Ion Cyclotron Electrostatic Instabilities*

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(Received 25 March 1965)

An investigation of the instabilities of longitudinal electrostatic oscillations in an infinite magnetized plasma at or near the ion cyclotron frequency has been made. This work extends that initiated by Harris. An approximate mechanical analysis of the coupling of the motion to the electrostatic field oscillations has been developed, which provides some degree of physical intuition and guides the more abstract dispersion-equation approach used in the suceeding analysis. The possible instabilities arising from the ion mo tion have been classified into two types. Type-A instabilities are characterized by (1) the circumstance that the electrostatic field propagates nearly transversely to the magnetic field, and (2) as the analysis shows, by the requirement that the transverse energy distribution must be peaked at other than zero energy. A monotonically decreasing transverse energy distribution is necessarily stable in this instance. Type-B instabilities arise when the oscillating electric field has a significant component along the magnetic field. Because of the effect of ion motion along the magnetic field, it is now possible for instabilities to arise when the transverse energy distribution decreases monotonically, unlike Type A. Both types of instabilities have been examined as to their dependence on the type of coupling to the plasma electrons, and the various instability criteria are stated. In these electron-ion instabilities a central feature is the coupling of the transverse ion motion to the motion of the electrons along field lines. The most rapidly growing modes are those in which electrons move one axial wavelength in one ion cyclotron period. For actual experimental plasmas, this axial wavelength may be too long to be supported in the experimental device. This can put severe restrictions on the occurrence of these particular instabilities. Another class of instabilities is of those which arise from a coupling of the ion distribution with itself, the electrons playing a passive role only. Of the several kinds of instabilities of this class discussed, the most important are those which arise from a double distribution of ions; e.g., a cold plasma in the presence of a hot ion plasma. The important feature of these instabilities is that they can occur for very low densities. The relevance of this analysis to several current experimental programs is discussed.

1. INTRODUCTION

THE instabilities of longitudinal modes of oscillation in a plasma with anisotropic velocity distributions have been studied by Harris[1,2] and since then have received increasing attention,[3–7] especially in connection with experiments related to the achievement of controlled thermonuclear reactions. Harris[1,2] obtained conditions for instability and growth rates for the unstable longitudinal modes in a magnetized plasma in which the temperatures along the magnetic-field lines were zero for both ions and electrons. Dnestrovsky, Kostomarov, and Pistunovich[5] have attempted to extend Harris' work to the case of nonzero temperature along the field, but their instability condition suffers from the use of an inconsistent approximation. It is our purpose here to present a more thorough examination of the several classes of electrostatic instabilities near cyclotron frequency in a magnetized plasma, and to obtain as much analytical understanding as we can regarding this class of interactions. For the most part, the areas in which detailed numerical work is necessary are left for future publication.

In the present paper we will examine the coupling of electrostatic oscillations to cyclotron motions from two points of view: first in terms of a mechanical solution to the equations of motion and then by means of the linearized Vlasov equations (i.e., ultimately in terms of a dispersion relation). In the former case, only an approximate analysis is attempted, the motivation being that of a search for physical insight rather than rigor.

On returning to the dispersion analysis, we examine the limits of Harris' work[1,2] and also analyze the results of Dnestrovsky, Kostomarov, and Pistunovich.[5] We then consider anew the problem of electrostatic instabilities near the cyclotron frequency with a somewhat modified plasma model, obtaining consistent instability criteria for several cases of interest. Specifically, considerable difference is discovered between cases in which the *electron* temperature along field lines is zero (Harris) or nonzero. Several modes of instability are identified and analyzed, and their relevance to current experiments is set forth. In particular, our results are used as a possible explanation of certain observations made in the

* Work performed under the auspices of the U. S. Atomic Energy Commission. A summary of the major results of this paper has appeared as Phys. Rev. Letters **13**, 603 (1964).

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[6] T. Kammash and W. Heckrotte, Phys. Rev. **131**, 2129 (1963); **133**, A 132 (1964).

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course of the Phoenix[8] and ALICE[9] experiments. The use of our results in connection with recent work on Table Top[10] is also discussed.

2. A MECHANICAL ANALYSIS OF THE ELECTROSTATIC COUPLING TO CYCLOTRON MOTIONS

In the absence of a magnetic field the mechanism for growth or damping of plasma oscillations is seen[11] as the result of the strong interaction possible between a wave and those particles moving very nearly at its phase velocity—particles trapped in the electrostatic troughs of the waves. This picture, which has often proved useful in guiding the intuition of plasma physicists, of course is also effective in the presence of a magnetic field for those motions directed along magnetic-field lines. On the other hand, the presence of a magnetic field also allows a coupling in the transverse direction between the electrostatic field and the cyclotron motions of the particles, and thus offers the possibility of an additional energy exchange between particles and waves via this mechanism. The present section is written in an attempt to gain a physical insight into the mechanics of the electrostatic cyclotron interaction comparable to that in the absence of a magnetic field.

Let us first consider the two-dimensional problem of a particle interacting with a *longitudinal electrostatic* wave propagating normal to a uniform magnetic field $\mathbf{B}_0$. Thus

$$\ddot{\mathbf{r}}=(Ze/m)[c^{-1}\dot{\mathbf{r}}\times\mathbf{B}_0+\mathbf{E}], \qquad (1)$$

where $\mathbf{r}$ is the instantaneous position of the particle and $\mathbf{E}$ is given by

$$\mathbf{E}=-k^{-1}\mathbf{k}E_0\cos[\mathbf{k}\cdot\mathbf{r}-\omega t+\phi^0], \qquad (2)$$

with $k\equiv|\mathbf{k}|$ and ϕ^0 an initial phase. Hence, taking the y axis along $\mathbf{k}$ and writing $\Omega\equiv|\mathbf{\Omega}|$, $\mathbf{\Omega}\equiv Ze\mathbf{B}_0/mc$, and setting $r=x\pm iy$, Eq. (1) takes the form

$$\ddot{r}+i\Omega\dot{r}=-i(ZeE_0/m)\cos\{\mathrm{Im}(kr)-\omega t+\phi^0\}. \qquad (3)$$

There is no loss of generality incurred by assuming ω, Ω, and ZeE_0 to be intrinsically positive, since the coordinate system (with $\mathbf{\Omega}$ along the positive z axis) may always be set up as outlined.

We may think of $\mathbf{E}$ as the y component of a vector $\mathbf{E}_0$, of magnitude $E_0\,\mathrm{sgn}(Z)$, which rotates in the same sense as the particle in the x-y plane. If we also choose our origin of time such that the particle velocity is in the negative y direction at $t=0$, then ϕ^0 is the angle measured in the sense of rotation by which the particle velocity leads $\mathbf{E}_0$ at $t=0$.

Let us now define the (complex) phase $\phi(\Omega t)$ which the particle's true velocity carries with respect to the unperturbed velocity, i.e., if $v_\perp$ is the magnitude of the initial velocity let ϕ be defined by[12]

$$\dot{r}\equiv-iv_\perp e^{-i(\phi+\Omega t)}. \qquad (4)$$

Hence by our previous choice of the zero of time, $\phi(0)=0$. Then

$$r=-iv_\perp\int_0^t dt' e^{-i[\phi(\Omega t')+\Omega t']}+r_0. \qquad (5)$$

Since $y(t=0)=\mathrm{Im}(r_0)=0$, if we now set $s\equiv\Omega t$, $\beta\equiv kv_\perp/\Omega$, $\nu\equiv\omega/\Omega$, and $\alpha\equiv ZeE_0/mv_\perp\Omega$, Eqs. (3) through (5) yield

$$\frac{d}{ds}e^{-i\phi}=\alpha e^{is}\cos\left\{-\beta\int_0^s ds'\cos[\phi(s')+s']-\nu s+\phi^0\right\}. \qquad (6)$$

If the perturbing electric field is small, we may expand ϕ in a power series in its amplitude, viz.:

$$\phi(s)=\alpha\phi_1(s)+\alpha^2\phi_2(s)+\cdots, \qquad (7)$$

so that upon substituting (7) into (6) and equating coefficients of the first power of α, we obtain

$$(d/ds)\phi_1=ie^{is}\cos\{-\beta\sin s-\nu s+\phi^0\}. \qquad (8)$$

We then ask for the quantity $\Delta_1\equiv\int_0^{2\pi}ds(d\phi_1/ds)$, since $\alpha\Delta_1$ is the net (complex) change in phase of the particle's velocity (with respect to its unperturbed value) in one cyclotron period. Note that $\mathrm{Re}(\alpha\Delta_1)$ gives the true phase change while $\mathrm{Im}(\alpha\Delta_1)$ gives the change in magnitude of the velocity, and thus the change in energy. Because of the definition of ϕ, Eq. (4), the particle has gained energy if $\mathrm{Im}(\Delta_1)>0$ and has done net work on the field if $\mathrm{Im}(\Delta_1)<0$.

Substituting (8) into the expression for Δ_1, the result is expressible in terms of Anger's functions,[13] viz.:

$$\Delta_1/2\pi=\mathbf{J}_\nu'(\beta)\sin(\phi^0-\nu\pi)-i\nu\beta^{-1}[\mathbf{J}_\nu(\beta)-\mathbf{J}_\nu(0)]\cos(\phi^0-\nu\pi), \qquad (9)$$

where

$$\mathbf{J}_\nu(\beta)\equiv\pi^{-1}\int_0^\pi ds\cos(\nu s-\beta\sin s). \qquad (10)$$

[8] L. G. Kuo, E. G. Murphy, M. Petravić, and D. R. Sweetman, Phys. Fluids 7, 988 (1964).
[9] C. C. Damm, A. H. Futch, F. Gordon, A. L. Hunt, E. C. Popp, R. F. Post, and J. F. Steinhaus, Nucl. Fusion 1, 280 (1961).
[10] W. A. Perkins and R. F. Post, Phys. Fluids 6, 1537 (1963).
[11] D. Bohm and E. P. Goss, Phys. Rev. 75, 1851 (1949).

[12] We define the phase in terms of the velocity rather than the position of the particle since we wish to leave out the effects of the guiding-center motion. The guiding-center velocity, which is proportional to the average electric field *seen by the particle* over a cyclotron period, depends only on the amplitude of the electric field at the times under consideration (e.g., it disappears when the field is turned off) and should not be considered in computing rates of energy transfer between particles and the wave for the purpose of establishing instability in the present situation.
[13] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1948) 2nd ed., pp. 309 ff.

When ν is equal to the positive integer l, (10) is Bessel's integral and (9) becomes

$$\Delta_1=(-)^l 2\pi[J_l'(\beta)\sin\phi - il\beta^{-1}J_l(\beta)\cos\phi], \quad (11)$$

and we have dropped the now superfluous superscript zero on ϕ. Thus in general there are two phase angles which are unchanged on the average: $\phi=0$ and $\phi=\pi$. Only one of these is stable, however, the net change being such as to tend to cancel the initial value of $\sin\phi$ in one case and to add to it in the other. [When $(-)^l J_l'(\beta)$ is negative, it is the position $\phi=0$ which is phase stable.] In addition, there are exceptional cases in which ϕ is unchanged, for if we define j_{lk}' as the kth extremum of J_l, then $\mathrm{Re}(\Delta_1)$ vanishes at those energies for which $\beta=j_{lk}'$, and *all* phase angles are stable.

The rate of energy transfer to the particle by the field is, as we noted, proportional to $\mathrm{Im}(\Delta_1)$. Thus, adiabatically, recalling the definitions of α and β and setting $a\equiv ZeE_0k/m\Omega^2$,

$$d(\tfrac{1}{2}\beta^2)/dt=(-)^{l+1}a\Omega lJ_l(\beta)\cos\phi. \quad (12)$$

(In using the adiabatic approximation we have had to assume that the time scale for energy exchange is long compared with the cyclotron period—essentially equivalent to assuming $a\ll 1$.) Similarly, the rate of net change of phase is

$$d\phi/dt=(-)^l a\Omega\beta^{-1}J_l'(\beta)\sin\phi. \quad (13)$$

We note that for particles in the vicinity of the phase-stable position, Eq. (12) indicates an energy *gain* if $J_l(\beta)J_l'(\beta)$ is positive, whereas these particles *lose* energy if this product is negative. However, if the distribution of particles over phase angle is initially homogeneous, the rate of energy exchange averaged over the whole distribution vanishes at $t=0$. Therefore we must look at the second derivative of $\frac{1}{2}\beta^2$ to find the direction of energy flow at small times. Thus from (12) and (13),

$$d^2(\tfrac{1}{2}\beta^2)/dt^2=a^2\Omega^2[\sin^2\phi+l\cos^2\phi]l\beta^{-1}J_l(\beta)J_l'(\beta). \quad (14)$$

Averaging over phase, one sees that the average energy transfer to those particles whose speed is characterized by β is of the same sign as $J_l(\beta)J_l'(\beta)$, i.e., exactly as for those particles near the phase stable positions. In the present picture, therefore, the phase stable position, $\phi=0$ or else $\phi=\pi$, plays essentially the same role as the position at the trough of the wave plays in the absence of a magnetic field.

Let $\langle\psi\rangle$ denote the average over initial transverse velocity of any function ψ. Then

$$\langle d^2(\tfrac{1}{2}\beta^2)/dt^2\rangle=-\tfrac{1}{4}a^2\Omega^2(1+l)\langle P_l(\beta)\rangle, \quad (15)$$

where

$$P_l(\beta)=-\tfrac{1}{2}[J_{l-1}^2(\beta)-J_{l+1}^2(\beta)] = -2l\beta^{-1}J_l(\beta)J_l'(\beta). \quad (16)$$

As we should expect from the above, we will find the quantity $\langle P_l(\beta)\rangle$ playing an important role in our later dispersion analysis. Here, however, its physical meaning is clear as the proportionality constant which determines the initial direction of average energy flow between the particles and an electrostatic wave propagating normal to the magnetic field. Energy flow is from or to the particles accordingly as $\langle P_l(\beta)\rangle$ is positive or negative. Thus if we are to have an instability driven by an excess of energy in the perpendicular motion of the ions, say, then $\langle P_l(\beta)\rangle$ must be positive for these particles, at least to the extent that the wavelength along the magnetic field is very long. The modification of this statement necessary when the wave propagates partially along the magnetic field will be considered shortly.

It is interesting to determine what kinds of distribution functions can lead to positive values of $\langle P_l(\beta)\rangle$, and thus to the possibility of instability. If in zero order the particles are distributed in transverse velocity according to the function $f_\perp(v_\perp)$, we have

$$\langle P_l(\beta)\rangle=2\pi\int_0^\infty v_\perp dv_\perp f_\perp(v_\perp)[-2l\beta^{-1}J_l(\beta)J_l'(\beta)]. \quad (17)$$

On substituting for $v_\perp$ in terms of β and integrating by parts,

$$\langle P_l(\beta)\rangle=2\pi l\left(\frac{\Omega}{k_\perp}\right)^2\int_0^\infty d\beta J_l^2(\beta)f_\perp'\left(\frac{\beta\Omega}{k_\perp}\right). \quad (18)$$

(We write $k_\perp$ instead of k in the above to conform to later notation.) Hence *only* distributions for which $f_\perp'$ is somewhere positive can give rise to positive $\langle P_l(\beta)\rangle$ and thus instabilities of the nature under discussion. In particular, any monotonically varying distribution function (e.g., Maxwellian) is stable since $f(v_\perp)$ must vanish as $v_\perp\to\infty$.

