathbf{g}. \tag{5}$$

Since $\mathbf{S}$ is a c-number matrix, the second term becomes $\mathbf{S}(\mathbf{X}'^*\mathbf{X}'^T)^T(\mathbf{S}^*)^T$. We seek new operators $\mathbf{X}'$ also having c-number commutators $\mathbf{g}'$.[2] Thus we have

$$[\mathbf{X}',\mathbf{X}'^\dagger]\equiv\mathbf{X}'(\mathbf{X}'^*)^T-(\mathbf{X}'^*\mathbf{X}'^T)^T=\mathbf{g}'. \tag{6}$$

Therefore, Eq. (5) reduces to

$$\mathbf{S}\mathbf{g}'(\mathbf{S}^*)^T=\mathbf{g} \quad\text{or}\quad \mathbf{S}\mathbf{g}'\mathbf{S}^\dagger=\mathbf{g}. \tag{7}$$

From this result we have

$$\mathbf{S}^{-1}=\mathbf{g}'\mathbf{S}^\dagger\mathbf{g}^{-1} \quad\text{and}\quad (\mathbf{S}^\dagger)^{-1}=\mathbf{g}^{-1}\mathbf{S}\mathbf{g}'. \tag{8}$$

Thus, we can immediately write the inverses of $\mathbf{S}$ and $\mathbf{S}^\dagger$ without tedious calculations.

By these results, the eigenvalue condition (3) now becomes

$$\mathbf{H}\mathbf{S}=\mathbf{g}^{-1}\mathbf{S}\mathbf{g}'\mathbf{\Omega}_H. \tag{9}$$

This is the matrix eigenvalue equation which determines $\mathbf{S}$ and $\mathbf{\Omega}_H$. In Eq. (18) we show that this may be written as a usual eigenvector equation.

It should be noted that in writing the Hamiltonian in the form of Eq. (1) that for the case of boson operators the column vector $\mathbf{X}$ will, in general, contain all the appropriate operators plus their adjoints. Thus, if the first n elements of $\mathbf{X}$ are independent annihilation operators and the $n+1$ to $2n$ elements are their corresponding creation operators, then $\mathbf{H}$ takes the form

$$\mathbf{H}=\begin{pmatrix}\mathbf{H}_{11} & \mathbf{H}_{12}\\ \mathbf{H}_{12}^* & \mathbf{H}_{11}^*\end{pmatrix}, \tag{10}$$

where $\mathbf{H}_{11}$ and $\mathbf{H}_{12}$ are n by n matrices. The metric for this choice of $\mathbf{X}$ is

$$\mathbf{g}=\begin{pmatrix}\mathbf{I} & 0\\ 0 & -\mathbf{I}\end{pmatrix}, \tag{11}$$

where $\mathbf{I}$ is the n by n identity matrix. If $x_i=\sum_j S_{ij}x_j'$ is to be consistent with $x_{n+i}=\sum_j S_{n+i,j}x_j'$ then the $\mathbf{S}$ matrix must also have the form

$$\mathbf{S}=\begin{pmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{12}^* & \mathbf{S}_{11}^*\end{pmatrix}. \tag{12}$$

From Eqs. (10) and (12) and the eigenvalue condition given by Eq. (9) it can be shown that the eigenvalues associated with the adjoint part of $\mathbf{X}$ are the same as those associated with the first n elements. Therefore the diagonalization of the $2n$ by $2n$ matrix (10) actually reduces to diagonalizing an n by n matrix because the secular equation is of order n in Ω where Ω is an eigenvalue. Furthermore, it can be shown that the eigenvalues are also real even though the matrix $\mathbf{g}\mathbf{H}$ is not, in general, Hermitian.

In most applications it turns out that the Hamiltonian does not contain terms coupling operators to their adjoints.

