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Electromagnetic Shocks and the Self-Annihilation of Intense Linearly Polarized Radiation in an Ideal Dielectric Material

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Maxwell's equations are studied for linearly polarized plane waves of intense electromagnetic radiation with the displacement field given by a nonlinear cubic function of the electric field. Analysis shows that *electromagnetic shocks*, surfaces of discontinuity for the electric and magnetic fields, are physically admissible and can indeed develop from an initially continuous field of radiation. Under ordinary conditions, the shocks proceed to "sweep up" and eventually dissipate all of the radiation field energy, giving rise theoretically to the complete self-annihilation of the electromagnetic radiation field. Under rather special conditions, a steady *electromagnetic shock wave train* (with some dissipation of the radiation field energy but only by the tail shock in the train) may possibly evolve dynamically. There is a rigorous mathematical correspondence between the theory here for intense linearly polarized electromagnetic plane waves in an ideal dielectric material and the theory of large-amplitude one-dimensional pressure waves in an ideal solid material.

INTRODUCTION

RECENTLY, Chiao, Garmire, and Townes[1] discussed the possible self-trapping and propagation without spreading of a very intense beam of electromagnetic radiation in an ideal dielectric material. Homogeneous, isotropic, nonconducting, and nonmagnetic, the ideal dielectric material supports a displacement field which is given effectively by a cubic function of the electric field[2]

$$D=\epsilon E+\eta E^3 \tag{1}$$

[ϵ and η = constants] in the case of intense and linearly polarized electromagnetic radiation. It is interesting that the self-trapping phenomenon represents the first *essentially nonlinear* mode of propagation that may be associated with an intense beam of electromagnetic radiation in a dielectric material. Nonlinear effects studied previously, such as the generation of second- and third-harmonic frequencies from an intense beam[3] and the mixing of frequencies between two intense beams,[4] have been analyzed and understood satisfactorily in terms of "pseudo-linear" electromagnetic theory, an effective source term computed by a perturbation-iteration procedure for each separate harmonic of the radiation field.[5] However, an essentially nonlinear solution of Maxwell's equations is required in order to get a theoretical description for self-trapping.

In the present paper we consider the essentially nonlinear modes of propagation which are predicted by Maxwell's equations with (1) for a very intense and linearly polarized one-dimensional *plane wave* of electromagnetic radiation in an ideal dielectric material.[6] It is shown here that *electromagnetic shocks*, surfaces of discontinuity for the electric and magnetic fields, can develop from an initially "smooth" electromagnetic field of radiation, electromagnetic shock formation being a consequence of wave-form distortional effects produced by nonlinearity in the displacement field [represented here by the ηE^3 term in (1)]. Furthermore, Maxwell's equations with (1) predict that the electromagnetic shocks can "sweep up" and eventually dissipate all of the radiation field energy, giving rise theoretically to the complete self-annihilation of the entire field of electromagnetic radiation in certain circumstances. Also following unambiguously from Maxwell's theory is a steady, essentially nonlinear mode of propagation for intense linearly polarized radiation which takes the form of an *electromagnetic shock wave train*. The latter mode of propagation exhibits dynamical stability and no internal dissipation of radiation field energy but, unlike the more natural self-annihilation phenomenon, requires rather special initial conditions. Whether these essentially nonlinear phenomena can be realized in appropriate experiments, say with well-focused beams of linearly polarized laser radiation and with more or less ideal dielectric materials [for

[1] R. Y. Chiao, E. Garmire, and C. H. Townes, Phys. Rev. Letters **13**, 479 (1964).

[2] For a comprehensive discussion of the theory and experimental status of nonlinear electric polarization effects, see: P. A. Franken and J. F. Ward, Rev. Mod. Phys. **35**, 23 (1963). The constant parameter η in Eq. (1) is related to electrostriction, high-frequency third-harmonic Kerr effects, and possibly nonlinearities associated with electronic polarization at very intense fields (of the order 10^8-10^9 V/cm).

[3] P. Franken, A. E. Hill, C. W. Peters, and G. Weinreich, Phys. Rev. Letters **7**, 118 (1961); R. W. Terhune, P. D. Maker, and C. M. Savage, *ibid.* **8**, 404 (1962). P. D. Maker and R. W. Terhune, Phys. Rev. **137**, A801 (1965).

[4] M. Bass, P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinrich, Phys. Rev. Letters **8**, 18 (1962).