When the wave propagates partially along the magnetic field, our analysis must be somewhat modified. First of all, there is a Doppler shift of the driving frequency as seen in the frame moving with the unperturbed velocity. Thus if $\mathbf{k}_\parallel$ and $\mathbf{k}_\perp$ are the components of $\mathbf{k}$ parallel and perpendicular to the magnetic field, respectively, and $n_\parallel=k_\parallel/k$, $n_\perp=k_\perp/k$, Eq. (6) becomes[14]

$$\frac{d}{ds}e^{-i\phi}=a\beta^{-1}n_\perp^2e^{is} \times\cos\left\{-\beta\int_0^s ds'\cos[\phi(s')+s']-\nu s+\zeta+\phi^0\right\}, \quad (19)$$

where we now write

$$\beta\equiv k_\perp v_\perp/\Omega, \quad \nu\equiv(\omega-k_\parallel v_\parallel)/\Omega, \quad \zeta=k_\parallel(z-v_\parallel t), \quad a\equiv ZeE_0k/m\Omega^2. \quad (20)$$

In addition we have the equation for the displacement

[14] Since the electric field is assumed longitudinal, the component of $\mathbf{E}$ perpendicular to $\mathbf{B}_0$ is proportional to $n_\perp$ and the component along $\mathbf{B}_0$ is proportional to $n_\parallel$.

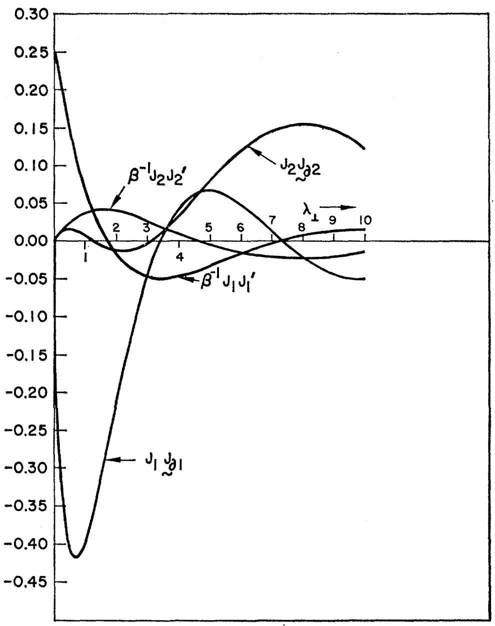

FIG. 1. $J_l(\beta)\mathbf{J}_{\partial l}(\beta)$ and $\beta^{-1}J_l(\beta)J_l'(\beta)$ versus $\lambda_\perp=\frac{1}{2}\beta^2$.

in the z direction relative to the unperturbed position,

$$\frac{d^2\zeta}{ds^2}=an_{\parallel}^2 \times\cos\left\{-\beta\int_0^s ds'\cos[\phi(s')+s']-\nu s+\zeta+\phi^0\right\}. \quad (21)$$

Again, we may expand the equations in powers of the perturbing electric field amplitude, writing

$$\phi=(a\beta^{-1}n_\perp^2)\phi_1+(a\beta^{-1}n_\perp^2)^2\phi_2+\cdots,$$
$$\zeta=(an_{\parallel}^2)\zeta_1+(an_{\parallel}^2)^2\zeta_2+\cdots.$$

Defining the net changes in ϕ_1 and ζ_1 over a cyclotron period,

$$\Delta_1\equiv\int_0^{2\pi}\left(\frac{d\phi_1}{ds}\right)ds,$$
$$\Delta_2\equiv\int_0^{2\pi}ds'\int_0^{s'}ds\left(\frac{d^2\zeta_1}{ds^2}\right)=\int_0^{2\pi}(2\pi-s)\left(\frac{d^2\zeta_1}{ds^2}\right)ds, \quad (22)$$

we find Δ_1 again given by Eq. (9) with ν now to be interpreted in terms of the Doppler-shifted frequency, and

$$\Delta_2/2\pi=[\pi\cos(\phi^0-\nu\pi)+\sin(\phi^0-\nu\pi)d/d\nu]\mathbf{J}_\nu(\beta). \quad (23)$$

We now wish to obtain the adiabatic time derivatives analogous to Eqs. (12) and (13) for the present context. Indeed, since Δ_1 is unchanged in form, Eq. (12) will again describe the average rate of change of perpendicular energy over a cyclotron period when the Doppler-shifted frequency of the electrostatic wave is an integral multiple of the cyclotron frequency. The phase of the particle's velocity with respect to that of the electrostatic field, however, is now changed by two effects. The first contribution, as before, comes from the real part of Δ_1, corresponding to a true phase change of the particle's transverse velocity with respect to its unperturbed phase. On the other hand, when the direction of propagation of the electric field is partially along the magnetic field, the net displacement of the particle with respect to its zero-order *longitudinal* position also contributes an equivalent *azimuthal* phase change $an_{\parallel}^2\Delta_2$. Thus (again dropping the superscript zero on ϕ),

$$d(\tfrac{1}{2}\beta^2)/dt=(-)^{l+1}an_\perp^2\Omega lJ_l(\beta)\cos\phi \quad (24)$$

$$d\phi/dt=(-)^l an_\perp^2\Omega\beta^{-1}J_l'(\beta)\sin\phi + (-)^l an_{\parallel}^2\Omega[\pi J_l(\beta)\cos\phi+\mathbf{J}_{\partial l}(\beta)\sin\phi], \quad (25)$$

where

$$\mathbf{J}_{\partial\nu}(\beta)\equiv\partial[\mathbf{J}_\nu(\beta)]/\partial\nu.$$

As before, $d(\frac{1}{2}\beta^2)/dt$ vanishes on averaging over ϕ initially, so that we differentiate (24) a second time, using both (24) and (25) in the result, obtaining

$$\langle d^2(\tfrac{1}{2}\beta^2)/dt^2\rangle=\tfrac{1}{2}a^2n_\perp^2\Omega^2 l[n_{\parallel}^2\langle J_l(\beta)\mathbf{J}_{\partial l}(\beta)\rangle + n_\perp^2(l+1)\langle\beta^{-1}J_l(\beta)J_l'(\beta)\rangle]. \quad (26)$$

Equation (26) is useful, for example, in whatever numerical analysis may have to be undertaken for the examination of a dispersion equation, in that the requirement that $\langle d^2(\frac{1}{2}\beta^2)/dt^2\rangle$ be initially negative eliminates certain parts of parameter space as possible regions of instability. However, we can no longer conclude, as we did from Eq. (15), that a system is stable whenever $\langle P_l(\beta)\rangle$ is negative. The presence of the extra term propportional to $n_{\parallel}^2$, which comes from the phase shift caused by an average parallel displacement of the perturbation allows $\langle d^2(\frac{1}{2}\beta^2)/dt^2\rangle$ to be negative even for monotonic distributions of transverse energy.

For purposes of illustration, the quantities $\langle J_l(\beta)\mathbf{J}_{\partial l}(\beta)\rangle$ and $\langle\beta^{-1}J_l(\beta)J_l'(\beta)\rangle$ have been computed numerically and plotted versus $\lambda_\perp\equiv k_\perp^2(\kappa T_\perp/m\Omega^2)$ for the case of a delta-function distribution at energy $\kappa T_\perp$. The results for the cases $l=1$, 2 are shown in Fig. 1.

3. THE DISPERSION RELATION

With the elementary ideas of the preceding section in mind, we may now proceed to a more detailed analysis of the total problem. Following the work of Bernstein[15] and Harris,[2] one can show that for perturbations of the form $\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$ of a steady-state infinite plasma in a uniform magnetic field $\mathbf{B}_0$, the linearized Vlasov equations yield the following general result for the

[15] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).

perturbed distribution of the jth species of particles[16]

$$f_j=-\frac{iZ_jeN_j}{m_j\Omega_j}\left\{\Phi\left[k_\perp\frac{\partial f_{0j}}{\partial v_\perp}I_{1j}+k_{\parallel}\frac{\partial f_{0j}}{\partial v_{\parallel}}I_{2j}\right]+\frac{1}{c}A_1\left[(k_{\parallel}v_{\parallel}-\omega)\frac{\partial f_{0j}}{\partial v_\perp}-k_{\parallel}v_\perp\frac{\partial f_{0j}}{\partial v_{\parallel}}\right]I_{1j}\right.$$
$$\left.+\frac{1}{c}A_2\left[(k_{\parallel}v_{\parallel}-\omega)\frac{\partial f_{0j}}{\partial v_\perp}-k_{\parallel}v_\perp\frac{\partial f_{0j}}{\partial v_{\parallel}}\right]I_{3j}+\frac{1}{c}A_3\left[k_\perp\left(v\frac{\partial f_{0j}}{\partial v_{\parallel}}-v_{\parallel}\frac{\partial f_{0j}}{\partial v_\perp}\right)I_{1j}-\omega\frac{\partial f_{0j}}{\partial v_{\parallel}}I_{2j}\right]\right\}. \quad (27)$$

Here Z_je, N_j, and m_j are, respectively, the charge, number-density, and mass for particles of species j; $\Omega_j=Z_jeB_0/m_jc$ is the (signed) cyclotron frequency; Φ is the electrostatic potential and $\mathbf{A}=(A_1,A_2,A_3)$ is the electromagnetic vector potential if $\mathbf{B}_0=(0,0,B_0)$ and $\mathbf{k}=(k_\perp,0,k_{\parallel})$; and $f_{0j}(v_\perp,v_{\parallel})$, normalized to unity, is the zero-order distribution of particles of type j, where $v_\perp=(v_1^2+v_2^2)^{1/2}$ is the magnitude of the perpendicular velocity and $v_{\parallel}$ the velocity along $\mathbf{B}_0$. The quantities $I_{\lambda j}$ are the integrals

$$I_{\lambda j}=\int_{Z_j^\infty}^{\phi}d\phi' G_j(\phi',\phi)\begin{Bmatrix}\cos\phi'\\ 1\\ \sin\phi'\end{Bmatrix}\quad\text{for}\quad\lambda=\begin{Bmatrix}1\\2\\3\end{Bmatrix}, \quad (28)$$

where

$$G_j(\phi',\phi)\equiv\exp\left[i\frac{k_{\parallel}v_{\parallel}-\omega}{\Omega_j}(\phi-\phi')+i\frac{k_\perp v_\perp}{\Omega_j}(\sin\phi-\sin\phi')\right]. \quad (29)$$

For longitudinal oscillations in a nonrelativistic plasma, the vector potentials may be set equal to zero and ω/k neglected next to the velocity of light, so that if (27) is inserted into Poisson's equation, we obtain a dispersion relation which may be written

$$D(\omega,\mathbf{k})\equiv1+\sum_j(\omega_{pj}^2/\Omega_j^2)F_j=0, \quad (30)$$

where

$$\omega_{pj}^2=4\pi N_jZ_j^2e^2/m_j,$$
$$F_j=i\frac{\Omega_j}{k^2}\int d^3\mathbf{v}\left[k_\perp\frac{\partial f_{0j}}{\partial v_\perp}I_{1j}+k_{\parallel}\frac{\partial f_{0j}}{\partial v_{\parallel}}I_{2j}\right], \quad (31)$$

and the sum in (30) is over particle species.

When f_{0j} is bi-Maxwellian in form, viz.,

$$f_{0j}=(2\pi\kappa T_{\perp j}/m_j)^{-1}(2\pi\kappa T_{\parallel j}/m_j)^{-1/2}\times\exp\left[-\frac{m_jv_\perp^2}{2\kappa T_{\perp j}}-\frac{m_jv_{\parallel}^2}{2\kappa T_{\parallel j}}\right], \quad (32)$$

then the prescription of Bernstein[17] may be followed, allowing one to obtain the expression

$$F_j=\int_0^\infty dx e^{i\omega_jx-\mu_jx^2-\lambda_j(1-\cos x)}[n_{\parallel}^2x+n_\perp^2\sin x], \quad (33)$$

where

$$\omega_j\equiv\omega/|\Omega_j|,\quad \mu_j\equiv\tfrac{1}{2}\kappa T_{\parallel j}k_{\parallel}^2/m_j\Omega_j^2,\ \mathrm{j}\ \lambda_j\equiv\kappa T_{\perp j}k_\perp^2/m_j\Omega_j^2,$$

and

$$n_{\parallel}^2=k_{\parallel}^2/k^2=1-n_\perp^2.$$

By a small modification of the derivation, one can also show that if

$$f_{0j}=f_{\perp j}(v_\perp)(2\pi\kappa T_{\parallel j}/m_j)^{-1/2}\exp\left(-\frac{m_jv_{\parallel j}^2}{2\kappa T_{\parallel j}}\right), \quad (34)$$

then

$$F_j=\left\langle\int_0^\infty dx e^{i\omega_jx-\mu_jx^2}\times J_0(2\beta_j\sin\tfrac{1}{2}x)[n_{\parallel}^2x+n_\perp^2\sin x]\right\rangle, \quad (35)$$

where

$$\beta_j\equiv|k_\perp v_\perp/\Omega_j|$$

and

$$\langle\psi(\beta_j)\rangle\equiv2\pi\int_0^\infty v_\perp dv_\perp f_{\perp j}(v_\perp)\psi(\beta_j).$$

When $f_\perp$ is Maxwellian, Eq. (33) is regained, whereas if $f_\perp$ is a delta function in perpendicular speed, the angular brackets in (35) drop away.

Finally, when f_{0j} is a Lorentzian in the parallel direction,

$$f_{0j}=f_{\perp j}(v_\perp)\frac{u_j/\pi}{u_j^2+v_{\parallel}^2}, \quad (36)$$

it is shown in Appendix A that one may write

$$F_j=\left\langle\left[n_{\parallel}^2\frac{\partial}{\partial\zeta_j}-n_\perp^2\frac{\zeta_j}{\beta_j}\frac{\partial}{\partial\beta_j}\right]\zeta_j^{-1}\Upsilon_{\zeta_j}(\beta_j)\right\rangle. \quad (37)$$

Here $\zeta_j\equiv\omega_j+i|k_{\parallel}u_j/\Omega_j|$, and

$$\Upsilon_\zeta(\beta)\equiv-i\zeta\int_0^\infty dx e^{i\zeta x}J_0(2\beta\sin\tfrac{1}{2}x) \quad (38a)$$
$$=\zeta\csc(\pi\zeta)\int_0^\pi dx\cos(\zeta x)J_0(2\beta\cos\tfrac{1}{2}x) \quad (38b)$$
$$=(\pi\zeta)\csc(\pi\zeta)J_\zeta(\beta)J_{-\zeta}(\beta) \quad (38c)$$
$$=1+\sum_{m=1}^\infty\frac{(2m)!}{2^{2m}(m!)^2}\frac{\beta^{2m}}{(\zeta^2-1^2)(\zeta^2-2^2)\cdots(\zeta^2-m^2)} \quad (38d)$$
$$=\sum_{k=-\infty}^{\infty}J_k^2(\beta)[\zeta/(\zeta-k)], \quad (38e)$$

[16] This result is derived only for $\mathrm{Im}(\omega)>0$, corresponding to the search for instabilities.

