

We get then

$$\bar{G}(i0, \gamma) = i \int d\{\mathbf{v}\} \langle 0 | -\gamma \delta L^I \sum_{n=1}^{\infty} \left(\frac{-1}{L-i0} \gamma J \right)^n P_0 | 0 \rangle, \quad (\text{A18})$$

where the prime means that all intermediate states should correspond to nonvanishing wave numbers $\{\mathbf{k}\} \neq 0$: Equation (A18) is precisely the complete expansion of the right side of Eq. (4.15), Ref. 3. The same calculation may be developed for the left side of (A8).

Energy Spectrum of a Simple Bose-Einstein Model*

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A soluble model of a Bose-Einstein system is examined, in which the interaction energy between particles is attractive as the momentum approaches zero. The model is based on one proposed by Bassichis and Foldy.

I. INTRODUCTION

THE second-quantized form of the Hamiltonian for a system of Bose-Einstein particles interacting via a two-body potential is given by

$$H = \sum_k \omega_k a_{k\omega}^* a_k + (1/2\Omega) \sum_{k_1, k_2, k_3, k_4} v(|\mathbf{k}_1 - \mathbf{k}_3|) \times a_{k_4}^* a_{k_3}^* a_{k_2} a_{k_1} \delta(\mathbf{k}_4 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_1), \quad (1)$$

where $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$ are the annihilation and creation operators for particles in the state of momentum \mathbf{k} , Ω is the volume occupied by the system, and $v(|\mathbf{k}|)$ is the Fourier transform of the two-body potential. The Bogoliubov¹ approximation consists of assuming that most of the particles are in the state $k=0$ and that it is thus permissible to replace a_0^* and a_0 by c numbers equal to \sqrt{N} , where N is the number of particles in the system. One then retains those portions of the interaction which are of order N . This procedure yields an energy spectrum

$$E(m_k) = \sum_{k \neq 0} m_k (\omega_k^2 + 2\omega_k v(k)N/\Omega)^{1/2} + E_0, \quad (2)$$

where m_k is the number of elementary excitations of momentum \mathbf{k} , and E_0 is the energy of the ground state. This result may be improved by using either a pseudo-potential^{2,3} or the single-particle Green's function.⁴ In addition, the number of particles in the state $k=0$ is taken to be N_0 rather than N , where N_0 is the average value of $a_0^* a_0$. The energy of an elementary excitation of momentum \mathbf{k} is then modified to

$$\epsilon_k = (\omega_k^2 + 8\pi\omega_k N_0 a / \Omega m)^{1/2} \quad (3)$$

where a is the S -wave scattering length. ($\hbar=1$). For

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¹ N. N. Bogoliubov, J. Phys. (USSR) **11**, 23 (1947).

² T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

³ T. T. Wu, Phys. Rev. **115**, 1390 (1959).

⁴ N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).

sufficiently small momenta this expression is real only if a is positive. In this case we obtain an acoustic dispersion, and hence a zero energy gap between the ground state and the lowest excited states. However, if a is positive these low-momenta excitations are unstable against decay into two or more excitations, which is not observed in liquid helium II. Unfortunately, a negative scattering length means a Bose-Einstein system cannot be dilute, which is an important assumption for the derivation of the above excitation spectrum. It is thus of interest to examine a soluble model in which the interaction energy between particles is attractive in the limit that the momentum approaches zero. The model we propose examining is one that was studied by Bassichis and Foldy⁵ earlier.

II. THE BASSICHIS AND FOLDY MODEL

This model consists of extracting from H of Eq. (1) only those terms involving three single-particle levels of momenta k , $-k$, and zero. Thus, we consider the Hamiltonian⁶

$$\begin{aligned} h = & \omega_k (n_k + n_{-k}) \\ & + \frac{v(k)}{\Omega} [n_0 (n_k + n_{-k}) + a_0^{*2} a_k a_{-k} + a_0^2 a_k^* a_{-k}^*] \\ & + \frac{v(0)}{2\Omega} [2n_0 (n_k + n_{-k}) + 2n_k n_{-k} + n_k^2 - n_k \\ & + n_{-k}^2 - n_{-k} + n_0^2 - n_0] + \frac{v(2k)}{\Omega} n_k n_{-k}, \quad (4) \end{aligned}$$

⁵ W. H. Bassichis and L. L. Foldy, Phys. Rev. **133**, A935 (1964).