B. Equations of Motion

In many cases, particularly in classical situations, mode coupling is described by equations of motion which relate the time or space derivative of $\mathbf{X}$ to itself. The relation between our direct treatment of the Hamiltonian matrix and the equations of motion is now established; the methods of Sec. II A then apply also to the equations of motion. However, in the usual case where a Hamiltonian is given, the equations of motion need not be written down. Consider the equations of motion

$$i\partial\mathbf{X}/\partial t=\mathbf{L}\mathbf{X}, \tag{13}$$

where $\mathbf{L}$ describes the coupling. Such an equation of motion can be obtained from the Hamiltonian $\mathcal{H}$ by the relations

$$i\partial\mathbf{X}/\partial t=[\mathbf{X},\mathcal{H}]=\mathbf{g}\mathbf{H}\mathbf{X}. \tag{14}$$

If we introduce new variables by $\mathbf{X}=\mathbf{S}\mathbf{X}'$, Eq. (13) is diagonalized if

$$\mathbf{L}\mathbf{S}=\mathbf{S}\mathbf{\Omega}_L, \tag{15}$$

where $\mathbf{\Omega}_L$ is the eigenvalue matrix of $\mathbf{L}$. By comparing Eqs. (13) and (14) we see that

$$\mathbf{L}=\mathbf{g}\mathbf{H}. \tag{16}$$

Therefore the eigenvalues of $\mathbf{L}$ are related to the eigenvalues of $\mathbf{H}$ by

$$\mathbf{\Omega}_L=\mathbf{g}\mathbf{\Omega}_H. \tag{17}$$

In the boson case where $\mathcal{H}$ contains terms which couple operators to their adjoints then $\mathbf{X}$ will consist of twice as many elements. In such cases Eqs. (14), (16), and (17) are multiplied by a factor of 2.

It is interesting to note that the matrix $\mathbf{g}$ also appears classically when one is dealing with modes which are the components of axial vectors. This occurs, for example, in the case of spin $\mathbf{S}$ in a magnetic field $\mathbf{H}$, where

[2] If we were to require that $\mathbf{g}'=\mathbf{g}$ this would be just the quantum-mechanical condition that the transformation be canonical, analogous to the classical condition that the Poisson brackets be invariant.

the energy has the form $\mathbf{S}\cdot\mathbf{H}$ while the equation of motion is $\dot{\mathbf{S}}\propto\mathbf{S}\times\mathbf{H}$. In this case Ω_L has eigenvalues which are negative relative to one another corresponding to oppositely precessing modes. The form of $\mathbf{g}$, however, ensures that the energy eigenvalues are all positive.

A diagonalization procedure based on the equations of motion has recently been described for noncommuting modes by R. L. Walker.[3]

III. PERTURBATION THEORY

Although the application of the theory developed in the preceding section is straightforward, it is often tedious. Also there are certain calculations for which one needs only certain elements of S. Therefore an approximate expression for S_{ij} itself is desirable.

For this purpose we first notice that the matrix equation (9) can be written as the eigenvector equation

$$\mathsf{gHS}_i=\lambda_i\mathsf{S}_i, \tag{18}$$

where S_i is the ith column of S, i.e.,

$$\mathsf{S}_i=\begin{bmatrix} S_{1i}\\ S_{2i}\\ \vdots \end{bmatrix} \tag{19}$$

and

$$\lambda_i=g_{ii}'\Omega_{ii}. \tag{20}$$

This holds when $\mathbf{g}'$ is diagonal, which is the case when the final modes are bosons. Since this is what we usually desire we shall consider this case. A nondiagonal $\mathbf{g}'$ is easily handled.

The usual perturbation theory[4] now applies. Some care is required since gH is not Hermitian in general. We consider the case in which the off-diagonal comcomponents of H are small and write

$$\mathsf{gH}=\mathsf{gH}^{(0)}+\mathsf{gH}^{(1)}, \tag{21}$$

where $\mathsf{H}^{(0)}$ is the diagonal part of H and $\mathsf{H}^{(1)}$ is the off-diagonal part. We also consider only the nondegenerate cases. The zeroth-order eigenvectors are chosen as

$$\mathsf{S}_i(0)=\begin{bmatrix} 0\\ \vdots\\ S_{ii}^{(0)}=1\\ \vdots\\ 0 \end{bmatrix}. \tag{22}$$

The zeroth-order eigenvalues are $\lambda_i^{(0)}=g_{ii}'H_{ii}^{(0)}$. The first-order eigenvectors are

$$\mathsf{S}_i^{(1)}=\sum_{j\neq i}(\mathsf{S}_j^{(0)}\mathsf{S}_j^{(0)T}\mathsf{gH}^{(1)}\mathsf{S}_i^{(0)}/(g_{ii}'H_{ii}^{(0)}-g_{jj}'H_{jj}^{(0)})). \tag{23}$$