[17] I. B. Bernstein, Ref. 15, Appendix II.

where $J_\zeta(\beta)$ is a Bessel function whose index ζ is in general complex. (Note, however, that $\mathrm{Im}(\zeta_j)=\mathrm{Im}(\omega_j)+|k_\parallel u_j/\Omega_j|>0$.) The integral representations (33), (35), and (37) are quite useful and often may be preferable to the infinite sums of Harris[2] and Dnestrovsky *et al.*[5] which, however, may be regained by use of (38e). In particular, when u_j is small so that Landau damping of waves travelling in the parallel direction is small, the representations (38b) and (38c) show very compactly the resonances at integral multiples of the cyclotron frequency via the $\csc(\pi\zeta)$ term. We also note that from (37) and (38d), only β_j^2 and ζ_j^2 enter into the determination of F_j.

4. SPECIAL LIMITS AND PRELIMINARY COMMENTS

Because of its simplicity we will be particularly concerned here with the case of a two-component plasma in which the electrons are represented by the bi-Maxwellian (32) and the ions by (36), especially when $f_{\perp i}(v_\perp)=(2\pi v_{\perp i})^{-1}\delta(v_\perp-v_{\perp i})$ or when $f_{\perp i}$ is Maxwellian. However, before we go into the details of these special cases, we will discuss a few of the features of the more general problem, particularly as it relates to the work of Harris[1,2] and Dnestrovsky, Kostomarov, and Pistunovich.[5] In the first place, we note that for instabilities associated with the ion cyclotron motion, there is a real qualitative difference in the solution of (30) obtained when the electron temperature in the parallel direction, $T_{\parallel e}$, is set equal to zero as opposed to the case when $T_{\parallel e}/T_{\perp i}$ is considered finite.[18] As shown by Harris,[1,2] the contribution of the ions to $D(\omega,\mathbf{k})$, Eq. (30), vanishes as $(m_e/m_i)^{1/2}$ compared with the electron contribution when $T_{\parallel e}/T_{\perp i}$ is first set equal to zero. On the other hand, we will find that this is no longer true (indeed, as in Sec. 6, even the opposite behavior can be exhibited) when the small-mass-ratio approximation is made for finite $T_{\parallel e}/T_{\perp i}$. To see our point most clearly, let us first assume that $(T_{\perp e}/T_{\perp i})\ll(m_i/m_e)$, so that the ion cyclotron radius is large compared to that of the electrons.[19] For the ion cyclotron instabilities in which we are interested, $k_\perp\rho_i\sim1$, where ρ_i is the ion cyclotron radius. Thus, for electrons whose distribution is bi-Maxwellian, $\lambda_e=\frac{1}{2}k_\perp^2\rho_e^2$ is negligible in (33). Then, introducing the so-called plasma dispersion function[20]

$$Z(\zeta)=2i\int_0^\infty dy\, e^{2i\zeta y-y^2}, \tag{39}$$

one may write

$$F_e\cong-\frac{n_\parallel^2}{4\mu_e}Z'\left(\frac{\omega_e}{\sqrt{4\mu_e}}\right)-\left(\frac{n_\perp^2}{\sqrt{4\mu_e}}\right)\frac{1}{2}\left[Z\left(\frac{\omega_e+1}{\sqrt{4\mu_e}}\right)-Z\left(\frac{\omega_e-1}{\sqrt{4\mu_e}}\right)\right].$$

In addition, both μ_e and ω_e are small, so that inserting the asymptotic form of Z for large argument[20] into the second term, and substituting the definitions of μ_e and ω_e in the first term, writing $v_{\parallel e}=(2\kappa T_{\parallel e}/m_e)^{1/2}$, we find

$$F_e\cong\frac{-\Omega_e^2}{(2\kappa T_{\parallel e}/m_e)k^2}Z'(\omega/k_\parallel v_{\parallel e})+n_\perp^2. \tag{40}$$

Putting (40) in (30), $n_\perp^2\omega_{pe}^2/\Omega_e^2$ can be neglected next to unity and our dispersion relation becomes

$$0=D(\omega,\mathbf{k})\cong1+k^{-2}d_e^{-2}\times[-\tfrac{1}{2}Z'(\omega/k_\parallel v_{\parallel e})]+\omega_{pi}^2\Omega_i^{-2}F_i, \tag{41}$$

where $d_e=(\kappa T_{\parallel e}/4\pi N_e e^2)^{1/2}$ is the Debye length corresponding to the temperature $T_{\parallel e}$. Note that both the real and imaginary parts of the term in square brackets are of less than unit magnitude when $\mathrm{Im}(\omega)>0$,[20] and that its imaginary part has the sign of $\mathrm{Re}(\omega)$.[21]

To obtain Harris' result for the electron term in (41), one merely lets $T_{\parallel e}\to0$, in which case the term takes on the value $-n_\parallel^2\omega_{pe}^2/\omega^2$. However, note that unless $n_\parallel^2$ is small,[22] the fact that $\omega\sim\Omega_i$ and $k\sim\rho_i^{-1}$ means that a condition for validity of Harris' analysis is that the parallel electron *velocity* be *small* compared to the ion velocity normal to the magnetic field. This is a severe restriction, although it may be nearly true under some of the experimental conditions in machines such Phoenix[8] and ALICE.[9] Nevertheless, even in these machines, a very small amount of electron heating would eliminate this possibility and it is desirable to examine the situation when the approximation is not made.

The equivalent of Eq. (41) using a series representation of (33) for F_i has also been studied by Dnestrovsky, Kostomarov, and Pistunovich[23] who do not make Harris' implicit assumption that parallel electron velocities are small compared to perpendicular ion velocities. Nevertheless, these authors do use the asymptotic form of $Z'(\zeta)$ for large argument in (41), in effect considering $n_\parallel=|k_\parallel/k|\ll(m_e/m_i)^{1/2}$ in order to make $\omega/k_\parallel v_{\parallel e}$ large. However, this approximation is not explicitly stated, and indeed is inconsistently used when they obtain as a result the instability criterion $\omega_{pe}>\frac{1}{2}\Omega_i$ together with the conclusion that the most

[18] We use $T_\perp$ somewhat loosely in this discussion. When $f_{\perp i}(v_\perp)$ is not Maxwellian, $\kappa T_{\perp i}$ is to be interpreted as a characteristic mean energy of the ion's perpendicular motion.

[19] This approximation, which was not made by Harris, nevertheless is appropriate to almost all systems of current physical interest in which the initial nonrelativistic approximations hold. Certainly this is true of realistic systems for which the parallel electron velocity is small compared to the transverse ion velocity, and this is the limit in which we will argue that Harris' results are valid.

[20] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

[21] The tables of Ref. 20 imply this. A proof is given following Eq. (13) of Ref. 7(a).

[22] Harris [Ref. 1(b)] eventually assumes $n_\parallel^2\sim1$ to obtain his instability criterion: $\omega_{pe}^2>\Omega_i^2$, unstable.

[23] Reference 5, Eq. (9). Note that their $i\pi^{1/2}W(\zeta)$ is our $Z(\zeta)$.

rapid growth rate is attained when $\omega_{pe}n_{\parallel}/\Omega_i$ is nearly an integer. (For then $\omega_{pe}/\Omega_i \gg (m_i/m_e)^{1/2} \gg 1$ is implicitly assumed!) In fact, since they eventually neglect the contribution of $\mathrm{Re}(F_i)$ in (41), we can see immediately that $\omega_{pe} > \frac{1}{2}\Omega_i$ cannot be a general condition for instability in their limit. Because the maximum value of $\mathrm{Re}[Z'(\zeta)]$ for $\mathrm{Im}(\zeta)>0$ is 0.57, it is necessary that $k^2d_e^2 > 0.28$. But $k\rho_i \sim 1$, so that a condition assuring that $\rho_i > \sim d_e$ is at least required for the validity of their results.

We will not go on to an analysis of the whole solution to (41) when the ions are bi-Maxwellian, but we do mention here that a number of theorems can be proved regarding regions in the complex ω plane in which no solutions can occur. The details are given in the references of Footnote 7, the principal results being that conditions of marginal instability $[\mathrm{Im}(\omega)=0^+]$ are possible only if both $l+\frac{1}{2} < \omega_i < l+1-T_{\parallel i}/T_{\perp i}$ (where $l=0, 1, 2, \cdots$) and $\omega_i < T_{\perp i}/T_{\parallel i} - 1$. Hence in particular the system is stable if $T_{\parallel i}/T_{\perp i} > \frac{1}{2}$.

For reasons of analytic simplicity, and in order to be free to consider distributions of perpendicular velocity that may more nearly correspond to experiment, we will drop further consideration of the system analyzed by Dnestrovsky, Kostomarov, and Pistunovich and will concentrate instead on the use of (37) for the ion term in (41). This then corresponds to a situation in which the electrons have a Maxwellian distribution of parallel velocity, the ions being Lorentzian in this direction. For simplicity, we will also assume that $a_i \equiv |k_{\parallel}u_i/\Omega_i|$ is small, and will then look for instabilities near integral multiples of the ion cyclotron frequency. Specifically, we write $\zeta_i \equiv \omega_i + ia_i = l+\eta$, where l is an integer and $\mathrm{Im}(\eta) = a_i + \mathrm{Im}(\omega_i) > 0$, and we consider $|\eta| \ll 1$. Because $-\omega^* = -\mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega)$ satisfies the dispersion relation if ω does, we need look for solutions only in the first quadrant of the complex ω plane.

Using (38b), one can easily show from Bessel's equation that

$$\frac{1}{\beta}\frac{\partial}{\partial\beta}\Upsilon_\zeta(\beta) = \csc(\pi\zeta) \times \int_0^\pi dx\,\sin(\zeta x)\,\sin x J_0(2\beta\cos\tfrac{1}{2}x) \quad (42)$$

so that substituting $l+\eta$ for ζ_i in (37), using (38b) for Υ_ζ and expanding in powers of η, one finds

$$F_i = (-)^l \left\langle \left[n_{\parallel}^2 \frac{\partial}{\partial\eta} - n_{\perp}^2 \frac{l+\eta}{\beta}\frac{\partial}{\partial\beta} \right] \csc\pi\eta \int_0^\pi dx\,\cos[(l+\eta)x] J_0(2\beta\cos\tfrac{1}{2}x) \right\rangle$$

$$= (-)^l \left\langle \frac{1}{\pi}\int_0^\pi dx J_0(2\beta\cos\tfrac{1}{2}x) \left\{ -\frac{n_{\parallel}^2\cos lx}{\eta^2} + \frac{n_{\perp}^2 \sin x\,\sin lx}{\eta} + \left[n_{\parallel}^2\left(\frac{\pi^2-3x^2}{6}\right) + n_{\perp}^2 x\,\sin x \right]\cos lx \right.\right.$$
$$\left.\left. + \eta\left[-n_{\parallel}^2\left(\frac{\pi^2-x^2}{3}\right)x + n_{\perp}^2\left(\frac{\pi^2-3x^2}{6}\right)\sin x \right]\sin lx + \cdots \right\}\right\rangle. \quad (43)$$

Using the identity $\sin x\,\sin lx = \frac{1}{2}[\cos(l-1)x - \cos(l+1)x]$, the first two terms may be integrated analytically,[24] obtaining

$$F_i = -n_{\parallel}^2 \langle J_l^2(\beta)\rangle \eta^{-2} + n_{\perp}^2 \langle P_l(\beta)\rangle \eta^{-1} + [n_{\perp}^2\langle C_l^{\perp}(\beta)\rangle - n_{\parallel}^2\langle C_l^{\parallel}(\beta)\rangle] + [n_{\perp}^2\langle D_l^{\perp}(\beta)\rangle + n_{\parallel}^2\langle D_l^{\parallel}(\beta)\rangle]\eta + \cdots, \quad (44)$$

where $J_l(\beta)$ is a Bessel function and $P_l(\beta)$ has been previously defined in Eq. (16), and where

$$C_l^{\perp}(\beta) \equiv (-)^l \pi^{-1} \int_0^\pi dx J_0[2\beta\cos(\tfrac{1}{2}x)] x\sin(x)\cos(lx),$$
$$C_l^{\parallel}(\beta) \equiv (-)^l \pi^{-1} \int_0^\pi dx J_0[2\beta\cos(\tfrac{1}{2}x)] \tfrac{1}{6}[3x^2-\pi^2]\cos(lx), \quad (45)$$

$$D_l^{\perp}(\beta) \equiv (-)^{l+1} \pi^{-1} \int_0^\pi dx J_0[2\beta\cos(\tfrac{1}{2}x)] \tfrac{1}{6}[3x^2-\pi^2]\sin(x)\sin(lx),$$
$$D_l^{\parallel}(\beta) \equiv (-)^{l+1} \pi^{-1} \int_0^\pi dx J_0[2\beta\cos(\tfrac{1}{2}x)] \tfrac{1}{3}[\pi^2-x^2] x\sin(lx). \quad (46)$$

When (44) is used in (41), we see that both the electron and the ion terms depend strongly on whether $n_{\parallel}$ is assumed small or not. In addition, and depending on the various parameters involved, the presence of the electrons may or may not be essential to the onset of instability. In the next section we will begin by focussing our attention on those instabilities in which both species of particles play an essential part.

[24] W. Gröbner and N. Hofreiter, *Integraltafel* (Springer-Verlag, Vienna, 1961), 3rd ed., Part 2, p. 201.

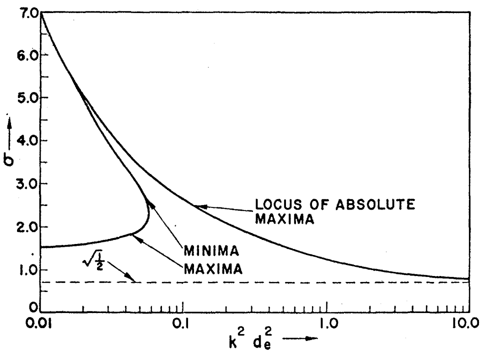

FIG. 2. Values of the parameter σ extremizing $\mathrm{Im}(\eta)$ versus $k^2d_e^2$ for Type-A electron-ion instabilities.

5. COUPLING BETWEEN UNLIKE PARTICLES (ELECTRON-ION INSTABILITIES)

We will find in this section, in addition to unstable modes such as those found by Harris[1,2] in the limit of very low electron temperature, that instabilities arising because of the interaction of the electron and ion terms in the dispersion relation also occur (in systems which can support long axial wavelengths) which depend on the finiteness of the electron temperature. In all of these cases, the tendency is for the direction cosine of the propagation vector to be small along magnetic field lines, typically $n_{\parallel}\sim v_{\perp i}/v_{\parallel e}$ where $v_{\perp i}$ is a characteristic transverse speed of the ions and $\frac{1}{2}m_ev_{\parallel e}^2=\kappa T_{\parallel e}$. Thus a typical electron tends to move about a wavelength in the axial direction in the same time it takes an ion to move a wavelength in the transverse direction, i.e., of the order of a cyclotron orbit diameter. The wavelength of these instabilities along the magnetic field lines, therefore, is usually very long.