⁶ In Eq. (4) we could retain the entire kinetic energy term without it affecting the ensuing discussion. The extra terms merely separate out.

where n_k , n_{-k} , and n_0 are the usual number operators

$$n_k = a_k^* a_k, \quad n_{-k} = a_{-k}^* a_{-k}, \quad n_0 = a_0^* a_0; \quad (5)$$

the total number operator for the system (which commutes with h) is given by

$$\mathfrak{N} = n_k + n_{-k} + n_0. \quad (6)$$

Thus if N is the total number of particles we may rewrite Eq. (4) as

$$h = \omega_k(n_k + n_{-k}) + (v(k)/\Omega) \times (n_0 n_k + n_0 n_{-k} + a_0^* a_k a_{-k} + a_0^2 a_k^* a_{-k}^*) + (v(2k)/\Omega) n_k n_{-k} + (1/2\Omega) v(0) N(N-1). \quad (7)$$

The last term on the right-hand side of Eq. (7) is a constant and may be dropped from further consideration. In applying the Bogoliubov approximation to Eq. (7) one neglects the term involving $v(2k)$. Inasmuch as we wish to make comparisons with these results we shall choose $v(2k)$ to be zero. The Bogoliubov approximation applied to the remaining terms gives rise⁵ to the analog of Eq. (2) for this model:

$$E(m_k, m_{-k}) = (m_k + m_{-k})(\omega_k^2 + 2\omega_k v(k) N_0/\Omega)^{1/2} + E_g, \quad (8)$$

$$E_g = (\omega_k^2 + 2\omega_k v(k) N_0/\Omega)^{1/2} - \omega_k - v(k) N_0/\Omega,$$

where N_0 is the average value of n_0 . Bassichis and Foldy compared these results to those obtained numerically with the aid of a computing machine. They considered a range of positive values of $v(k) N_0/\Omega$ and found excellent agreement for the cases (a) $m_k = m_{-k} = 0$ and (b) $m_k = m_{-k} = 1$. The situation we wish to consider is that for which $v(k) < 0$ and $\omega_k \rightarrow 0$, where clearly Eq. (8) is not valid.

III. ENERGY SPECTRUM OF h FOR A SPECIAL CASE

It is convenient to introduce operators which have the same commutation relations as the components of angular momenta,

$$L_+ = \sqrt{2}(a_k^* a_0 + a_0^* a_{-k}),$$

$$L_- = \sqrt{2}(a_0^* a_k + a_{-k}^* a_0), \quad (9)$$

$$L_3 = n_k - n_{-k},$$

and the "total angular momentum operator"

$$L^2 = L_3^2 + \frac{1}{2}(L_+ L_- + L_- L_+) = n_k^2 + n_{-k}^2 + n_k + n_{-k} + 2n_0 + 2[n_0 n_k + n_0 n_{-k} - 2n_k n_{-k} + a_0^* a_k a_{-k} + a_0^2 a_k^* a_{-k}^*]. \quad (10)$$

These operators satisfy the commutation relations

$$[L^2, L_+] = [L^2, L_-] = [L^2, L_3] = 0,$$

$$[L_3, L_{\pm}] = \pm L_{\pm}, \quad (11)$$

$$[L_+, L_-] = 2L_3.$$

The Hamiltonian h [still omitting the last two terms in

Eq. (7)] may then be written as

$$h = -[\omega_k + v(k)/2\Omega] n_0 - [v(k)/2\Omega] L_3^2 + [v(k)/2\Omega] L^2 + [\omega_k - v(k)/2\Omega] N. \quad (12)$$

We are interested in the case where $v(k)$ is negative. Furthermore, if we choose $\omega_k = -v(k)/2\Omega = \rho |v(k)|/2N$, ($\rho = N/\Omega$), we see that ω_k is of order $1/N$ times the strength of the interaction and hence is very close to zero. This is exactly the situation in which we are interested. From here on we will set $g \equiv v(k)/\Omega$, and $\omega_k = -g/2$, with the understanding that $g < 0$. Omitting the last term of Eq. (12), (which is a constant), we thus have

$$h = \frac{1}{2}g(L^2 - L_3^2). \quad (13)$$

The number operator \mathfrak{N} commutes with L^2 and L_3 . Hence all three may be simultaneously diagonalized. If we denote the eigenvalues of L^2 by $l(l+1)$ and L_3 by m , then the energy spectrum is