Therefore, we obtain

$$S_{ij}^{(1)}=-g_{ii}H_{ij}^{(1)}/(g_{ii}'H_{ii}^{(0)}-g_{jj}'H_{jj}^{(0)});\quad S_{ii}^{(0)}=0. \tag{24}$$

The first-order correction to the eigenvalues is zero since $\mathsf{H}^{(1)}$ has no diagonal elements. But the second-order correction is

$$\lambda_i^{(2)}=\sum_{j\neq i}(H_{ij}H_{jk}g_{ii}g_{jj}/(g_{ii}'H_{ii}^{(0)}-g_{jj}'H_{jj}^{(0)})). \tag{25}$$

IV. ANTIFERROMAGNETIC MAGNON-PHONON INTERACTION

A. Simple Antiferromagnet

Let us take as our simple antiferromagnet a two-sublattice system in which sublattice i is exchange-coupled to sublattice j, both having uniaxial anisotropy. The Hamiltonian is then

$$\mathcal{H}_{\mathrm{AF}}=2J\sum_{i,j}\mathbf{S}_i\cdot\mathbf{S}_j-K(\textstyle\sum_i(S_i^z)^2+\sum_j(S_j^z)^2). \tag{26}$$

The spin fields of each sublattice are quantized into magnons by the transformations

$$S_i^x=(S/2N)^{1/2}\textstyle\sum_k(a_k+a_{-k}^\dagger)e^{i\mathbf{k}\cdot\mathbf{r}_i};\quad S_j^x=(S/2N)^{1/2}\sum_k(b_k+b_{-k}^\dagger)e^{i\mathbf{k}\cdot\mathbf{r}_j}, \tag{27a}$$

$$S_i^y=-i(S/2N)^{1/2}\textstyle\sum_k(a_k-a_{-k}^\dagger)e^{i\mathbf{k}\cdot\mathbf{r}_i};\quad S_j^y=i(S/2N)^{1/2}\sum_k(b_k-b_{-k}^\dagger)e^{i\mathbf{k}\cdot\mathbf{r}_j}, \tag{27b}$$

and

$$S_i^z=S-(1/N)\sum_{kk'}a_k^\dagger a_{k'}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i}, \tag{27c}$$

$$S_j^z=-S+(1/N)\sum_{kk'}b_k^\dagger b_{k'}e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_j}, \tag{27d}$$

where

$$[a_k,a_k^\dagger]=1$$

and

$$[a_k,a_{-k}]=[a_k^\dagger,a_{-k}^\dagger]=0,$$

with similar relations for the b_k's. The resulting Hamiltonian is

$$\mathcal{H}_{\mathrm{AF}}=\textstyle\sum_k[A_ka_k^\dagger a_k+A_kb_k^\dagger b_k+B_ka_kb_{-k}+B_ka_k^\dagger b_{-k}^\dagger], \tag{28}$$

where

$$A_k=2SJz+2SK\equiv\gamma(H_E+H_A)\equiv\gamma H_E(1+\alpha), \tag{29}$$

$$B_k=2SJz\gamma_k\equiv\gamma H_E\gamma_k. \tag{30}$$

Here z is the number of nearest neighbors and $\gamma_k\equiv(1/z)\sum_\delta\exp(i\mathbf{k}\cdot\boldsymbol{\delta})$ where $\boldsymbol{\delta}$ is the vector to the nearest neighbor.

Equation (28) shows us that the magnons on the individual sublattices are coupled together. This coupling is removed by employing the theory of Sec. II. The diagonalizing transformation is

$$\begin{bmatrix} a_k\\ b_{-k}^\dagger\end{bmatrix}=\begin{bmatrix} u_k & -v_k\\ -v_k & u_k\end{bmatrix}\begin{bmatrix}\alpha_k\\ \beta_{-k}^\dagger\end{bmatrix}, \tag{31}$$

where

$$u_k\equiv((A_k+\Omega_k)/2\Omega_k)^{1/2},\quad v_k\equiv((A_k-\Omega_k)/2\Omega_k)^{1/2}, \tag{32}$$

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[3] L. R. Walker, *Magnetism I*, edited by G. T. Rado and H. Suhl (Academic Press Inc., New York, 1963), p. 312.