We will find two distinct kinds of electron-ion instability—one class which depends sensitively on the form of the distribution of ion velocities normal to the magnetic field, and another class which does not. In accord with the mechanical picture previously given, the first of these is an interaction in which the ions couple only to the transverse electric field, whereas in the second case the instabilities depend on the ability of the ions to move longitudinally in perturbation. As we have seen, the first class of unstable modes should occur only when the distribution of perpendicular energy has a hump in it (for otherwise the energy flow is *from* the wave *to* the ions at small times) and for purposes of reference we will call these Type-A modes. We will study these primarily for the case in which the transverse distribution of ion energies is a delta function. The other class of electron-ion modes, which we will call Type B, occur because the ions can shift their phase with respect to the electric field by moving longitudinally. These are restricted in occurrence by energy flow considerations only to the extent that Eq. (26) be negative. Type-B modes will be studied here primarily when the distribution of ion transverse velocities is Maxwellian.

A. Type-A Electron-Ion Instabilities

If we assume that ionic velocities are small compared to electronic velocities and choose $n_{\parallel}\sim v_{\perp i}/v_{\parallel e}$ so that $\sigma\equiv\omega/k_{\parallel}v_{\parallel e}$ is of order unity near cyclotron resonance, then as long as $n_{\parallel}^2\sim(v_{\perp i}/v_{\parallel e})^2\ll|\eta|\ll 1$, only the second term of (44) contributes appreciably to F_i and (41) becomes

$$0=D(\omega,\mathbf{k})\cong 1+k^{-2}d_e^{-2}[-\tfrac{1}{2}Z'(\sigma)]+\omega_{pi}^2\Omega_i^{-2}\langle P_l\rangle\eta^{-1}, \quad (47)$$

whence

$$\eta=\frac{\omega_{pi}^2}{\Omega_i^2}\frac{\langle P_l\rangle}{1+k^{-2}d_e^{-2}[-\frac{1}{2}Z'(\sigma)]}. \quad (48)$$

Because the electron term contributes a *positive* imaginary part to the denominator, instabilities occur only when $\langle P_l\rangle$ is positive, confirming the result already found in Sec. 2 for purely transversely coupled ions. As we have seen, $\langle P_l\rangle$ can be positive only in the case of a humped distribution of transverse energy, and in particular if we have a delta function distribution of perpendicular energy at energy $\kappa T_{\perp}$, $\langle P_l\rangle=P_l(k_{\perp}\rho_i)$ where $\rho_i=(2\kappa T_{\perp}/m_i\Omega_i^2)^{1/2}$ is the ion cyclotron radius. Thus, from Eq. (16), instability is possible only if $k_{\perp}\rho_i>j_l'$ where j_l' is the first maximum of J_l. For example, $j_l'=1.841$, 3.054, 4.201, 5.317 for $l=1$, 2, 3, 4. (It can be shown[25] that in general $j_l'>l$ and that asymptotically for large l, we have $j_l'\sim l+0.808l^{1/3}$.[26])

To the present small $n_{\parallel}$ approximation, we may replace $k_{\perp}$ in $\langle P_l\rangle$ by k. Then we may choose $n_{\parallel}$ (which now appears only in σ and, through the ion Landau damping term a_i, in η) so as to maximize the growth rate.[27] If for the moment we assume the damping is small in order to get an idea of the magnitude of the growth rate, we may choose $\sigma(\cong l\Omega_i/k_{\parallel}v_{\parallel e})$ so as to maximize $\mathrm{Im}(\eta)$. Figure 2 shows those values of σ extremizing $\mathrm{Im}(\eta)$ plotted versus $k^2d_e^2$. When $k^2d_e^2<0.06$, three roots appear although for larger values only one root occurs. In the former case, the extrema correspond to two maxima and one minimum, whereas for larger $k^2d_e^2$ the single solution maximizes $\mathrm{Im}(\eta)$. In both cases, however, the same curve is the locus of *absolute* maxima, so that for unrestricted σ, the extra roots at small $k^2d_e^2$ may be ignored. No maxima occur for $\sigma<(1/2)^{1/2}$.

When $k^2d_e^2$ is large, $\sigma\cong(1/2)^{1/2}$, $\mathrm{Im}[-\frac{1}{2}Z'(\sigma)]\cong(\pi/2e)^{1/2}$, and

$$\eta\cong\omega_{pi}^2\Omega_i^{-2}[-1+i(\pi/2e)^{1/2}k^{-2}d_e^{-2}]\langle P_l\rangle, \qquad k^2d_e^2\gg 1. \quad (49)$$

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[25] G. N. Watson, Ref. 13, p. 485.

[26] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., 1953), Vol. II, p. 1565.

[27] Recall that growth rate $=\mathrm{Im}(\omega_i)=\Omega_i[\mathrm{Im}(\eta)=a_i]=\Omega_i\,\mathrm{Im}(\eta)-n_{\parallel}ku_i$.

On the other hand, when $k^2d_e^2$ is small, the maximum of the imaginary part in (48) occurs for σ such that the real part of the denominator vanishes. (This is the regime which should most nearly correspond to the work of Dnestrovsky, Kostomarov, and Pistunovich.[5]) Through use of asymptotic relation[20]

$$-\tfrac{1}{2}Z'(\sigma)\sim i\pi^{\frac{1}{2}}\sigma e^{-\sigma^2}-\frac{1}{2\sigma^2}\left[1+\frac{3}{2\sigma^2}+\cdots\right],$$

the real part of the denominator in (48) vanishes for $\sigma\cong(2k^2d_e^2)^{-1/2}$, corresponding to $n_{\parallel}\omega_{pe}=l\Omega_i$, and[28]

$$\eta=i\omega_{pi}^2\Omega_i^{-2}\langle P_l\rangle(2/\pi)^{1/2}k^3d_e^3 \times\exp(\tfrac{1}{2}k^{-2}d_e^{-2}),\quad k^2d_e^2\ll 1. \quad (50)$$

Thus the growth rate appears to *increase* exponentially with $k^{-2}d_e^{-2}$. Of course this cannot be true indefinitely, for the approximation $|\eta|\ll 1$ will soon break down. In particular, one difficulty will be that the electron contribution to $\mathrm{Im}[\tilde{D}(\omega,\mathbf{k})]$, which we retained in (47), now becomes exponentially small, whereas we have already thrown out imaginary contributions of first order in η. More practically, however, in a given experimental system it becomes more and more difficult to fit in the required long axial wavelengths (which are proportional to σ) for the larger values of σ.

In any event, the large growth rates may occur in infinite systems when $2k^2d_e^2<\mathrm{Max}\{\mathrm{Re}[Z'(\sigma)]\}=0.57$. Hence $k^2d_e^2=0.28$ effectively defines the stability boundary in the sense that large growth rates are possible when $k^2d_e^2$ can be made smaller than this value. The transition from "small" to "large" values of $\mathrm{Im}(\eta)$ is exponential, as shown by (50), although the actual limiting growth rate is determined by other considerations. On the other hand, k cannot be made so small that $\langle P_l\rangle$ becomes negative, and for a delta-function distribution this means that $k\rho_i>j_1'=1.841$. Thus we have the condition for "strongly" unstable growth in an *infinite* system:

$$3.4d_e<\rho_i,\quad \text{unstable}. \quad (51)$$

We can also obtain an instability condition for the more slowly growing modes which occur when (51) is violated. We write,[27] using (48),

$$\mathrm{Im}(\omega)=l\Omega_i\sigma^{-1}\{-\omega_{pi}^2\Omega_i^{-2}l^{-1}\langle P_l\rangle \times\sigma\,\mathrm{Im}[1-\tfrac{1}{2}k^{-2}d_e^{-2}Z'(\sigma)]^{-1}-u_i/v_{\parallel e}\},$$

where we have used $\sigma\cong l\Omega_i/n_{\parallel}kv_{\parallel e}$ to eliminate $n_{\parallel}$. At the stability boundary, the maximum of the first term in braces just cancels the second. Moreover, when $2k^2d_e^2$ becomes larger than 0.57,

$$-\sigma\,\mathrm{Im}[1-\tfrac{1}{2}k^{-2}d_e^{-2}Z'(\sigma)]^{-1}\cong-\tfrac{1}{2}k^{-2}d_e^{-2}\sigma\,\mathrm{Im}Z'(\sigma),$$

which maximizes when $\sigma=1$, giving

$$\mathrm{Im}(\omega)\cong l\Omega_i\left\{\left(\frac{2\pi^{1/2}}{e}\right)\left(\frac{\omega_{pi}^2}{\Omega_i^2}\right)^2\frac{T_{\perp i}}{T_{\parallel e}}\frac{\langle P_l\rangle}{lk^2\rho_i^2}-\frac{u_i}{v_{\parallel e}}\right\}. \quad (52)$$

Putting in the maximum value of $l^{-1}k^{-2}\rho_i^{-2}P_l(k\rho_i)$ ($=0.016$; $l=1$, $k\rho_i=2.4$) for the delta-function distribution, we find that to have growth in the range $3.4d_e>\rho_i$,

$$(\omega_{pi}^2/\Omega_i^2)^2\gtrsim 50(m_e/m_i)^{1/2} \times[(T_{\parallel e}T_{\parallel i})^{1/2}/T_{\perp i}],\quad\text{unstable}. \quad (53)$$

Let us return to the point raised above regarding the suppression of long wavelengths in the axial direction due to the finite length of any real laboratory apparatus. (A qualitative discussion of the applicability of such criteria to this and other cases is given in Appendix B.) We note that it is unusual to find any experiment in which a machine is more than, say, 10 cyclotron-orbit diameters in length. Thus if $\lambda_{\parallel}=2\pi/k_{\parallel}$ and we choose $k=j_1'\rho_i^{-1}$ corresponding to the boundary of instability when (51) just begins to be satisfied, then the smallest wavelength $\lambda_{\parallel}$ which is unstable is $\lambda_{\parallel}=(\pi j_1'/n_{\parallel})(2\rho_i)$, so that for practical systems $\pi j_1'\lesssim 10n_{\parallel}$. More generally, if we take L to be the maximum longitudinal wavelength supportable by the system in units of cyclotron-orbit diameters, we must have $\sigma<(L/\pi)(v_{\perp i}/v_{\parallel e})$ for the instability to occur. With $L\lesssim 10$ practically, σ will be limited to rather small values unless the ions are *very* much more energetic than the electrons. Because $k^2d_e^2$ cannot get very large (for rapid growth), this means that we are in fact usually below the lower branch in Fig. 2 rather than near the absolute maximum of $\mathrm{Im}(\eta)$. In any event, we are in a regime in which the smallest possible $n_{\parallel}$ gives the greatest instability.

If we are limited as above to small σ, $-\tfrac{1}{2}Z'(\sigma)\cong 1+i\pi^{1/2}\sigma$ and (48) becomes for a delta-function distribution[18]

$$\eta=\frac{T_{\parallel e}}{T_{\perp i}}\frac{\tfrac{1}{2}\beta^2}{1+x^2\beta^2}\left[-1+\frac{i\pi^{1/2}\sigma}{1+x^2\beta^2}\right]P_l(\beta), \quad (54)$$

where $x\equiv d_e/\rho_i$ and $\beta\equiv k_{\perp}\rho_i\cong k\rho_i$. Setting

$$a_i=k_{\parallel}u_i/\Omega_i=2\pi u_i/\lambda_{\parallel}\Omega_i=(\pi/L)(u_i/v_{\perp i}),\quad\text{for}\quad l=1,$$

$$\mathrm{Im}(\omega)=\Omega_i\left\{\frac{L}{2\pi^{1/2}}\frac{T_{\parallel e}}{T_{\perp i}}\frac{\beta^2P_1(\beta)}{(1+x^2\beta^2)^2}\frac{v_{\perp i}}{v_{\parallel e}}-\frac{\pi}{L}\frac{u_i}{v_{\perp i}}\right\}. \quad (55)$$

Now $\beta^2P_1(\beta)$ is an oscillating function of nearly constant amplitude, so that $\beta^2P_1(\beta)[1+x^2\beta^2]^{-2}$ is a maximum near the first maximum of $\beta^2P_1(\beta)$ (since x cannot be too large if we are to have instability). Thus $\beta^2\cong 7.7$ and

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[28] A convenient approximation for computing the maximum value of $\mathrm{Im}(\eta)$ from (54) is

$$\mathrm{Max}[\mathrm{Im}(\eta)]=\omega_{pi}^2\Omega_i^{-2}\langle P_l\rangle(\pi/2e)^{1/2}k^{-2}d_e^{-2},\quad 2k^2d_e^2>0.57,$$
$$=\omega_{pi}^2\Omega_i^{-2}\langle P_l\rangle\{-\mathrm{Re}[Z'(\sigma_0)]/\mathrm{Im}[Z'(\sigma_0)]\},\quad 2k^2d_e^2<0.57,$$

where σ_0 is determined from $\tfrac{1}{2}\mathrm{Re}[Z'(\sigma_0)]=k^2d_e^2$. The worst error occurs when $k^2d_e^2$ is near unity, where the true value is about $\tfrac{3}{4}$ that given by this approximate form.

$\beta^2 P_1(\beta) \cong 0.76$, and

$$\operatorname{Im}(\omega) \cong \frac{\pi\Omega_i}{L}\left(\frac{m_e T_{\parallel e}}{m_i T_{\perp i}}\right)^{1/2} \times\left[\frac{0.068L^2}{(1+7.7d_e^2\rho_i^{-2})^2}-\left(\frac{m_i T_{\parallel i}}{m_e T_{\parallel e}}\right)^{1/2}\right], \quad (56)$$

using $\frac{1}{2}m_i u_i^2 = \kappa T_{\parallel i}$. Hence we have the instability criterion for finite machines:

$$L^2 > 15(m_i T_{\parallel i}/m_e T_{\parallel e})(1+7.7d_e^2\rho_i^{-2}), \quad \text{unstable}. \quad (57)$$

Inequality (57) is a severe limitation on the possibility of finding such instabilities in ordinary experiments.

We may summarize this section by observing that although an infinite system is unstable with rather large growth rates when (51) is satisfied [Eq. (50) begins to break down when $\operatorname{Im}(\omega)$ approaches the smaller of ω_{pi} or Ω_i], or with smaller growth rates when (53) is satisfied, these instabilities occur for very long axial wavelengths. In a finite machine L cyclotron orbit diameters in length, the suppression of long wavelengths leads to the stability criterion (57) which is much less severe, with growth rates being given by Eq. (56).

The principal approximation of the analysis is that $(v_{\perp i}/v_{\parallel e})^2 \ll |\eta| \ll 1$, and from Eq. (54) this means that at the condition of maximum growth $(m_e/m_i)(T_{\perp i}/T_{\parallel e}) \ll 0.38(T_{\parallel e}/T_{\perp i})(1+7.7d_e^2/\rho_i^2) \ll 1$. It is the first of these inequalities that is the most critical, since that is what suppressed the first term in the expansion (44), and we will consider in the next section what happens when this term is taken into account. When the second inequality begins to be violated, the failure is due primarily to the relatively large real frequency shift below resonance, and an expansion in powers of $\operatorname{Im}(\eta)$ may be used instead,[29] or the problem may be attacked numerically. It is unlikely that a material change in the conclusions would occur, however.