[5] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, Phys. Rev. **127**, 1918 (1962); N. Bloembergen and P. S. Pershan, *ibid.* **128**, 606 (1962), and works cited therein.

[6] A critical beam power level (actually attainable in the optical region with well-focused laser radiation) is derived by Chiao *et al.*[1] as a necessary condition for a sustained (steady) self-trapped beam of radiation, involving an important transverse spatial variation of the electromagnetic field. Since an ordinary intense beam of electromagnetic radiation that is not self-trapped propagates like a one-dimensional plane wave to within secondary diffraction effects at the beam's boundary, the considerations in the present paper logically precede analysis of multi-dimensional nonlinear propagation (such as the conjectured self-trapping phenomenon), while being manifestly independent of beam power level (or beam diameter).

which (1) is a good approximation to the actual displacement field], is still uncertain at the present time. Yet the underlying mathematical theory worked out here is rigorous on all counts and certainly worthy of consideration in connection with future experiments.

GENERAL THEORY FOR SIMPLE WAVES AND ELECTROMAGNETIC SHOCKS

The displacement field and the electric field are mutually parallel and everywhere perpendicular to the direction of propagation for a one-dimensional electromagnetic plane wave in an ideal (homogeneous, isotropic, nonmagnetic, and nonconducting) dielectric material. Letting the x axis lie parallel to the direction of propagation, Maxwell's equations reduce to

$$\partial E/\partial x = -\mu_0(\partial H/\partial t), \quad -\partial H/\partial x = \partial D/\partial t \tag{2}$$

and combine to give

$$\partial^2 E/\partial x^2 = \mu_0(\partial^2 D/\partial t^2) \tag{3}$$

for linearly polarized radiation.[7] Simple wave solutions to Eq. (3) with (1), solutions that represent wave propagation only in the $+x$ direction, are expressed implicitly by the form

$$E = F[v(E)t - x], \tag{4}$$

where F is a continuous and twice-differentiable function of the indicated argument, and the local (so-called characteristic) velocity of propagation is

$$v(E) = [\mu_0(\epsilon + 3\eta E^2)]^{-1/2}. \tag{5}$$

That the form (4) rigorously satisfies Eq. (3) with (1) can be proved by differentiating (4) directly and using the chain rule to get the simple wave relation

$$(\partial E/\partial t) + v(E)(\partial E/\partial x) = 0 \tag{6}$$

which is readily shown to satisfy (3) with the velocity of propagation given by (5).

[7] Throughout this paper, our analysis is concerned exclusively with linearly polarized electromagnetic radiation, the form usually of practical relevancy to very intense (ruby) laser beams [e.g., J. H. Brunton, Appl. Opt. **3**, 1241 (1964)], in particular, a beam produced by a Kerr-cell Q-switched ruby laser. Circularly polarized and other more complicated forms of electromagnetic radiation are generally associated with distinctly different essentially nonlinear wave phenomena. Indeed, by putting the vector generalization of (1) $\mathbf{D} = (\epsilon + \eta|\mathbf{E}|^2)\mathbf{E}$ into Maxwell's equations, one obtains solutions for circularly polarized running waves $\mathbf{E} = (0, A\cos\phi, A\sin\phi)$ with $A = \text{constant}$ and $\phi = \omega\{t - [\mu_0(\epsilon + \eta A^2)]^{\frac{1}{2}}x\}$ and solutions for circularly polarized standing waves $\mathbf{E} = [0, A(x)\cos\omega t, A(x)\sin\omega t]$, $d^2A/dx^2 + \mu_0\omega^2(\epsilon A + \eta A^3) = 0$ which exhibit no time-dependent wave-form distortional effects or dynamical features similar to those for linearly polarized radiation. Moreover, one readily obtains essentially nonlinear solutions that are "hyperelliptically" polarized (that is, with forms of polarization nonexistent in linear Maxwell theory), as exemplified by the standing wave solution $\mathbf{E} = [0, A(t)\cos kx, A(t)\sin kx]$, $\mu_0(d^2(\epsilon A + \eta A^3)/dt^2) + k^2 A = 0$. Although special attention must be given to the polarization in nonlinear electromagnetic radiation theory, linearly polarized radiation is "maximal" in exhibiting new and essentially nonlinear dynamical features.