[4] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed.

and

$$\Omega_k=\gamma[H_A(H_A+2H_E)+H_E^2(1-\gamma_k^2)]^{1/2}. \tag{33}$$

Under this transformation Eq. (28) becomes

$$\mathcal{H}_{\mathrm{AF}}=\sum_k \Omega_k(\alpha_k^\dagger\alpha_k+\beta_k\beta_k^\dagger)\,. \tag{34}$$

B. Magnon-Phonon Interaction

Let us now investigate the effect of lattice vibrations on our antiferromagnetic modes. These modes can interact in a number of ways. First of all, if the lattice parameters change, the dipolar energy between the spins will change. Similarly, since the exchange integral J is a function of the lattice parameters, the exchange energy will also change (exchangestriction). Another source of magnon-phonon coupling arises from the fact that the anisotropy constants are also functions of the lattice parameters (magnetostriction). Whatever the source of this coupling, one usually represents it by a power series expansion in the strain and spin variables. Therefore we take as our total Hamiltonian

$$\mathcal{H}=\mathcal{H}_{\mathrm{AF}}+\mathcal{H}_p+\mathcal{H}_{\mathrm{ME}}. \tag{35}$$

Here $\mathcal{H}_{\mathrm{AF}}$ is given by Eq. (34). Assuming that the lattice vibrations have already been quantized into phonons with amplitudes c_{qs}, propagation vectors $\mathbf{q}$, and polarizations s, then $\mathcal{H}_p$ is

$$\mathcal{H}_p=\sum_{q,s}\omega_{qs}c_{qs}^\dagger c_{qs}. \tag{36}$$

For the magnetoelastic interaction we take

$$\mathcal{H}_{\mathrm{ME}}=G\sum_i(S_i^xS_i^z\epsilon_i^{xz}+S_i^yS_i^z\epsilon_i^{yz}) +G\sum_j(S_j^xS_j^z\epsilon_j^{xz}+S_j^yS_j^z\epsilon_j^{yz})\,, \tag{37}$$

where G is the magnetoelastic constant and ϵ is the strain. In this expression x, y, and z refer to crystallographic axes. It will be assumed that the sublattice magnetizations are directed parallel and anti-parallel to the crystal z axis. The strain is related to the phonon operators by

$$\epsilon_i^{\mu\nu}=\tfrac{1}{2}\sum_{q,s}\frac{[(\mathbf{q}\cdot\hat{x}_\mu)(\hat{\epsilon}_s\cdot\hat{x}_\nu)+(\mathbf{q}\cdot\hat{x}_\nu)(\hat{\epsilon}_s\cdot\hat{x}_\mu)]}{(2NM\omega_{qs})^{1/2}} \times(c_{qs}-c_{-qs}^\dagger)e^{i\mathbf{q}\cdot\mathbf{r}_i}. \tag{38}$$

If the phonon polarizations are defined as in Fig. 1, the total Hamiltonian then becomes

$$\mathcal{H}=\sum_{k,s}[\Omega_k\alpha_k^\dagger\alpha_k+\Omega_k\beta_k^\dagger\beta_k+\omega_{ks}c_{ks}^\dagger c_{ks} +D_{ks}c_{ks}\alpha_{-k}+D_{ks}^*c_{ks}\alpha_k^\dagger+D_{ks}c_{ks}^\dagger\alpha_k +D_{ks}^*c_{ks}^\dagger\alpha_{-k}^\dagger-E_{ks}c_{ks}\beta_k^\dagger-E_{ks}^*c_{ks}\beta_{-k} -E_{ks}c_{ks}^\dagger\beta_{-k}^\dagger-E_{ks}^*c_{ks}^\dagger\beta_k]\,, \tag{39}$$

where

$$D_{ks}\equiv\tfrac{1}{4}GSu_k(S/M\omega_{ks})^{1/2} \times[(\mathbf{k}\cdot\hat{r}^-)(\hat{\epsilon}_s\cdot\hat{z})+(\mathbf{k}\cdot\hat{z})(\hat{\epsilon}_s\cdot\hat{r}^-)] \tag{40}$$