B. Type-B Electron-Ion Instabilities

When the distribution of transverse ion velocities is a monotonic function of the speed, so that $\langle P_l \rangle$ is negative and Type-A instabilities are proscribed, the coupling provided by the *first* term in the expansion (44) can still give rise to other forms of instability. The first of these is the instability originally studied by Harris[1,2] and by Dnestrovsky *et al.*,[5] in which the electrons make a large negative real contribution to the dispersion relation—overcoming the Laplacian contribution represented by unity in Eq. (41). In order for this to occur, however, σ must be larger than 0.925 and as we observed in Sec. 4 this is rarely possible experimentally, at least if one tries to use the expedient of Dnestrovsky *et al.* Unless the electrons are truly cold, as Harris originally assumed, the long axial wavelengths required cannot be supported in a practical experiment.

Semiquantitatively, when $\sigma(\cong l\Omega_i/k_\parallel v_{\parallel e})$ can be made larger than about 3 or 4, the asymptotic form of $Z'(\sigma)$ can be substituted in and the dispersion relation, Eq. (41), becomes

$$1-n_\parallel^2\omega_{pe}^2/(l^2\Omega_i^2)-n_\parallel^2\langle J_l^2\rangle\eta^{-2}\cong 0. \quad (58)$$

Thus instability is possible when $n_\parallel^2\omega_{pe}^2 > l^2\Omega_i^2$. As we noted, however, in order to justify use of the asymptotic form for $Z'(\sigma)$, we require $\sigma^2 \gtrsim 10$ or $l^2\Omega_i^2 \gtrsim 10k_\parallel^2 v_{\parallel e}^2$. Hence $\omega_{pe}^2 \gtrsim 10k^2 v_{\parallel e}^2$ or, for instability,

$$\omega_{pi}^2/\Omega_i^2 \gtrsim 10(k\rho_i)^2(T_{\parallel e}/T_{\perp i}). \quad (59)$$

Using the smallest permissible $k\rho_i$ for a given system allows one to estimate its susceptibility to unstable Harris modes from (59). One must keep in mind that (59) is not sufficient, however, and that this instability only goes when the electron temperature is so small that σ can be made at least larger than unity, despite the finite machine length. We will not try to refine these results here, but defer further discussion of cold-electron plasmas to the specific experiments covered in Sec. 8.

Another instability can occur if one capitalizes on the dissipative rather than the reactive contribution of the electrons to $D(\omega,\mathbf{k})$—indeed this is just what was done in obtaining the Type-A instabilities, except that we used the second rather than the first term of (44) to provide the coupling to the ions. We consider these modes next.

Because of the restriction of σ to small values in most important cases, we will confine our analysis to the circumstances in which $Z'(\sigma)$ can be represented by the first two terms in its power series expansion in σ. However, since the instabilities will again tend to come in soonest when $n_\parallel$ is as small as possible within the limits imposed by finite geometry, both of the first two terms in (44) are important and (41) becomes

$$0=D(\omega,\mathbf{k})\cong 1+k^{-2}d_e^{-2}[1+i\pi^{1/2}\sigma] +\omega_{pi}^2\Omega_i^{-2}[-n_\parallel^2\langle J_l^2\rangle\eta^{-2}+n_\perp^2\langle P_l\rangle\eta^{-1}],$$

so that, for the unstable solution,

$$\frac{1}{\eta}=\frac{1}{2\omega_{pi}^2\Omega_i^{-2}n_\parallel^2\langle J_l^2\rangle}[\omega_{pi}^2\Omega_i^{-2}n_\perp^2\langle P_l\rangle -\{[\omega_{pi}^2\Omega_i^{-2}n_\perp^2\langle P_l\rangle]^2+4\omega_{pi}^2\Omega_i^{-2}n_\parallel^2\langle J_l^2\rangle \times[1+k^{-2}d_e^{-2}(1+i\pi^{1/2}\sigma)]\}^{1/2}], \quad (60)$$

where the principal value of the square root is to be taken. Moreover, if the distribution of transverse velocities is Maxwellian and we define $\rho_i^2 \equiv 2\kappa T_{\perp i}/m_i\Omega_i^2$ as before, then

$$\langle P_l\rangle = -2lk_\perp^{-2}\rho_i^{-2}b_l, \quad \langle J_l^2\rangle = b_l; \quad b_l = e^{-\frac{1}{2}k_\perp^2\rho_i^2}I_l(\tfrac{1}{2}k_\perp^2\rho_i^2). \quad (61)$$

[29] See Sec. 6 for an analysis of Type-A ion-ion instabilities using this approach.

When $T_{\perp i}/T_{\parallel e}$ is large, the last term in the square root dominates and

$$\eta \cong n_{\parallel} k \rho_i \left[\frac{\frac{1}{2} b_l (T_{\parallel e}/T_{\perp i})}{1+k^2 d_e^2}\right]^{1/2} \left[-1+i\frac{L\pi^{1/2}\sigma}{1+k^2 d_e^2}\right]. \quad (62)$$

Thus, for $l=1$,

$$\mathrm{Im}(\omega) = n_{\parallel} k (2\kappa T_{\perp i}/m_i)^{1/2} \times \left\{\frac{\sigma}{4}\left[\frac{\pi b_1 (T_{\parallel e}/T_{\perp i})}{(1+k^2 d_e^2)^3}\right]^{1/2} - \left(\frac{T_{\parallel i}}{T_{\perp i}}\right)^{1/2}\right\}, \quad (63)$$

and on the boundary of instability σ takes on its maximum value:

$$(L/\pi)(m_e T_{\perp i}/m_i T_{\parallel e})^{1/2}.$$

The quantity $\alpha \equiv b_1(1+k^2 d_e^2)^{-3}$ maximizes when b_1 maximizes if $d_e^2 \ll \rho_i^2$, and for the reverse inequality α maximizes when $k^2 d_e^2 = \frac{1}{2}$. Hence, writing α_m for the maximum value of α, this means

$$\begin{aligned} \alpha_m &= (1/27)(\rho_i^2/d_e^2), \quad && \rho_i^2 \ll d_e^2, \\ \alpha_m &= 0.219, && \rho_i^2 \gg d_e^2. \end{aligned} \quad (64)$$

Therefore, we have the instability condition from (63)

$$\alpha_m L^2 > 16\pi (m_i T_{\parallel i}/m_e T_{\perp i}), \quad \text{unstable}. \quad (65)$$

Again, it is very difficult to find an experiment in which the length of the machine is sufficiently great that these modes become offensive.

6. INTERACTIONS OF A SINGLE DISTRIBUTION WITH ITSELF (ION-ION INSTABILITIES)

We have seen in the previous section that microinstabilities can arise through the coupling of the ion cyclotron motion to an electrostatic wave which in turn couples either reactively (Harris modes) or dissipatively to the longitudinal motions of the electrons in a plasma. It is also possible for the wave to couple into the parallel motion of the ions, with the electrons playing only a passive role. Such instabilities we call ion-ion instabilities to distinguish them from the previously considered electron-ion modes. Specifically, we characterize the ion-ion instabilities by the requirement that only one species of particles makes other than a nonresonant *positive* real contribution to the dispersion relation, Eq. (30).[30]

When neither the electron temperature nor $n_{\parallel}$ is extremely small, σ is small in Eq. (41) and the dispersion relation takes the form

$$0 = D(\omega,\mathbf{k}) = 1 + k^{-2} d_e^{-2} + \omega_{pi}^2 \Omega_i^{-2} F_i. \quad (66)$$

One may again use the expansion (44) for F_i and try to determine the unstable solutions to (66), but it is now necessary to keep at least the first three terms in order to find *any* solution for other than purely real η, and it appears to be necessary to keep at least the first four terms for an adequate exploration of the whole problem. Thus one begins to worry about the effects of an early termination of the series at all, and in order to present a more convincing analysis we seek an alternative to keeping a large number of terms in the representation (44). The methods we will adopt will vary with the cases examined.

As in the section on electron-ion instabilities, we also find that ion-ion instabilities depend on the two classes of interaction corresponding to the two driving mechanisms discussed in Sec. 2. We will again denote these as Type A or Type B, our nomenclature depending upon which driving mechanism is of principal importance.

A. Type A Ion-Ion Instabilities

The Type A ion-ion instability, as the Type A electron-ion instability, is driven by an interaction which is primarily between a purely transversely propagating electric wave and the cyclotron motions of the particles. Formally, we may separate out this mode by letting $n_{\parallel}^2$ vanish, so that from (37)

$$0 = D(k,\zeta) = 1 + k^{-2} d_e^{-2} - \omega_{pi}^2 \Omega_i^{-2} \langle \beta^{-1} \partial \Upsilon_\zeta(\beta)/\partial\beta \rangle. \quad (67)$$

Again one can easily show[31] that no unstable solution exists for Type-A modes when the distribution of transverse ion energies is monotonic (e.g., Maxwellian). Hence we now restrict our attention to a delta-function distribution at energy $\kappa T_{\perp i}$.

The method of attack which we will use in the present case is equivalent to writing $\zeta = \nu + i\mu$ and expanding the dispersion function in powers of μ.[32,33] If one then keeps only terms through first order, both real and imaginary

[30] It is clear that electron-electron instabilities are covered in a discussion of ion-ion instabilities by making the appropriate change in cyclotron and plasma frequencies. See Sec. 6C.

[31] The proof depends on showing that $\mathrm{Im}(D)$ is nonvanishing. To see this most easily, observe that application of the identity $1 = \sum_{n=-\infty}^{\infty} J_n^2(\beta)$ to (38e) allows one to write $\Upsilon_\zeta = \sum_{n=-\infty}^{\infty} n^2 \times (\zeta^2 - n^2)^{-1} J_n^2(\beta)$, after which the proof becomes trivial. For a transverse Maxwellian, an equivalent proof is also noted by Harris [Ref. 1(b)], the special case of a more general result later obtained by Hall and Heckrotte [Ref. 7(a), footnote 6].

[32] Strictly speaking, in this approximation a solution exists only if $D(k,\nu)=0$ and [cf. Eq. (75)] $G_\nu(\lambda,\nu)=0^+$, since the electrons contribute a positive imaginary part although we take it vanishingly small here. Thus if the curve of $G_\nu(\lambda,\nu)$ touches the axis tangentially from below, but does not cross, there is no instability.

[33] It is interesting as an aside to note that if we define $k(\nu)$ by the requirement that $D(k,\nu)$ vanish, then $0 = dD(k,\nu)/d\nu = D_\nu(k,\nu) + D_k(k,\nu)dk/d\nu$. Hence in order to guarantee the vanishing of $D_\nu(k,\nu)$, it is sufficient that either $D_k(k,\nu)$ or $dk/d\nu$ vanish (in such a way that the product goes to 0^+). Since $d\nu/dk = \Omega_i^{-1} d\omega/dk$, the vanishing of $dk/d\nu$ can be interpreted as the group velocity of the waves becoming infinite. It is at this point that the direction of energy propagation changes direction, i.e., this is the transition point between "forward" and "backward" waves, and thus it corresponds to the condition at which the feedback to a disturbance generated "upstream" goes from negative to positive. Instabilities of such nature are well known in electrical engineering technology, and have been used, for example, in the development of the so-called "backward-wave" oscillators. We will not make explicit use of this result here, however.

parts of the dispersion relation may be solved (in part numerically) for the conditions of instability.

Let us define $\lambda=\frac{1}{2}\beta^2$ and set

$$v_\zeta(\lambda)\equiv J_\zeta(\beta)J_{-\zeta}(\beta)$$
$$=\pi^{-1}\int_0^\pi dx\cos(\zeta x)J_0[2\beta\cos(\tfrac{1}{2}x)]. \quad (68)$$

If a prime is used to denote differentiation with respect to λ, our dispersion relation may then be rewritten

$$\Omega_i^2\omega_{pi}^{-2}+(T_{\perp i}/T_{\parallel e})\lambda^{-1}=\pi\zeta\csc(\pi\zeta)v_\zeta'(\lambda)\equiv G(\lambda,\zeta). \quad (69)$$

Equation (69), except for the second term on the left, is equivalent to that considered by Gross[34] and subsequently by Sen[35] and Harris.[1,2] Although in the previous work[2] instability was thought to be possible when $\beta>1.84$ (the first *maximum* of J_1) corresponding to $\lambda>1.69$, we find that in fact the requirement is $\beta>3.83$ (the first *zero* of J_1), or $\lambda>7.34$.[36]

If we recall Eq. (42) or, equivalently, note the relation

$$\zeta v_\zeta'(\lambda)=\tfrac{1}{2}[v_{\zeta+1}(\lambda)-v_{\zeta-1}(\lambda)], \quad (70)$$

we may multiply (69) through by $\sin(\pi\zeta)$, set $\zeta=\nu+i\mu$, and separate real and imaginary parts, obtaining the pair of equations

$$\Omega_i^2\omega_{pi}^{-2}+(T_{\perp i}/T_{\parallel e})\lambda^{-1}=-\int_0^\pi dx\sin x\sin(\nu x)\cosh(\mu x)J_0[2\beta\cos(\tfrac{1}{2}x)]\Big/\sin(\pi\nu)\cosh(\pi\mu)$$
$$=-\int_0^\pi dx\sin x\cos(\nu x)\sinh(\mu x)J_0[2\beta\cos(\tfrac{1}{2}x)]\Big/\cos(\pi\nu)\sinh(\pi\mu). \quad (71)$$

Since the term in the middle in (71) has a magnitude less than $\pi|\csc(\pi\nu)|$ and the last term is less than $\pi|\sec\pi\nu|$ in magnitude, a necessary condition for instability is that

$$\Omega_i^2\omega_{pi}^{-2}+(T_{\perp i}/T_{\parallel e})\lambda^{-1}<\pi 2^{1/2}. \quad (72)$$

From the equality involving the two integrals in (71) one can show, with a little trigonometry, that

$$\mu^{-1}\int_{-\pi}^{\pi} dx\sin x\sin\nu(\pi-x)\sinh\mu(\pi+x)\times J_0[2\beta\cos(\tfrac{1}{2}x)]=0, \quad (73)$$

which in the limit $\mu\to 0$ becomes

$$\int_{-\pi}^{\pi} dx(\pi+x)\sin x\sin\nu(\pi-x)J_0[2\beta\cos(\tfrac{1}{2}x)]=0. \quad (74)$$

Similarly, taking the limit $\mu\to 0$ of (73) directly,

$$\Omega_i^2\omega_{pi}^{-2}+(T_{\perp i}/T_{\parallel e})\lambda^{-1}$$
$$=-\csc(\pi\nu)\int_0^\pi dx\sin x\sin\nu xJ_0[2\beta\cos(\tfrac{1}{2}x)]$$
$$=-\pi^{-1}\sec(\pi\nu)\int_0^\pi dxx\sin x\cos\nu xJ_0[2\beta\cos(\tfrac{1}{2}x)]. \quad (75)$$

In numerical work, for example, the first integral may be used with (74) when $l+\frac{1}{4}<\nu<l+\frac{3}{4}$ and the second when $l-\frac{1}{4}<\nu<l+\frac{1}{4}$, where l is an integer, in order to determine conditions of marginal instability.