In addition to perfectly continuous simple wave solutions of the form (4), Eq. (3) with (1) also admits *discontinuous* solutions that represent physically admissible wave propagation in the $+x$ direction. Such solutions are composed of simple waves connected by *electromagnetic shock* surfaces of discontinuity, exemplified say at $x = v_s t$ by a finite jump in the electric field

$$E = E_i \quad \text{for} \quad x > v_s t$$
$$= E_f \quad \text{for} \quad x < v_s t, \tag{7}$$

where the (instantaneous) shock velocity of propagation is given by

$$v_s = \{\mu_0(\epsilon + \eta[E_i^2 + E_i E_f + E_f^2])\}^{-1/2}. \tag{8}$$

To show the admissibility of the shock jump condition (7) with the shock velocity (8), we seek a *local* solution to Eq. (3) of the form

$$E = S(x - v_s t) \tag{9}$$

in the neighborhood of the plane $x = v_s t$. By putting (9) into Eq. (3), we obtain the necessary condition on S

$$S - \mu_0 v_s^2(\epsilon S + \eta S^3) = \text{constant}, \tag{10}$$

so that if we impose the jump condition (7) by setting $S(0+) \equiv E_i$ and $S(0-) \equiv E_f$, then (8) follows from (10) if $E_f \neq E_i$. In the limiting case of an infinitesimal discontinuity with $E_f \to E_i$, the shock velocity (8) approaches the ordinary characteristic velocity (5). For finite shock discontinuities, it is also possible for E_i to remain (exactly) constant with time but only if $v_s = v(E_i)$, requiring the special jump condition $E_f = -2E_i$. Ordinarily the latter condition cannot be maintained, and the electromagnetic shock overtakes the $E = \text{constant}$ characteristics of the simple wave radiation field, the quantities E_i and E_f (and therefore v_s and the functional form of S) changing slowly with time. Nevertheless, the shock jump condition (7) with (8) holds good approximately at any instant of time, despite gradual dynamical changes in E_i, E_f, and v_s.

Let us investigate the effect of an electromagnetic shock on the flow and conservation of radiation field energy. Normally the field energy density

$$U = \tfrac{1}{2}\epsilon E^2 + \tfrac{3}{4}\eta E^4 + \tfrac{1}{2}\mu_0 H^2 \tag{11}$$

satisfies the conservation law

$$\partial U/\partial t + \partial(EH)/\partial x = 0 \tag{12}$$

by virtue of Maxwell's equations (2) with (1) if the electric and magnetic fields are continuous (and therefore necessarily differentiable in a homogeneous medium) at x and t. In the special case of a shock discontinuity, however, the energy conservation law (12) breaks down at the shock front, and the *rate of field energy absorption* by the shock is expressed as the finite quantity

$$\dot{\mathcal{E}}_s = v_s(U_i - U_f) - E_i H_i + E_f H_f, \tag{13}$$

where the subscripts i and f refer to field conditions before and after the shock front, respectively. Using the jump condition for the magnetic field

$$H_i - H_f = (\mu_0 v_s)^{-1}(E_i - E_f) \tag{14}$$

derived from (7), (9), and the first Maxwell equation in (2), expression (13) is evaluated algebraically:

$$\begin{aligned}\dot{\mathcal{E}}_s &= \tfrac{1}{2}v_s\epsilon(E_i^2 - E_f^2) + \tfrac{3}{4}v_s\eta(E_i^4 - E_f^4) \\ &\quad + \tfrac{1}{2}v_s\mu_0(H_i^2 - H_f^2) - E_iH_i + E_fH_f \\ &= \tfrac{1}{2}v_s(E_i^2 - E_f^2)[\epsilon + \tfrac{3}{2}\eta(E_i^2 + E_f^2)] - \tfrac{1}{2}(\mu_0 v_s)^{-1} \\ &\quad \times (E_i^2 - E_f^2) \\ &= \tfrac{1}{4}v_s\eta(E_i^2 - E_f^2)(E_i - E_f)^2 \text{ ergs/cm}^2\text{ sec}.\end{aligned} \tag{15}$$

Thus, the electromagnetic shock can act (at least formally) as a source or sink for radiation field energy, depending on the relative magnitudes of E_i^2 and E_f^2. The former possibility, that the shock acts as a source for radiation field energy, is clearly inadmissible on physical grounds, and hence from (15) we obtain an additional necessary condition for a physically realizable electromagnetic shock,

$$E_f^2 \leqslant E_i^2, \tag{16}$$

in order to have a non-negative rate of field energy absorption, $\dot{\mathcal{E}}_s \geqslant 0$.