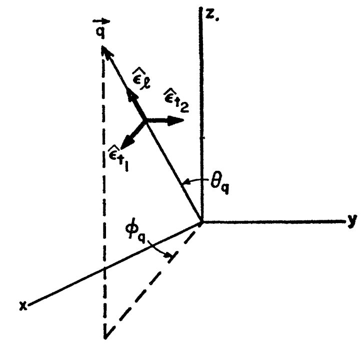

FIG. 1. Orientation of phonon propagation and polarization vectors.

and

$$E_{ks}=(v_k/u_k)D_{ks}. \tag{41}$$

Then $\mathcal{H}$ can be written in the form of Eq. (1) with

$$\mathbf{X}=\begin{bmatrix}\alpha_k\\ \alpha_{-k}^\dagger\\ \beta_k\\ \beta_{-k}^\dagger\\ c_{ks}\\ c_{-ks}^\dagger\end{bmatrix} \tag{42}$$

and

$$\mathbf{H}=\begin{bmatrix}\Omega_k & 0 & 0 & 0 & D_{ks}^* & -D_{ks}^*\\ 0 & \Omega_k & 0 & 0 & D_{ks} & -D_{ks}\\ 0 & 0 & \Omega_k & 0 & -E_{ks} & E_{ks}\\ 0 & 0 & 0 & \Omega_k & -E_{ks}^* & E_{ks}^*\\ D_{ks} & D_{ks}^* & -E_{ks}^* & -E_{ks} & \omega_{ks} & 0\\ -D_{ks} & -D_{ks}^* & E_{ks}^* & E_{ks} & 0 & \omega_{ks}\end{bmatrix}. \tag{43}$$

With the magnon-phonon coupling expressed in this form we can apply the perturbation theory results of Sec. III to determine the new coupled modes.

C. Parametric Excitation of Phonons

One particular phonomenon which depends upon a knowledge of the normal modes of the system is parametric excitation at high-power levels. It is a well-known phenomenon in ferromagnetic insulators that if a magnetic field, oscillating at a frequency comparable to or greater than required for magnetic resonance, is applied transverse to or parallel to the saturating field, spin waves are parametrically excited when this field exceeds a certain critical value.[5,6] It has also been observed that if one drives the system at a frequency far below the resonant frequency it is possible to parametrically excite phonons.[7,8] In antiferromagnetic materials only the so-called second-order Suhl transverse spin-wave instability has been observed.[9] Part of

[5] H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).

[6] E. Schlömann, J. J. Green, and U. Milano, J. Appl. Phys. **31**, 386S (1960).

[7] B. A. Auld, R. E. Tokheim, and D. K. Winslow, J. Appl. Phys. 34, 2281 (1963).

[8] R. L. Comstock and R. C. LeCraw, Phys. Rev. Letters **10**, 219 (1963).

[9] A. J. Heeger, Phys. Rev. **131**, 608 (1963).

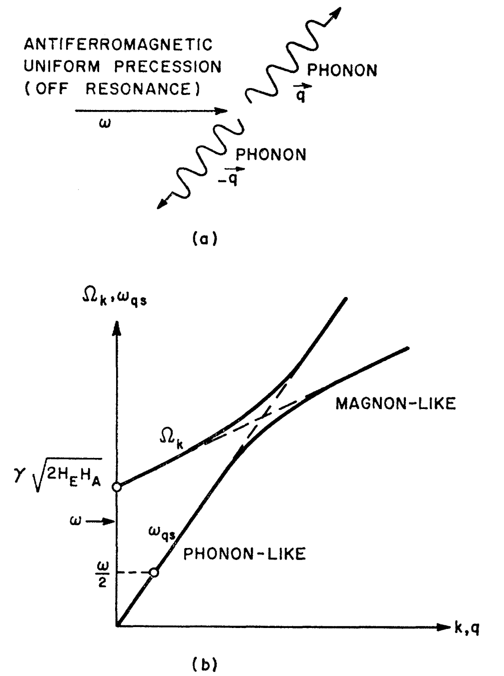

FIG. 2. (a) Schematic representation of parametric phonon excitation. (b) Dynamics of the parametric process in relation to the normal-mode dispersion diagram.

the reason for this is the practical difficulty in obtaining high-power sources at frequencies comparable to the antiferromagnetic resonance frequencies. The question of low-frequency phonon instabilities is therefore of importance.