$$E = \frac{1}{2}g[l(l+1) - m^2]. \quad (14)$$

Below, we show that l can take on the positive values N , $N-2$, $N-4$, etc., and m as usual is restricted to $-l \leq m \leq l$ in integer steps. We note that the momentum operator for this model is

$$P = \mathbf{k}L_3. \quad (15)$$

Thus $m=0$, corresponds to a state of zero momentum, while $m = \pm 1$ to one of momentum $\pm \mathbf{k}$, etc. Inasmuch as $g < 0$, the ground state is given by $l=N$ and $m=0$:

$$E_{\text{gnd}} = \frac{1}{2}gN(N+1). \quad (16)$$

The first excited state of momentum k (or $-k$) has the energy

$$E_k = \frac{1}{2}gN(N+1) - g/2.$$

Thus

$$E_k - E_{\text{gnd}} = -g/2 = |g|/2. \quad (17)$$

This energy difference is of order $1/N$ times the ground-state energy per particle and hence is essentially zero. Thus there is no energy gap between the first excited state and the ground state. However, the first doubly excited state of zero momentum has the energy

$$E(k, -k) = \frac{1}{2}g(N-2)(N-1) \quad (18)$$

and

$$E(k, -k) - E_{\text{gnd}} = -2gN = 2|gN|. \quad (19)$$

Now gN is of the order of the ground-state energy per particle, and thus Eq. (19) represents an energy gap as $\omega_k \rightarrow 0$. In this model there is therefore a big qualitative difference between a doubly excited state of momentum $2k$ and one of momentum zero. Below, we will further show that the state of momentum k (or $2k$) may be generated from the ground state by operating on it with the density operator ρ_k (or ρ_k^2). Thus Feynman's⁷ approximation is exact in this situa-

⁷ R. P. Feynman, Phys. Rev. **94**, 262 (1954).

tion. However, the first excited state of zero momentum can *not* be generated by the application of $\rho_k \rho_{-k}$.

IV. EIGENFUNCTIONS OF h FOR $\omega_k = -g/2$

Let ψ_l^m be an eigenfunction of L^2 , L_3 , and \mathfrak{H} with eigenvalues $l(l+1)$, m and l , respectively. Then

$$\psi_l^{-l} = (l!)^{-1/2} (a_{-k}^*)^l |0\rangle, \quad (20)$$

where $|0\rangle$ is the state of zero particles. ψ_l^m may then be constructed with the aid of the raising operator L_+ . Thus

$$\psi_l^m = \left[\frac{1}{(2l)!} \frac{(l-m)!^{-1/2}}{(l+m)!} \right] (L_+)^{l+m} \psi_l^{-l}. \quad (21)$$

Inasmuch as L_+ commutes with \mathfrak{H} this is still an eigenfunction of \mathfrak{H} with eigenvalue l . In order to increase the number of particles without changing l or m we introduce the operator K ,

$$K = a_0^{*2}/2 - a_k^* a_{-k}^*, \quad (22)$$

K commutes with L_3 , L_{\pm} (and hence L^2). It increases the number of particles by two. Let $\Psi_l^m(N)$ be an eigenfunction of L^2 , L_3 , and \mathfrak{H} with eigenvalues $l(l+1)$, m and N , respectively. Then

$$\Psi_l^m(N) = C(K)^{(N-l)/2} \psi_l^m, \quad (23)$$

where C is a constant of normalization. Since the eigenfunctions of h and \mathfrak{H} must be homogeneous polynomials of degree N in the a_0^* , $a_{\pm k}^*$ we see that l is restricted to the values $N, N-2, N-4$, etc.

Now the density operator ρ_k is given by

$$\begin{aligned} \rho_k &= \int e^{ik \cdot x} \psi^*(x) \psi(x) d^3x \\ &= \sum_q a_{k+q}^* a_{-q}. \end{aligned} \quad (24)$$

We note that when acting on the ground state $\Psi_{N^0}(N)$, it has the same effect as $L_+/\sqrt{2}$. However, L_+ acting on this state generates $\Psi_N^1(N)$ which is the first excited state of momentum k . Similarly, ρ_k^2 acting on $\Psi_{N^0}(N)$ is the same as $L_+^2/2$ acting on $\Psi_{N^0}(N)$ which generates $\Psi_N^2(N)$. On the other hand, $\rho_k \rho_{-k}$ is the same as $L_+ L_- / 2$ acting on $\Psi_{N^0}(N)$ which gives back $\Psi_{N^0}(N)$. This justifies the statement made at the end of Sec. III.