The preceding analysis closely parallels the classical theory of shock waves in an ideal fluid.[8] As in classical shock-wave theory, the electromagnetic shock is regarded here as a surface of discontinuity without internal structure, and the energy-dissipation mechanism that works within the shock is not treated explicitly by our theory. Nevertheless, it is obvious from physical considerations that finite conductivity effects and the modification they render to the second Maxwell equation in (2) can account for detailed (finite) shock structure and the associated energy dissipation mechanism. That Maxwell's equations (2) are valid *outside* the electromagnetic shock is however sufficient in itself to give the over-all rate of field energy absorption (15), although not the rate at which this "shock energy" is then spent irreversibly within the shock.

FORMATION OF ELECTROMAGNETIC SHOCKS AND THE SELF-ANNIHILATION OF RADIATION FIELDS

Subject to initial or boundary conditions of experimental interest, solutions for the electromagnetic field given implicitly by the form (4) can develop shock discontinuities for certain finite values of x and t, like the simple wave solutions of other nonlinear hyperbolic partial differential equations.[9] Once formed, the electromagnetic shocks usually proceed to overrun the simple-wave domains, "sweeping up" and eventually dissipating all of the radiation field energy and thereby giving rise to the complete self-annihilation of the electromagnetic field. This extraordinary phenomenon is illustrated in the following paragraphs for two cases of practical importance involving monochromatic radiation.

Case 1. Monochromatic electromagnetic radiation at normal incidence to a semi-infinite dielectric for $t \geqslant 0$ [with no delay in the nonlinear electric polarization, represented in (1) by the ηE^3 term]; we have the pure harmonic boundary condition on the semi-infinite domain $x \geqslant 0$,

$$E = \hat{E}\sin\omega t \quad \text{at} \quad x = 0 \quad \text{for} \quad t \geqslant 0 \tag{17}$$

with no electromagnetic field initially in the domain $x \geqslant 0$, $E = 0$ at $t = 0$. Adjusted for the boundary condition (17), the form (4) vanishes for negative values of the argument, while for positive values, (4) gives the electromagnetic field implicity as

$$E = \hat{E}\sin\omega[t - (x/v(E))] \quad \text{for} \quad t \geqslant x/v(E) \geqslant 0. \tag{18}$$

Some elementary analysis applied to (18) demonstrates that electromagnetic shock discontinuities develop from this simple wave solution, the first electromagnetic shock appearing at about $t \lesssim \pi/\omega[(1 + (3\eta/\epsilon)\hat{E}^2)^{1/2} - 1]$, that is, the earliest time at which two $E =$ constant characteristics intersect. Using the method of characteristics in conjunction with the shock conditions (7) and (8), the space-time evolution of the radiation field can be worked out numerically for all positive values of t, as shown in Fig. 1 for the numerically convenient parameter value $\hat{E}^2 = \epsilon/\eta$, the space-time evolution being qualitatively similar for all values of $\hat{E}^2$. In Fig. 1, positive acceleration of the first electromagnetic shock is apparent during the phase of formation with $t \lesssim \pi/\omega$; the shock velocity of propagation increases monotonically and approaches $(\mu_0\epsilon)^{-1/2}$ in an asymptotic fashion as the electromagnetic shock overruns $E =$ constant characteristics, absorbing and eventually dissipating radiation-field energy. The complete self-annihilation of the radiation field is clearly evident.

Case 2. Monochromatic electromagnetic radiation propagating in the $+x$ direction at $t = 0$ through an

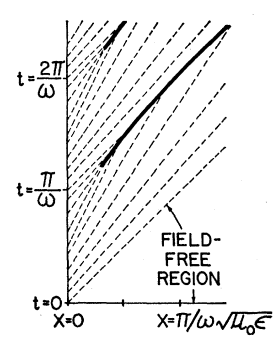

FIG. 1. Space-time evolution of the electromagnetic radiation field for *Case 1* with $\hat{E}^2 = \epsilon/\eta$. Dashed lines represent the $E =$ constant characteristics while heavy solid lines represent the electromagnetic shocks.

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[8] See, for example: R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948).

[9] For instance: P. D. Lax, J. Math. Phys. **5**, 611 (1964), and works cited therein.