In antiferromagnets characterized only by isotropic exchange and uniaxial anistropy it is not possible to excite phonons by parallel pumping because the z component of the magnetization does not have a time-varying part. When dipolar interactions are introduced in the absence of an external field the individual spins on the sublattices process elliptically but in such a way that the z component is still time independent.[10] Only by adding an external field to such a system can a time-varying z component of magnetization be produced. However, the normal modes of such a system are extremely complicated. Therefore we shall consider only the excitation of phonons by transverse pumping.

The lowest order transverse process is that in which the antiferromagnetic uniform precession, driven far below resonances, parametrically excites a $\mathbf{q}$, $-\mathbf{q}$ pair of phonons. This is illustrated schematically in Fig. 2.

[10] R. M. White, Ph.D. dissertation, Stanford University, Stanford, California, 1964 (unpublished).

The interaction responsible for this process is again the magnetoelastic interaction. However, since this is a three-boson process it must arise from terms of the form $(S_i^x)^2\epsilon_i^{xx}$, etc. As a particular case let us calculate the instability threshold for a longitudinal phonon propagating in the xz plane (i.e., $\phi_q=0$ in Fig. 1). The pertinent terms in the Hamiltonian are then

$$\mathcal{H}=G[\textstyle\sum_i (S_i^x)^2\epsilon_i^{xx}+\sum_j (S_j^x)^2\epsilon_j^{xx}]\,. \tag{44}$$

Using Eqs. (27), (31) and the perturbation results for the transformation which diagonalizes Eq. (43), we obtain terms of the following form:

$$\mathcal{H}=\tfrac{1}{2}f_{ql}\alpha_0' c_{ql}'^{\dagger}c_{-ql}'^{\dagger}+\text{c.c.}-\tfrac{1}{2}f_{ql}\beta_0' c_{ql}'^{\dagger}c_{-ql}'^{\dagger}+\text{c.c.}\,, \tag{45}$$

where

$$f_{ql}=\frac{G^2S^2\omega_{ql}\sin^3\theta_q\cos\theta_q}{2Mv_l^2\gamma H_A}\left[\frac{S}{N}\frac{H_E}{(2H_EH_A)^{1/2}}\right]^{1/2}, \tag{46}$$

and α_0', β_0', and c_{ql}' are the new coupled-mode operators. The sum over $\mathbf{q}$ has vanished because there is only one longitudinal mode at half the pump frequency, $\omega_{ql}=\frac{1}{2}\omega$.

The rate at which the number of phonons builds up through the excitation of the α_0 mode is, by perturbation theory,

$$(dn_{ql}/dt)_{\alpha_0}=2|f_{ql}|^2 n_{\alpha_0}n_{ql}/\eta_{ql}\,, \tag{47}$$

where η_{ql} is the relaxation frequency of the longitudinal phonon. The rate at which they relax back to the thermal value $\bar{n}_{ql}$ is described by

$$(dn_{ql}/dt)_{\text{relax}}=-2\eta_{ql}(n_{ql}-\bar{n}_{ql})\,. \tag{48}$$

Parametric growth will therefore result when

$$n_{\alpha_0}>\eta_{ql}^2/|f_{ql}|^2. \tag{49}$$

Expressing the phonon relaxation frequency by its "Q", $\eta_{ql}=\omega_{ql}/Q_{ql}$, the threshold of the driving magnetic field is

$$h_{1\text{crit}}=\text{Min}\left\{\frac{2Mv_l^2 g\mu_B H_A(2H_EH_A)^{1/2}}{G^2S^3Q_{ql}\sin^3\theta_q\cos\theta_q}\right\}. \tag{50}$$

The minimum threshold occurs for that longitudinal phonon which makes an angle $\theta=60°$ with the z axis. For the typical number $M\simeq10^{-22}$ g, $v_l\simeq5\times10^5$ cm/sec, $H_A\simeq10^3$ Oe, $H_E\simeq10^6$ Oe, and $G\simeq10^{-13}$ erg this reduces to $h_{1\text{crit}}\simeq10^4\,Q_{ql}^{-1}$. Since acoustic Q's are of the order of 10^3 such an instability should be relatively easy to produce. This instability will be characterized by a sudden increase in the imaginary part of the susceptibility χ'' or equivalently by a sudden increase in the effective acoustic Q.