V. AVERAGE VALUE OF n_0

Inasmuch as the Bogoliubov approximation fails for this model, it is of interest to ask if the state $k=0$ is macroscopically occupied. Thus we would like to calculate the average number of particles in this level, both for the ground state ($\Psi_{N^0}(N)$) and the first doubly excited state of zero momentum ($\Psi_{N-2^0}(N)$).

In principle we could use the form of the eigenfunctions given by Eq. (23). However, these are difficult to

handle and it is preferable first to transform to "spherical coordinates" where the derivatives of the Legendre polynomials are well known.

As a first step in this procedure we introduce "rectangular" annihilation and creation operators b_i and b_i^* ($i=1, 2, 3$).

$$\begin{aligned} a_k^* &= -(b_1^* + i b_2^*)/\sqrt{2}, \\ a_{-k}^* &= (b_1^* - i b_2^*)/\sqrt{2}, \\ a_0^* &= b_3^*. \end{aligned} \quad (25)$$

The Hermitian conjugate of Eqs. (25) then gives the relationships between the a_k and the b_i . The b_i and b_i^* satisfy the commutation relations $[b_i, b_j^*] = \delta_{ij}$ and zero otherwise. These commutation relations are satisfied by setting $b_j^* = x_j$ and $b_j = \partial/\partial x_j$. However, since it involves very little effort to formalize this transformation we will do so. To this end we introduce the eigenfunctions of the annihilation operator⁸

$$\begin{aligned} \Omega(x_1^*, x_2^*, x_3^*) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \frac{(x_1^* b_1^*)^{n_1}}{n_1!} \\ &\quad \times \frac{(x_2^* b_2^*)^{n_2} (x_3^* b_3^*)^{n_3}}{n_2! n_3!} |0\rangle, \end{aligned} \quad (26)$$

where the x_i are complex variables. Thus

$$b_i \Omega = x_i^* \Omega. \quad (27)$$

Let $\Psi_{\alpha}(b_i^*)$ be a wave function in Fock space given by

$$\begin{aligned} \Psi_{\alpha} &= \sum_{n_1, n_2, n_3} C_{\alpha}(n_1, n_2, n_3) \\ &\quad \times \frac{(b_1^*)^{n_1} (b_2^*)^{n_2} (b_3^*)^{n_3}}{(n_1!)^{1/2} (n_2!)^{1/2} (n_3!)^{1/2}} |0\rangle. \end{aligned} \quad (28)$$

We define the corresponding wave function $\varphi_{\alpha}(x_i)$ by

$$\begin{aligned} \varphi_{\alpha}(x_i) &= \sum_{n_1, n_2, n_3} C_{\alpha}(n_1, n_2, n_3) \\ &\quad \times \frac{(x_1^*)^{n_1} (x_2^*)^{n_2} (x_3^*)^{n_3}}{(n_1!)^{1/2} (n_2!)^{1/2} (n_3!)^{1/2}}. \end{aligned} \quad (29)$$

Then Ψ_{α} and φ_{α} are related by the transformation

$$\begin{aligned} \Psi_{\alpha}(b_i^*) &= \int \frac{d^2 x_1 d^2 x_2 d^2 x_3}{(\pi)^3} \\ &\quad \times \exp[-(|x_1|^2 + |x_2|^2 + |x_3|^2)] \varphi_{\alpha}(x_i) \Omega(x_i^* b_i^*), \end{aligned} \quad (30)$$

where $\int d^2 x_i$ means the integral over the entire complex

⁸ These functions and the transformation following, [Eq. (32)], have been discussed by R. Glauber, in another connection: Phys. Rev. **131**, 2766 (1963). See also V. Bargmann, Commun. Pure Appl. Math. **14**, 187 (1961); Proc. Natl. Acad. Sci. U. S. **48**, 199 (1962).

plane, i.e.,

$$\int d^2x_i = \int_0^\infty d\rho_i \rho_i \int_0^{2\pi} d\alpha_i, \quad (31)$$

$$x_i = \rho_i e^{i\alpha_i}.$$

The previously mentioned relations are now established via the equations

$$b_i^* \Psi_\alpha = \int \left[\prod_{i=1}^3 d^2x_i / (\pi)^3 \right] \times \exp \left[- \sum_{i=1}^3 |x_i|^2 \right] (x_i \varphi_\alpha(x_i)) \Omega(x_i^* b_i^*),$$

and

$$b_i \Psi_\alpha = \int \left[\prod_{i=1}^3 d^2x_i / (\pi)^3 \right] \times \exp \left[- \sum_{i=1}^3 |x_i|^2 \right] \left(\frac{\partial}{\partial x_i} \varphi_\alpha(x_i) \right) \Omega(x_i^* b_i^*). \quad (32)$$