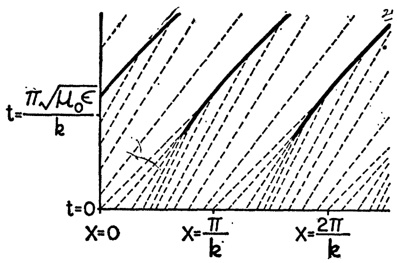

FIG. 2. Space-time evolution of the electromagnetic radiation field for *Case 2* with $\hat{E}^2=\epsilon/\eta$. Dashed lines represent the $E=$ constant characteristics while heavy solid lines represent the electromagnetic shocks.

unbounded dielectric (sudden activation or "switching on" of the nonlinear electric polarization); we have the pure harmonic initial condition for the unbounded domain $-\infty<x<+\infty$,

$$E=\hat{E}\sin kx \quad \text{at} \quad t=0. \tag{19}$$

Adjusted for the initial condition (19), the form (4) gives the electromagnetic field implicitly as

$$E=\hat{E}\sin k[x-v(E)t] \quad \text{for} \quad t\geqslant 0. \tag{20}$$

Like solution (18), the simple wave solution (20) develops electromagnetic shock discontinuities, but in the case of the latter solution an infinite number of uniformly spaced shocks appear simultaneously at about $t\gtrsim\pi(\mu_0\epsilon)^{1/2}/2k[(1+(3\eta/\epsilon)\hat{E}^2)^{1/2}-1]$. Figure 2 shows the space-time evolution of the radiation field for the parameter value $\hat{E}^2=\epsilon/\eta$. Again positive acceleration of the electromagnetic shocks is apparent during the phase of formation with $t\gtrsim\pi(\mu_0\epsilon)^{1/2}/2k$; the shock velocity increases and asymptotically approaches $(\mu_0\epsilon)^{-1/2}$ as $E=$constant characteristics are overrun and the radiation field energy is absorbed and eventually dissipated. Once again the complete self-annihilation of the radiation field is clearly evident.

The preceding cases are typical for an intense, linearly polarized electromagnetic plane wave of radiation in an ideal dielectric material with the displacement field given by (1). Electromagnetic shocks generally develop from the continuous simple wave solutions of Eq. (3) with (1), and the shocks then proceed to "sweep up" and eventually dissipate all of the radiation field energy. Hence the theory predicts the extraordinary phenomenon of complete self-annihilation of the electromagnetic radiation field.

THE ELECTROMAGNETIC SHOCK WAVE TRAIN

Let us now consider the possibility of steady, time-independent wave-forms composed entirely of piecewise-constant electric and magnetic fields through space-time domains separated by electromagnetic shocks, an *electromagnetic shock wave train*, without any simple wave of the form (4) being present in the radiation field. Whereas, the shock formation and self-annihilation phenomenon discussed in the section above appears to be rather general in theory, the development of a shock wave train would require very special initial conditions that are much less likely to be realizable in an experiment. Nevertheless, the electromagnetic shock wave train, being wholly consistent with Maxwell's equations and exhibiting dynamical stability, is of considerable theoretical interest.

Consider a solution to Eq. (3) that takes the form (9) for all values of $x\leqslant x_1$ with $\lim_{x\to-\infty}E=0$. Condition (10) then gives

$$S=+E^*, \quad -E^*, \quad \text{or} \quad 0, \tag{21}$$

where

$$v_s=v_s^*\equiv[\mu_0(\epsilon+\eta(E^*)^2)]^{-1/2} \tag{22}$$

and E^* is a disposable constant electric field amplitude, assuming that S is not identically equal to zero for all values of $(x-v_st)$. Thus, the solution is composed entirely of domains in which the electric and magnetic fields are piecewise-constant, in accordance with (21). Furthermore, Eqs. (21) and (16) admit an arbitrary number of electromagnetic shock discontinuities in the field function S provided that $|\Delta S|=2E^*$ across a shock, but if the electromagnetic field shocks to $S=0$, say at $(x-v_st)=x_0$, then $S\equiv 0$ for all $x<(x_0+v_st)$ by virtue of (16). Let us suppose that a perfect absorber of radiation exists at $x=x_1$ so that the *electromagnetic shock wave train* propagates at a constant velocity (22) with $E=\pm E^*$ throughout the finite spatial region $(x_0+v_st)\leqslant x\leqslant x_1$. Equation (15) shows radiation field energy is absorbed and dissipated only by the tail shock at $(x-v_st)=x_0$, for which $\dot{\mathcal{E}}_s=\frac{1}{4}v_s^*\eta(E^*)^4$. Moreover, linearization of Eq. (3) for the perturbed solution $E=\pm E^*+\theta(x,t)$ with $|\theta|\ll E^*$ gives the elementary wave equation

$$[v(E^*)]^2(\partial^2\theta/\partial x^2)-(\partial^2\theta/\partial t^2)=0, \tag{23}$$

implying that small perturbations in the field propagate without growth in amplitude at the characteristic velocity (5) with $E^2=(E^*)^2$. Therefore, an electromagnetic shock wave train exhibits dynamical stability.