When $\nu=l$, Eq. (74) requires $\lambda=\lambda_{lk}$, the kth zero of v_l' (i.e., $\lambda_{lk}=\frac{1}{2}\beta_{lk}^2$ where β_{lk} is the kth positive zero of $dJ_l^2(\beta)/\partial\beta$). It is convenient, therefore, to classify solutions in general by that branch of the curve $\lambda=\Lambda_k(\nu)$ satisfying (74) which satisfies $\Lambda_k(l)=\lambda_{lk}$.[37] The functions $\Lambda_k(\nu)$ for $1\leq k\leq 4$ were obtained from numerical solution of (74) and the corresponding values of $G(\lambda,\nu)$ [equal to the integrals in (75)] were also computed along these curves. When $k=1$ and 3, $G(\lambda,\nu)$ is negative along $\Lambda_k(\nu)$ so that neither of these cases lead to instability. Figure 3 shows $G(\lambda,\nu)$ along Λ_2 and Λ_4, and $\Lambda_2(\nu)$ is also shown there. Cross plotting the results against λ^{-1} and determining the envelope of its intersection with the

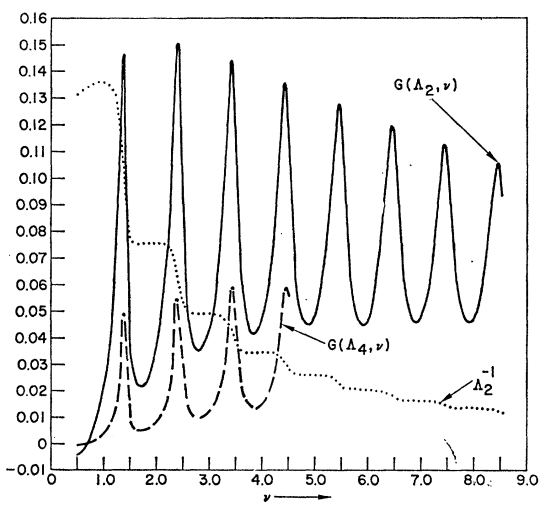

FIG. 3. Showing $G(\lambda,\nu)$ along $\lambda=\Lambda_2(\nu)$ and $\lambda=\Lambda_4(\nu)$, and showing $\Lambda_2^{-1}(\nu)$.

[34] E. P. Gross, Phys. Rev. 82, 232 (1951).
[35] H. K. Sen, Phys. Rev. 88, 816 (1952).
[36] An exception is the special case of zero frequency [Re(ζ)=0] for which instabilities first occur when $\beta>2.40$ (the first zero of J_0). See Appendix C.
[37] We have not investigated $v_\nu'(\lambda)$ completely in order to show that the solutions $\Lambda_k(\nu)$ yield all of the zeros of $v_\nu'(\lambda)$ for nonintegral ν, but this seems a reasonable conjecture.

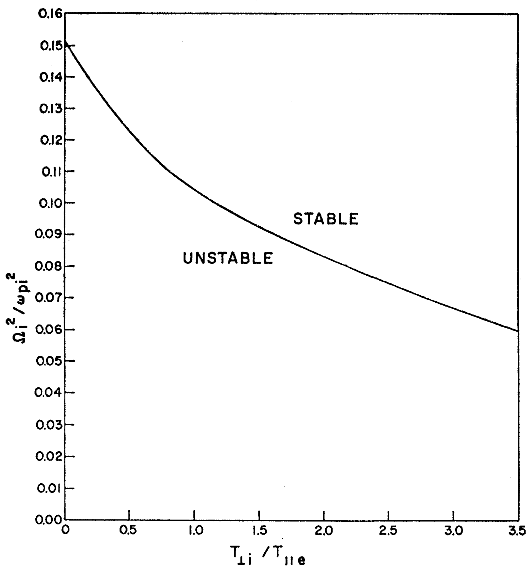

FIG. 4. Showing the instability boundary for Type-A ion-ion instabilities satisfying Eq. (75).

straight lines $a+b\lambda^{-1}$ allows the determination of the instability boundary, Fig. 4. From this we see that a necessary condition for instability is

$$\omega_{pi}^2>6.6\Omega_i^2. \tag{76}$$

B. Type-B Ion-Ion Instabilities

As in the case of electron-ion instability, we may separate out the Type B ion-ion modes by choosing a distribution function for which the Type A modes cannot exist. Thus if the transverse distribution of ion energies is Maxwellian at temperature $T_{\perp i}$, and if we define

$$\lambda_\perp\equiv\kappa T_{\perp i}k_\perp^2/m_i\Omega_i^2=\tfrac{1}{2}k_\perp^2\rho_i^2,$$
$$\lambda_{\parallel}\equiv\tfrac{1}{2}k_{\parallel}^2\rho_i^2=n_{\parallel}^2\lambda_\perp/n_\perp^2,$$

and

$$\bar{\Upsilon}_\zeta(\lambda_\perp)\equiv\lambda_\perp^{-1}\int_0^\infty\beta d\beta\exp(-\beta^2/2\lambda_\perp)\Upsilon_\zeta(\beta)$$
$$=\zeta\csc(\pi\zeta)\int_0^\pi dx e^{-\lambda_\perp(1+\cos x)}\cos(\zeta x), \tag{77}$$

then from (37) and (66)

$$\Omega_i^2\omega_{pi}^{-2}+(T_{\perp i}/T_{\parallel e})(\lambda_\perp+\lambda_\parallel)^{-1}$$
$$=[\bar{\Upsilon}_\zeta(\lambda_\perp)-1](\lambda_\perp+\lambda_\parallel)^{-1}-n_\parallel^2\partial[\zeta^{-1}\bar{\Upsilon}_\zeta(\lambda_\perp)]/\partial\zeta. \tag{78}$$

Equation (78) is intrinsically more complicated than the comparable result (69) obtained for Type-A instabilities because of the presence of the additional free parameter $\lambda_\parallel$(or $n_\parallel^2$). Indeed, if we were to pursue the previous approach and assume $\mathrm{Im}(\zeta)=\mu$ to be small at this point (which cannot be justified anyway in the present finite $n_\parallel$ case unless $T_{\parallel i}/T_{\perp i}$ is very small) it seems to be possible to find a very large range of unstable conditions, for much of which the growth rate could be easily compensated by adding a little Landau damping. Hence before any further approximations are made, we will call to mind some general restrictions regarding possible solutions to (78). Several theorems of a general nature have been explicitly obtained in the references of footnote 7 for the case of a bi-Maxwellian distribution of ions, and the methods described there can also be used for our Maxwellian-Lorentzian distribution, Eq. (36), to show from the behavior of the imaginary part of (78) that

(a) solutions may occur only when $l-\frac{1}{2}<\nu\equiv\mathrm{Re}(\zeta)<l$, where l is an integer;

(b) marginally unstable solutions may occur only if $T_{\parallel i}/T_{\perp i}<\frac{1}{2}$; and

(c) $\mathrm{Im}(\bar{\Upsilon}_\zeta)<0$.

Recalling that $\mu=\mathrm{Im}(\omega/\Omega_i)+a_i$, we note that $\mu^2=\alpha\lambda_\parallel$ where $\alpha=(2T_{\parallel i}/T_{\perp i})[1+\mathrm{Im}(\omega/k_\parallel u_i)]^2$. Clearly, condition (b) above implies that solutions to (78) may occur only when $\alpha<1$. Secondly, however, the imaginary part of (78) may be rewritten

$$\lambda_\parallel=\Lambda(\zeta,\lambda_\perp)\equiv\mathrm{Im}\{\bar{\Upsilon}_\zeta(\lambda_\perp)\}/\mathrm{Im}\{\partial[\zeta^{-1}\bar{\Upsilon}_\zeta(\lambda_\perp)]/\partial\zeta\} \tag{79}$$

and the impossibility of solving this equation unless $\mu^2<\lambda_\parallel$ is equivalent to the fact that $\Lambda(\zeta,\lambda_\perp)$ is negative unless $\mu^2>\Lambda$. Thus Λ changes sign *only* discontinuously, and since (for $\mu>a_i>0$) both numerator and denominator of (79) are continuous and bounded, changes in sign of Λ occur only at zeros of the denominator. Recalling (77) and condition (c), this means that solutions occur only inside regions of the ζ-plane bounded by the real axis and those curves for which

$$0^-=G(\lambda_\perp;\mu,\nu)\equiv[1-\cos(2\pi\nu)\cosh(2\pi\mu)]\int_{-1}^1 dx\exp[-\lambda_\perp(1+\cos\pi x)](1-x^2)\sin[\pi\nu(1-x)]\frac{\sinh[\pi\mu(1-x)]}{\pi\mu(1-x)}$$
$$+2\sin(2\pi\nu)\frac{\sinh(2\pi\mu)}{2\pi\mu}\int_{-1}^1 dx\exp[-\lambda_\perp(1+\cos\pi x)](1+x)\cos[\pi\nu(1-x)]\cosh[\pi\mu(1-x)]. \tag{80}$$

It is clear from condition (a) above that these boundaries are closed curves lying within the strips $l-\frac{1}{2}<\mathrm{Re}(\zeta)<l$.

Equation (80) has been solved numerically for $l=1$, and typical results are shown in Fig. 5. Before discussing these results further, however, it is useful to go back and

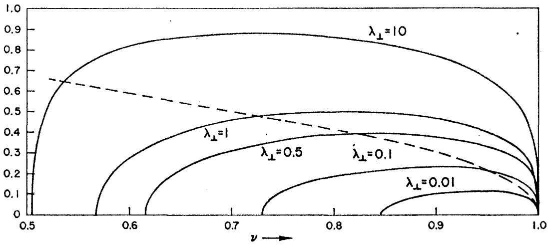

FIG. 5. Curves of μ versus ν along $G(\lambda_\perp;\mu,\nu)=0$ for various $\lambda_\perp$. The dashed line corresponds to the condition at which $\mathrm{Re}\{-\partial(\zeta^{-1}\bar{\Upsilon}_\xi)/\partial\zeta\}$ changes sign.

examine (78) again from a slightly different point of view.

If we use our previous definition, $\mu^2=\alpha\lambda_{\parallel}$, to eliminate $\lambda_{\parallel}$ (and $n_{\parallel}^2$) from (78), we obtain

$$\alpha=\frac{\mu^2\{\Omega_i^2\omega_{pi}^{-2}+\partial[\zeta^{-1}\bar{\Upsilon}_\zeta(\lambda_\perp)]/\partial\zeta\}}{\bar{\Upsilon}_\zeta(\lambda_\perp)-1-T_{\perp i}/T_{\parallel e}-\lambda_\perp\Omega_i^2\omega_{pi}^{-2}}. \tag{81}$$

Note that $\mathrm{Im}(\omega)$ and $k_{\parallel}$ now both appear *only* in the ratio $\mathrm{Im}(\omega)/k_{\parallel}u_i$, and then only in the parameter α. Hence in the unstable regime the growth rate is proportional to $k_{\parallel}u_i$ and the fastest growth occurs for large $k_{\parallel}$. Depending upon what process ultimately limits the smallest supportable wavelength, these growth rates may become very large indeed.

In general, in order to solve (81) one should look for α_M, the largest value of α consistent with those fixed parameters $\Omega_i^2\omega_{pi}^{-2}$ and $T_{\perp i}/T_{\parallel e}$ appropriate to a given system. Such systems are then seen to be unstable for all ion-temperature ratios $T_{\parallel i}/T_{\perp i}<\frac{1}{2}\alpha_M$. Carrying out such a program numerically, on the other hand, is a difficult problem and beyond the scope intended for the present paper. However, if we ask for the more modest result determining the marginally unstable condition when $T_{\parallel i}/T_{\perp i}$ is very small, the numerical work is simplified considerably, for solutions then become possible only when the numerator of (81) vanishes.[38] Such instabilities have very short axial wavelengths, as is implied by the relation between μ^2 and $\lambda_{\parallel}$, and our resulting equation

$$\Omega_i^2\omega_{pi}^{-2}+\partial[\zeta^{-1}\bar{\Upsilon}_\zeta(\lambda_\perp)]/\partial\zeta=0 \tag{82}$$

as well could have been obtained directly from (78) by taking the limit $\lambda_{\parallel}\to\infty$.

The imaginary part of (82) is satisfied along our previously determined curves $G(\lambda_\perp;\mu,\nu)=0$ (cf. Fig. 5), which we now see to have a dual significance: not only do they enclose all regions of possible instability, but they also determine the limiting condition of marginal stability when $T_{\parallel i}/T_{\perp i}\ll 1$. The real part of $-\partial[\zeta^{-1}\bar{\Upsilon}_\zeta(\lambda_\perp)]/\partial\zeta$ has been computed numerically along the curves $G(\lambda_\perp;\mu,\nu)=0$, typical results being shown in Fig. 6. Instabilities of this kind are thus seen to be possible (for $l-\frac{1}{2}<\nu<l$) when

$$\omega_{pi}^2>0.70\Omega_i^2. \tag{83}$$

The tendency of instabilities to occur near half-integral multiples of the cyclotron frequency in this limit is again notable. In the present case the unstable frequencies lie just *above* the half-multiples whereas for the Type-A modes they were found just below. An additional distinction is that here the lowest unstable frequencies lie near $\frac{1}{2}\Omega_i$ while Type-A modes began to be unstable near $\frac{3}{2}\Omega_i$.

C. Electron-Electron Instabilities

Electron-electron instabilities are really just a special case of ion-ion instabilities, since these modes also satisfy an equation of the form (66). The difference is that there is now no contribution from the opposite species analogous to the term $k^{-2}d_e^{-2}$ formerly contributed by the electrons, since the ions are effectively immobilized in virtue of their large mass. Thus a prescription for transforming the analyses of ion-ion instabilities over to the electron-electron case is to first set $T_{\parallel e}=\infty$ in all of the former formulas, and then change all subscripts i to e. Hence from Sec. 6A we find that Type-A electron-electron instabilities occur when $\omega_{pe}^2>6.6\Omega_e^2$, and from Sec. 6B, unstable Type-B modes occur in the limit $T_{\parallel e}/T_{\perp e}\ll 1$ when $\omega_{pe}^2>0.7\Omega_e^2$.

7. DOUBLE DISTRIBUTIONS

It is sometimes true that a group of particles with anisotropic velocity distribution is found—or placed—within an experimental system in which there is another group of the same species of particles, usually colder, whose velocity distribution is more or less isotropic. Such a condition might well be realized, for example, in

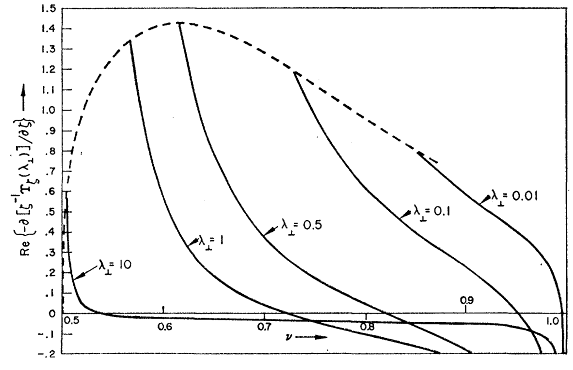

FIG. 6. $\mathrm{Re}\{-\partial[\zeta^{-1}\bar{\Upsilon}_\zeta(\lambda_\perp)]/\partial\zeta\}$ versus ν for various $\lambda_\perp$ along $G(\lambda_\perp;\mu,\nu)=0$. The dashed line corresponds to the condition at which μ changes sign.