Equations (32) follow upon expanding both sides in a power series in the b_i^* . Furthermore, the inner product of two states Ψ_α and Ψ_β is given in terms of φ_α and φ_β by

$$\langle \Psi_\alpha | \Psi_\beta \rangle = \int \left[\prod_{i=1}^3 d^2x_i / (\pi)^3 \right] \times \exp \left[- \sum_{i=1}^3 |x_i|^2 \right] (\varphi_\alpha(x_i)^* \varphi_\beta(x_i)). \quad (33)$$

The angular-momentum operators L_k , when acting on the φ_α are now written in the usual form

$$L_k = -i \epsilon_{klm} (x_l (\partial / \partial x_k) - x_k (\partial / \partial x_l)), \quad (34)$$

where ϵ_{klm} is the completely anti-symmetric tensor equal to 0, ± 1 . The operator K acting on the φ_α is just $\frac{1}{2} r^2$, i.e.,

$$K \varphi_\alpha = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \varphi_\alpha. \quad (35)$$

It is thus desirable to make the customary transformation to spherical coordinates r, θ, φ . In terms of these the number operator and n_0 are given by

$$\mathfrak{N} = r \partial / \partial r$$

and

$$n_0 = x_3 \partial / \partial x_3 = (\cos^2 \theta) r \partial / \partial r - \cos \theta \sin \theta \partial / \partial \theta. \quad (36)$$

Furthermore, if $\Phi_l^m(N)$ is the transformed eigenfunction corresponding to $\Psi_l^m(N)$ of Eq. (23), then

$$\Phi_l^m(N) = D r^N Y_l^m(\theta, \varphi), \quad (37)$$

where D is a constant of normalization, and Y_l^m is the usual spherical harmonic. If we set $z = \cos \theta$, then

$$n_0 \Phi_l^m(N) = [N z^2 + z(1-z^2) \partial / \partial z] \Phi_l^m(N). \quad (38)$$

For both the ground state and first doubly excited state $m=0$. Thus the spherical harmonics involved in Eq. (37) reduces to the Legendre polynomials $P_l^0(z)$. They satisfy the following relationships:

$$N z^2 P_l^0(z) = N \left[\frac{(l+1)(l+2)}{(2l+1)(2l+3)} P_{l+2}^0 + \frac{(2l(l+1)-1)}{(2l+3)(2l-1)} P_l^0 + \frac{l(l-1)}{(2l+1)(2l-1)} P_{l-2}^0 \right]; \quad (39)$$

and

$$z(1-z^2) \frac{d}{dz} P_l^0(z) = \frac{l(l+1)}{(2l+1)} \left[\frac{(2l+1)}{(2l+3)(2l-1)} P_l^0 + \frac{(l-1)}{(2l-1)} P_{l-2}^0 - \frac{(l+2)}{(2l+3)} P_{l+2}^0 \right]. \quad (40)$$

Since we are interested in the expectation value of n_0 , we need only retain the terms involving P_l^0 on the right-hand sides of Eqs. (39) and (40). The terms involving P_{l+2}^0 and P_{l-2}^0 , when multiplied by r^N , give rise to the eigenfunctions $\Phi_{l+2}^0(N)$ and $\Phi_{l-2}^0(N)$ which are orthogonal to $\Phi_l^0(N)$. Thus the sum of the coefficients of P_l^0 on the right-hand sides of Eqs. (39) and (40) is the expectation value of n_0 in the state $\Psi_l^0(N)$. Thus

$$\langle \Psi_l^0(N) | n_0 | \Psi_l^0(N) \rangle = \frac{[(2N+1)(l+1)l - N]}{(2l+3)(2l-1)}. \quad (41)$$

For the ground state, $l=N$; while for the first excited state with $m=0$, $l=N-2$. In both cases $\langle n_0 \rangle \simeq N/2$. For the most highly excited state with $m=0$, we have $l=0$ and $\langle n_0 \rangle = N/3$. Thus, the single-particle level with $k=0$ is always macroscopically occupied even though the Bogoliubov approximation is not valid.