If an electromagnetic shock wave train were to develop from an intense and linearly polarized beam of radiation in a suitable dielectric material, then the radiation would propagate at the special velocity (22). It is relatively easy to measure the transit time for an intense laser beam through ten meters or so of suitable dielectric material, photomultipliers and fast electronic circuitry determining the transit time accurately to within a couple of nanoseconds. Such an experimental measurement would give the velocity of propagation accurately to about 5% and enable one to discern propagation of the radiation at the special shock velocity (22) and to infer the existence of an electromagnetic shock wave train.

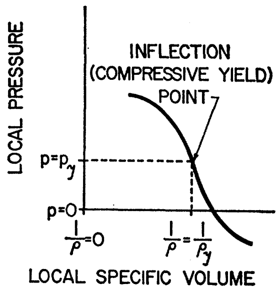

FIG. 3. Pressure-versus-specific volume diagram for an ideal solid of Type CA-CX in the Duvall[10] classification, the functional dependence represented by Eq. (24) here.

A REMARKABLE PHYSICAL ANALOGY

There is a rigorous mathematical correspondence, and therefore an illuminating physical analogy, between the theory presented here for intense linearly polarized electromagnetic plane waves in an ideal dielectric material and the theory of large-amplitude one-dimensional pressure waves in an ideal solid material.[10] With the latter theory already established on an experimental basis,[11] the mathematical correspondence provides some additional insight regarding the new and essentially nonlinear electromagnetic phenomena conjectured theoretically in the preceding sections.

We consider an ideal solid material for which the local specific volume $1/\rho$ is given effectively by a cubic function of the local pressure p,

$$1/\rho=(1/\rho_y)-a(p-p_y)-b(p-p_y)^3 \qquad (24)$$

[a, b, ρ_y, and p_y=constants] in case of extremely high pressure waves (see Fig. 3). Equations expressing mass continuity and momentum conservation take the standard form for ideal one-dimensional unsteady flow[12] and combine, with rigorous elimination of the local velocity field, to give the governing dynamical equation[13]

$$\partial^2(1/\rho)/\partial t^2+\partial^2 p/\partial\psi^2=0\,, \qquad (25)$$

where ψ is a Lagrangian mass coordinate, a constant value of ψ referring to a fixed element of the solid. Since a formal mathematical correspondence between Eq. (3) with (1) and Eq. (25) with (24) can be set up, as shown in Table I, it follows that the two nonlinear wave theories are mathematically equivalent.

TABLE I. Mathematical correspondence between the two nonlinear wave theories, one for large-amplitude mechanical pressure waves in an ideal solid, the other for intense linearly polarized electromagnetic waves in an ideal dielectric.

Quantity in the theory of large-amplitude pressure waves in an ideal solid		Quantity in the theory of intense electromagnetic waves in an ideal dielectric
$a^{\frac{1}{2}}\psi$	$\longleftrightarrow$	$(\mu_0\epsilon)^{\frac{1}{2}}x$
t	$\longleftrightarrow$	t
$(b/a)^{\frac{1}{2}}(p-p_y)$	$\longleftrightarrow$	$(\eta/\epsilon)^{\frac{1}{2}}E$
$(b/a^3)^{\frac{1}{2}}(1/\rho_y-1/\rho)$	$\longleftrightarrow$	$(\eta/\epsilon^3)^{\frac{1}{2}}D$

For the theory of large-amplitude pressure waves in a solid, there is an extensive literature[10,11] concerned with the formation of shock discontinuities, their propagation, stability and dynamical decay behavior in a pressure field of simple waves. We find qualitative evidence in support of the essentially nonlinear dynamical phenomena discussed above for linearly polarized electromagnetic radiation [for example, the existence and dynamical stability of rarefaction shocks with $p_f<p_i$ provided that $(p_f-p_y)^2\leqslant(p_i-p_y)^2$, in correspondence with our condition (16)]. A more detailed comparison of the two theories is encumbered at the present time by considerable dissimilarity in the initial and boundary conditions germane to specific problems of practical interest.

ACKNOWLEDGMENTS

The author is grateful to Dr. Robert J. Bell and William L. Rollwitz for assistance in locating some of the pertinent references.

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