[38] Special mention should be made of the circumstance for which $\mu\ll 1$, $|\nu-l|\ll 1$, where the equations need special handling. It may be easily verified, however, that no solutions to (78) are possible in this neighborhood.

many experiments[8,9,39,40] which inject energetic particles into a machine and ionize or dissociate them on a background plasma. Or it may be that the high-energy electrons generated in a high-compression mirror machine[10] are contaminated with a low-temperature plasma that has somehow leaked in from regions which have escaped the heating cycle. In any event, the presence of a cold plasma can provide a dissipative medium to which the hotter, and more ordered, group can give up its energy via electrostatic instabilities.

Let us consider the case in which the transverse energy of the cold (c) ions is much less than that of the hotter (h) group. Then $\rho_c \ll \rho_h$ (ρ = cyclotron radius) so that to a good approximation $k_\perp \rho_c$ may be considered to vanish for those waves that can couple appreciably to the hot ions. Then, if the distribution of cold ions is taken to be Maxwellian in the parallel direction at temperature $T_{\parallel c}$ and we set $\kappa T_{\parallel c}=\frac{1}{2}m_i v_{\parallel c}^2$, we obtain from (33) and (39)

$$F_c=-\left\{n_\parallel^2\Omega^2 k_\parallel^{-2}v_{\parallel c}^{-2}Z'\left(\frac{\omega}{k_\parallel v_{\parallel c}}\right)+\tfrac{1}{2}\Omega k_\parallel^{-1}v_{\parallel c}^{-1}n_\perp^2\left[Z\left(\frac{\omega+\Omega}{k_\parallel v_{\parallel c}}\right)-Z\left(\frac{\omega-\Omega}{k_\parallel v_{\parallel c}}\right)\right]\right\}. \quad (84)$$

If we let ω_p be the total plasma frequency due to the densities of both hot and cold groups, and take q_c and $q_h=1-q_c$ to be the fractions of cold and hot ions, rerespectively, the dispersion equation, Eq. (30), becomes

$$0=D(\omega,\mathbf{k})=1+\omega_p^2\Omega^{-2}[q_cF_c+q_hF_h]. \quad (85)$$

The extra term $k^{-2}d_e^{-2}$, which should be added in general for computations near ion cyclotron frequency, has been dropped for simplicity in the present analysis, the assumption being that $T_{\perp h}/T_{\parallel e}\ll\Omega_i^2\omega_{pi}^{-2}$ for this case. The inclusion of this term would present no special problem. Equation (85) is also appropriate as it stands to the study of the stability of double distributions at the electron cyclotron frequency.

The expression for the cold-ion contribution to the dispersion equation shows that these particles participate dissipatively through the imaginary part of Eq. (85). These particles can also make a *negative real* contribution, corresponding to a reactive participation, and this can also play a part in instability.

Because of the presence of F_c, it is again possible for instabilities to appear close to cyclotron resonance, as in the case of the electron-ion modes, rather than requiring the rather large frequency shifts that were found in Sec. 6 when a single distribution has to couple to itself.

[39] J. L. Dunlap, C. F. Barnett, R. A. Dandl, and H. Postma, Nucl. Fusion, Suppl. 1, 233 (1962); and P. R. Bell, G. G. Kelley, N. W. Lazar, and R. J. Mackin, Jr., *ibid.* 1, 251 (1962).

[40] G. F. Bogdanov, I. N. Golovin, Yu. Kucheryaev, and D. A. Panov, Nucl. Fusion Suppl. 1, 215 (1962); and A. E. Bazhanova, V. T. Karpukhin, A. N. Kharkov, and V. I. Pistunovich, *ibid.* 1, 227 (1962).

Thus the contribution to the dispersion relation of the driving particles may again be handled satisfactorily by use of the expansion (44), and (85) becomes

$$q_h\{n_\parallel^2\langle J_l^2\rangle\eta^{-2}-n_\perp^2\langle P_l\rangle\eta^{-1}\}\cong\Omega^2\omega_p^{-2}-q_c\Omega^2k^{-2}v_{\parallel c}^{-2}Z'(\omega/k_\parallel v_{\parallel c})-\tfrac{1}{2}q_cn_\perp^2\Omega k_\parallel^{-1}v_{\parallel c}^{-1}\times\{Z[(\omega+\Omega)/k_\parallel v_{\parallel c}]-Z[(\omega-\Omega)/k_\parallel v_{\parallel c}]\}. \quad (86)$$

The cold-ion terms on the right of (86) are of importance (for $\omega\cong l\Omega$) only when $k_\parallel v_{\parallel c}\lesssim l\Omega$ or when $|\omega-\Omega|\sim k_\parallel v_{\parallel c}$, the latter case differing from the former only near the fundamental resonance ($l=1$). However when $k_\parallel v_{\parallel c}\sim\Omega$ the hot-ion Landau damping becomes large ($a_h\sim v_{\parallel h}/v_{\parallel c}$) and the coupling to the hot ions is lost (unless $v_{\parallel h}\ll v_{\parallel c}$, a circumstance of considerably limited physical importance). Thus in both cases we are interested in the limit $k_\parallel v_{\parallel c}\ll l\Omega$, the difference being whether $\delta\equiv(\omega-\Omega)/k_\parallel v_{\parallel c}$ is or is not of lower order than $\omega/k_\parallel v_{\parallel c}$. Of these, the former cases appear to be of considerable importance since such modes can be shown to be unstable *even at low densities*, where $\omega_p^2\ll\Omega^2$, and therefore of possible relevance to several current experiments.[8–10,39,40] In the following we will concentrate on these low-density instabilities, considering the behavior at higher harmonics only briefly at the end.

A. Low-Density Instabilities of Double Distributions

When $l=1$, $|\omega-\Omega|\ll\Omega$, and $k_\parallel v_{\parallel c}\ll\Omega$, the only term of importance contributed by the cold ions in (86) is that containing $Z(\delta)$. Thus if we define $y^2\equiv T_{\parallel h}/T_{\parallel c}$ and recall that $a_h\equiv k_\parallel v_{\parallel h}/\Omega$, we may make use of the identity $\eta=a_h(\delta+iy)/y$ to obtain

$$q_h\{n_\parallel^2\langle J_1^2\rangle y^2a_h^{-2}(\delta+iy)^{-2}-n_\perp^2\langle P_1\rangle ya_h^{-1}(\delta+iy)^{-1}\}=\Omega^2\omega_p^{-2}+\tfrac{1}{2}q_cn_\perp^2ya_h^{-1}Z(\delta). \quad (87)$$

Separating this result into its real and imaginary parts on the boundary of instability, with a little algebra one finds

$$\begin{aligned}\frac{q_cn_\perp^2}{2a_h}\{y\,\mathrm{Re}[-Z(\delta)]-\delta\,\mathrm{Im}[Z(\delta)]\}-\frac{\Omega^2}{\omega_p^2}&=\frac{q_hy^2n_\parallel^2\langle J_1^2\rangle}{a_h^2(\delta^2+y^2)},\\ \langle P_1\rangle-\frac{q_c(\delta^2+y^2)}{2q_hy}\,\mathrm{Im}[Z(\delta)]&=\frac{2\delta yn_\parallel^2\langle J_1^2\rangle}{a_hn_\perp^2(\delta^2+y^2)}.\end{aligned} \quad (88)$$

Probably all situations of physical interest are those for which $y>1$. Because $0\leq\delta\,\mathrm{Im}[Z(\delta)]/\Omega_e[-Z(\delta)]\leq\frac{1}{2}\pi^{1/2}$, for all δ, one readily finds that solutions for $y>0.866$ occur only for $\delta>0$. But if $\delta>0$, then $\langle P_i\rangle$ must be positive, so that only Type-A modes can occur.

When $\langle P_1\rangle$ is positive, the least possible density for which instability can set in occurs for the smallest admissible $n_\parallel$. In the limit of small $n_\parallel$ Eqs. (87) or (88)

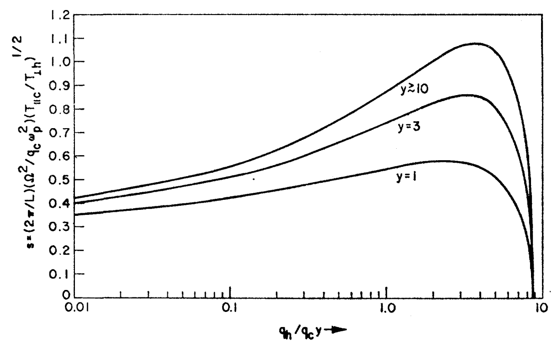

FIG. 7. The instability boundary for double distributions for Type-A coupling. The unstable regions lie below the curves.

can be rewritten

$$\sigma=-[s+Z(\delta)](\delta+iy) \tag{89}$$

where

$$s\equiv 2\Omega k_{\parallel}v_{\parallel c}/q_c\omega_p^2, \quad \sigma\equiv 2\langle P_1\rangle q_h/q_c, \tag{90}$$

it being required that $n_{\parallel}\ll kv_{\parallel c}/\Omega$. (In this approximation *only* Type-A modes are possible, since solutions to (89) exist for positive y and s only when $\sigma>0$.)

Now let us assume that, for given y, we have found a particular solution, $s=s_1$ and $\sigma=\sigma_1$, to (88). Clearly then, to the extent $k_{\parallel}$ is arbitrary, a solution exists for all values of ω_p^2 since we only need choose the appropriate new value of $k_{\parallel}$, keeping all other parameters fixed. However, it is again true that $k_{\parallel}$ cannot be made so small that the required longitudinal wavelength cannot occur in the finite geometry of a given experimental assembly.[41] Therefore if L is the longest supportable wavelength in units of the hot-ion cyclotron orbit diameter, $k_{\parallel}>\pi\Omega/Lv_{\perp h}$, and the minimum unstable density is attained for $(2\pi/L)(\Omega^2/q_c\omega_p^2)(T_{\parallel c}/T_{\perp h})^{1/2}=s_1$.

In a similar fashion we note that $k_{\perp}$ appears only in σ and if we now keep the density of cold plasma fixed, so that s is held fixed also, unstable values of $k_{\perp}$ are related to the hot plasma density through the expression $q_h=\sigma_1/(\sigma_1+2\langle P_1\rangle)$. Since $\langle P_1\rangle$ is free to take on all values less than its maximum by a suitable choice of $k_{\perp}$, all densities greater than that for which $q_h=\sigma_1/(\sigma_1+2\langle P_1\rangle_{\max})$ are unstable.

By use of a delta function at energy $\kappa T_{\perp h}$ for the transverse distribution of hot ions (so that $\langle P_1\rangle_{\max}=0.101$), Eq. (89) has been solved numerically for several values of y. The resulting instability boundaries, in terms of the parameters $s=(2\pi/L)(\Omega^2/q_c\omega_p^2)(T_{\parallel c}/T_{\perp h})^{1/2}$ and $q_h/q_c y=\sigma/2y\langle P_1\rangle_{\max}$, have been plotted in Fig. 7. Because the curves asymptote to zero on the left, there exists a smallest value of q_h for which instability can exist. Similarly, the system stabilizes when q_c becomes small or when the *total* density drops below some critical value characteristic of the remaining parameters. However, depending upon the values of L and the temperature ratios, the densities corresponding to the onset of instability may be very small.

It is of interest to note that this mode shares with the Type-A ion-ion instability (but to a much lesser degree) the peculiar feature of a shift toward *higher* frequencies than that of cyclotron resonance. This is contrary to the tendency of most unstable electrostatic cyclotron modes, e.g., the Type-A electron-ion instabilities of Sec. 6A and *all* Type B instabilities (which can *only* exhibit negative shifts[7b]). The mathematical origin of this behavior in the present case is the ability of the *cold* ions to contribute simultaneously both an imaginary (dissipative) and a negative real (reactive) part to the dispersion equation. Semiquantitatively this positive shift is here of order $(\pi/L)(T_{\parallel c}/T_{\perp h})^{1/2}\Omega$. Thus for systems of sufficient uniformity of magnetic field, this feature may prove useful as a partial aid to experimental identification.

B. Other Double Distribution Instabilities

When $\omega-\Omega$ is not small, i.e., near the higher harmonics of the cyclotron frequency, for small $k_{\parallel}v_{\parallel c}/\Omega$ Eq. (86) becomes

$$q_h\{n_{\parallel}^2\langle J_l^2\rangle\eta^{-2}-n_{\perp}^2\langle P_l\rangle\eta^{-1}\}=\Omega^2\omega_p^{-2} -q_c(l^2-n_{\parallel}^2)/[l^2(l^2-1)], \tag{91}$$

whence

$$\eta=2n_{\parallel}^2\langle J_l^2\rangle\Bigg/\left(n_{\perp}^2\langle P_l\rangle-\left\{n_{\perp}^4\langle P_l\rangle^2-\frac{4q_c n_{\parallel}^2}{q_h}\langle J_l^2\rangle\left[\frac{l^2-n_{\parallel}^2}{l^2(l^2-1)}-\frac{\Omega^2}{q_c\omega_p^2}\right]\right\}^{1/2}\right). \tag{92}$$

No purely Type-A instabilities ($n_{\parallel}=0$) exist in this limit. However, the least possible cold-plasma density for which the instability can set in occurs when $\langle P_l\rangle$ can be made to vanish for finite $k_{\perp}$ (i.e., such that $\langle J_l^2\rangle$ does not also vanish). This is only possible when the distribution in transverse energy of the hot particles has somewhere a positive slope, and thus the Type-A nature is important when it occurs. However, the first term on the left of (91) is also necessary and instability *can* occur even for a purely Type-B condition where $\langle P_l\rangle$ is negative.

When $\langle P_l\rangle$ is zero, the instability condition is

$$q_c w_p^2>l^2(l^2-1)\Omega^2/(l^2-n_{\parallel}^2)\cong l^2\Omega^2. \tag{93}$$

When $\langle P_l\rangle$ cannot be made to vanish, a restriction on hot-particle density is also imposed, the system apparently stabilizing again above some maximum hot plasma density if the density of cold plasma stays fixed. We leave the further investigation of this case to the reader.

[41] See Appendix B for the reasoning by which this procedure is justified in the present case, and which differs from the reasoning applicable to the case of electron-ion instabilities.

8. EXPERIMENTAL APPLICATIONS

It is useful to attempt to discuss some current experiments in order to determine their susceptibility to the types of instabilities we have analyzed and to see if certain observations can be explained on the basis of our results. In this vein let us first look at some recent measurements made on the Phoenix high-energy neutral injection mirror machine at Culham.[8] The effects with which we will be concerned have also been noted on the ALICE machine at Livermore,[42] an experiment quite similar to Phoenix,[9] and although there is insufficient information to make a detailed comparison, the observations of the Culham and Livermore groups appear to be in substantial agreement. Moreover, although we shall not refer in specific detail to the Ogra[40] and DCX[39] experiments, many of our remarks in relation to Phoenix can be applied to these machines as well.

A characteristic feature of experiments such as those currently reported on Phoenix (and one which will undoubtedly be eliminated either naturally or intentionally as these experiments progress) is that typical electron velocities are of the same order as the transverse velocities of the hot ions of the plasma. Thus, since a relatively small amount of electron heating should change the situation greatly, it is probable that the only instances in which the electrostatic electron-ion instabilities will be a serious problem is in the current stages of such experiments.[43]

Because in these experiments $\sigma(\cong l\Omega_i/k_{\parallel}v_{\parallel e})$ is not necessarily small compared with unity, the finite length limitations we obtained with that approximation, i.e., conditions (57) and (65), are too severe. On the other hand, it is beginning to stretch the parameters a little in order to require that σ be as large as 3 or 4 so that the infinite geometry criteria become applicable. Nevertheless the results in this limit should be at least semiquantitative, and conditions (51) [or (53)] and (59) can be expected to be reasonable approximations.[44]

Of particular interest in the way of experimental observations is that of noise at the ion cyclotron frequency, the onset of which is interpreted as indicating the presence of an instability. The relevant information with respect to Phoenix is presented in Fig. 8, which is a modification of a similar diagram given by the Culham group.[45] Two regions of interest are indicated, the first being one in which noise is continuously observed, and the second in which the noise is discontinuous, coming in intermittent bursts. There is no sharp line between the two regimes, but instead there is an overlap region in which the noise is sometimes continuous, sometimes discontinuous.

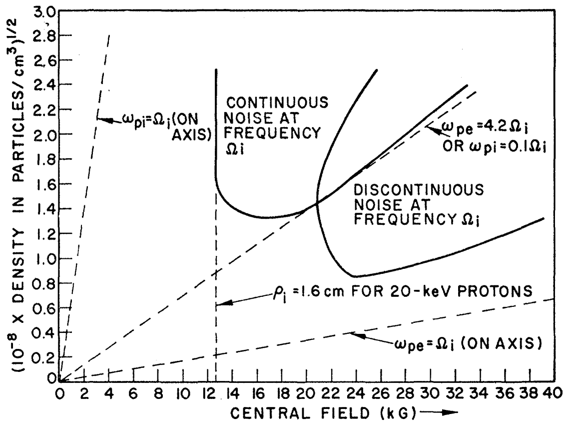

FIG. 8. Regions of instability in the Phoenix experiment, plotted versus magnetic field and square root of the density (see Ref. 45).

The presence of two regions, interpreted as being continuously or intermittently unstable, is very suggestive if the instability criterion is density-dependent. If the plasma density is at a value at which instabilities should occur, one would indeed expect to find noise generated more or less continuously. However, if the system is somewhat below the critical density, normal fluctuations can be expected to occasionally raise the density to the point at which the condition for instability is satisfied. Noise would then be generated until the fluctuation abates and the density again drops below the critical value. The closer the density is to the critical value, the more often one should find such bursts of noise and the longer each burst should endure. This qualitative aspect also seems to correspond, at least roughly, with the experimental facts.[42] Moreover, the failure to observe the noise at small magnetic fields may be explained in terms of the large ion cyclotron radii then obtaining, and the inability of the machine to support the required long wavelengths in its finite geometry.

We have drawn in a dashed line in Fig. 8 corresponding to $\omega_{pe}=4.2\Omega_i$ (on the axis of the Phoenix machine), this curve being the one which bounds the observed region of continuous noise. If we interpret this line in terms of the Type-A electron-ion instability, we find that $3.4d_e=\rho_i$ for electrons of about 30 eV, assuming 20-keV perpendicular energy for the protons. Since the electron temperature in Phoenix has been estimated to be of the order of 10 eV,[46] such a result is consistent

[42] C. C. Damm and A. H. Futch (private communication).

[43] This statement is made in the context of application to the programs of controlled thermonuclear research. Certain effects, for example, occurring in Van Allen-belt plasmas may require an understanding of electron-ion instabilities for their explanation. The authors are indebted to Dr. Theodore Northrop for pointing out this application.

[44] In any event, in order to make a really good analysis, numerical computations would be required that are too specific for inclusion here. In addition to references already cited, a series of calculations have recently been made by Soper and Harris [G. K. Soper, U. S. Atomic Energy Commission Report No. ORNL-3696, 1964 (unpublished); G. K. Soper and E. G. Harris, Phys. Fluids 8, 984 (1965)]. The choices of parameters in this work were directed by the experiments on the Phoenix, ALICE, DCX, and Ogra machines.

[45] Taken, with changes of scale, from Fig. 9 of Ref. 8.

[46] Gioietta Kuo-Petravić (private communication).

within the experimental and theoretical accuracies of the comparison. On the other hand, if we interpret these instabilities in terms of the Type-B electron-ion modes, we have to say something about how the minimum wavenumber varies with magnetic field. However, since (except for the smaller magnetic fields) the overall size of the plasma does not vary appreciably with magnetic field, we would expect that the instability condition (59) should be essentially independent of B—not the behavior indicated by the experimental data of Fig. 8. Nevertheless, the Phoenix parameters are such that (59) is not far from being satisfied and we cannot rule out the possibility that, say, the two separate regimes of Fig. 8 correspond to Type-A modes in the continuous case and Type-B modes in the intermittent case. But in such an event the reason for the intermittency of the Type-B instabilities would not be explained, and the simplest picture would seem to be the one given first in which Type-A modes are taken to be the cause of instability in both regimes.[47]

As has been previously noted, both Phoenix and ALICE are on the borderline as far as electron-ion instabilities are concerned, and only a very slight heating of the electrons would make it impossible for the coupling to occur (within the compass of a linearized theory) because of the finite length of these machines. Moreover, from the point of view of avoiding electrostatic instabilities altogether, it would seem that heating the electrons to a temperature of 100 eV or so (so that an initial density perturbation of the electrons would be dispersed throughout the length of the system in a time somewhat less than the cyclotron period of the ions) is about optimum. Not only would the coupling to the electrons be lost, but the passive contribution of these particles to the dispersion relation would be about as large as possible, tending to stabilize the ion-ion or double distribution modes. (The critical density for onset of instability would be increased by the factor $1+k^{-2}d_e^{-2}$.)

Another system where our instability computations may have immediate relevance is the Table Top[10] experiment in which noise is observed at the electron cyclotron frequency. Up to the present, no satisfactory explanation in terms of known instabilities has been possible for the observations[48] at the relatively low densities characterizing the experiment ($\omega_{pe}^2 \sim \Omega_e^2/40$). However, recent observations have suggested that a relatively large amount of cold plasma may be present as a contaminant when the instabilities are found to occur, and the evidence is increasing that the double-distribution modes of Sec. 7 may be able to account for the results. For example, if a longitudinal wavelength of about 10 cm can be supported in the Table Top plasma, $L\cong 100$ if we take the experimental value $T_{\perp h}=10$–25 keV for the hot electrons. Thus a cold plasma contaminant at a temperature less than about 100 eV, if present in appreciable fraction, would easily put the system within the region of instability given in Fig. 7.

ACKNOWLEDGMENTS

The authors have had the benefit of useful conversations and correspondence with a number of physicists which have helped to clarify our own understanding of the problems of this paper. We would particularly like to acknowledge our debt to the following: S. Bodner, C. C. Damm, A. H. Futch, A. L. Gardner, L. G. Kuo-Petravić, T. Northrop, R. F. Post, and Y. Shima.

APPENDIX A. EVALUATION OF CERTAIN INTEGRALS

In obtaining the dispersion equations for the modes of oscillation of a plasma in a magnetic field, certain averages of the characteristic function $I_{\lambda j}$, Eq. (28), are needed. In particular, if the distribution function is dependent (in zero order) only on the *magnitude* of the velocity normal to the magnetic field, the perturbed distribution function is given by (27). Then, since (27) is to be used in Poisson's equation, the averages

$$K_{\lambda j}\equiv\frac{1}{2\pi}\int_0^{2\pi} d\phi I_{\lambda j} \tag{A1}$$

are needed. Dropping the subscript j and writing $\nu'\equiv(\omega-k_{\parallel}v_{\parallel})/\Omega$, $\beta'\equiv k_\perp v_\perp/\Omega$, we have for $\lambda=\{1,2,3\}$ that

$$\begin{aligned}K_\lambda&=(2\pi)^{-1}\int_0^{2\pi}d\phi\int_{Z\infty}^{\phi}d\phi'\{\cos\phi',1,\sin\phi'\}\exp[-i\nu'(\phi-\phi')+i\beta'(\sin\phi-\sin\phi')]\\&=(2\pi)^{-1}\int_0^{2\pi}d\phi\int_{Z\infty}^{0}d\psi\exp(i\nu'\psi)\left\{i(\beta')^{-1}\frac{\partial}{\partial\psi},1,\sin(\phi+\psi)\right\}\exp\{i\beta'[\sin\phi-\sin(\phi+\psi)]\}\\&=\int_{Z\infty}^{0}d\psi\exp(i\nu'\psi)(2\pi)^{-1}\int_{-\pi}^{\pi}d\psi'\left\{i(\beta'^{-1}\frac{\partial}{d\psi}),1,\tfrac{1}{2}i\left[(\beta')^{-1}\operatorname{ctn}(\tfrac{1}{2}\psi)\frac{\partial}{\partial\psi'}+\frac{\partial}{\partial\beta'}\right]\right\}\exp\{-2i\beta'\sin(\tfrac{1}{2}\psi)\cos\psi'\}.\end{aligned}$$

[47] For still another discussion, see G. Rowlands and L. G. Kuo, unpublished memo, Culham Laboratory, United Kingdom Atomic Energy Agency, 1964 (unpublished). Also, it may be necessary to take into account still another kind of interaction. One of us (L. S. H., to be published) has computed a related instability occurring when electrons oscillating in a longitudinal space-charge well (set up, say, by ambipolar diffusion of a plasma through the ends of a mirror machine) couple to the ion cyclotron motion. There is some experimental evidence (A. Gardner, private communication) that this instability is of importance in ALICE, and it may also be of importance in Phoenix. In particular, observations in Phoenix of oscillations at half the ion cyclotron fre quency may be due to instabilities of this type.

[48] W. A. Perkins and W. L. Barr, Bull. Am. Phys. Soc. **10**, 204 (1965); and W. A. Perkins (private communication).

Using the identity[49]

$$J_n(z)=\pi^{-1}\int_0^\pi \exp\{iz\cos x-\tfrac{1}{2}n\pi i\}\cos nx\,dx$$

and setting $x=\psi\,\mathrm{sgn}Z=\psi\,\mathrm{sgn}\Omega$, $\nu=\nu'\,\mathrm{sgn}\Omega$, and $\beta=\beta'\,\mathrm{sgn}\Omega$, then

$$K_\lambda=\mathrm{sgn}(\Omega)\int_0^\infty dx\,e^{i\nu x}\left\{-i\beta^{-1}\frac{\partial}{\partial x},\,-1,\,\tfrac{1}{2}i\,\mathrm{sgn}(\Omega)\frac{\partial}{\partial\beta}\right\}J_0(2\beta\sin\tfrac{1}{2}x). \tag{A2}$$

If we define[50]

$$\Upsilon_\nu(\beta)\equiv -i\nu\int_0^\infty dx\,e^{i\nu x}J_0(2\beta\sin\tfrac{1}{2}x)\equiv\nu Q(\beta,\nu) \tag{A3}$$

for $\mathrm{Im}(\nu)>0$, then

$$\begin{aligned} iK_2\,\mathrm{sgn}\Omega&=Q(\beta,\nu)\,,\\ K_3&=-\tfrac{1}{2}[\partial/\partial\beta]Q(\beta,\nu)\,. \end{aligned} \tag{A4}$$

Judicious integrations by parts also allow terms involving K_1 arising from integrating (27) over velocity to be expressed simply in terms of Q functions. For example, if as in the text we only need use (31) for F, we are able to write

$$\begin{aligned} F=&-\int_{-\infty}^\infty dv_{\parallel}\int_0^\infty dv_\perp\left(\frac{\partial f_0}{\partial v_\perp}\right)i\Omega^2k^{-2}\int_0^\infty dx\,\exp[i|\Omega|^{-1}(\omega-k_{\parallel}v_{\parallel})]\\ &\times\cos(\tfrac{1}{2}x)k_\perp v_\perp|\Omega|^{-1}J_1(2k_\perp v_\perp|\Omega|^{-1}\sin\tfrac{1}{2}x)+2\pi\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty dv_{\parallel}\left(\frac{\partial f_0}{\partial v_{\parallel}}\right)|\Omega|k_{\parallel}k^{-2}Q\left[\frac{k_\perp v_\perp}{|\Omega|},\frac{\omega-k_{\parallel}v_{\parallel}}{\Omega}\right]\\ =&\int_{-\infty}^\infty dv_{\parallel}\int_0^\infty dv_\perp f_0 i\Omega^2k^{-2}\int_0^\infty dx\,\exp[i|\Omega|^{-1}(\omega-k_{\parallel}v_{\parallel})]\cos\tfrac{1}{2}x\\ &\times(\partial/\partial v_\perp)[k_\perp v_\perp|\Omega|^{-1}J_1(2k_\perp v_\perp|\Omega|^{-1}\sin\tfrac{1}{2}x)]-2\pi\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty dv_{\parallel}f_0|\Omega|k_{\parallel}k^{-2}(\partial/\partial v_{\parallel})Q\left[\frac{k_\perp v_\perp}{|\Omega|},\frac{\omega-k_{\parallel}v_{\parallel}}{\Omega}\right], \end{aligned}$$

or

$$F=2\pi\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty dv_{\parallel}f_0\left\{n_{\parallel}^2\left(\frac{\partial}{\partial\nu}\right)Q(\beta,\nu)+\tfrac{1}{2}n_\perp^2[Q(\beta,\,\nu+1)-Q(\beta,\,\nu-1)]\right\}, \tag{A5}$$

where $n_{\parallel}^2=k_{\parallel}^2/k^2$, $n_\perp^2=k_\perp^2/k^2=1-n_{\parallel}^2$, $\nu=|\Omega|^{-1}(\omega-k_{\parallel}v_{\parallel})$, and $\beta=k_\perp v_\perp/|\Omega|$. We will show below that

$$\tfrac{1}{2}[Q(\beta,\,\nu+1)-Q(\beta,\,\nu-1)]=-\nu\beta^{-1}(\partial/\partial\beta)Q(\beta,\nu) \tag{A6}$$

whence, on converting to the use of the Υ function,[51]

$$F=2\pi\int_0^\infty v_\perp dv_\perp\int_{-\infty}^\infty dv_{\parallel}f_0(v_{\parallel},v_\perp)\left[n_{\parallel}^2\left(\frac{\partial}{\partial\nu}\right)-n_\perp^2\nu\beta^{-1}\left(\frac{\partial}{\partial\beta}\right)\right]\nu^{-1}\Upsilon_\nu(\beta)\,. \tag{A7}$$

For large values of $|\nu|$, $\Upsilon_\nu(\beta)\to 1$ so that if $v_{\parallel}f(v_{\parallel},v_\perp)\to 0$ as $|v_{\parallel}|\to\infty$, we can complete the contour of integration in the complex $v_{\parallel}$ plane by including the infinite semicircle below (above) the real axis if $k_{\parallel}$ is positive (negative). Then, since ω has a *positive* imaginary part and the poles of $\nu^{-1}\Upsilon_\nu(\beta)$ all lie on the real ν axis [cf. Eq. (38d)], all of the poles of $\nu^{-1}\Upsilon$ lie outside the contour in the $v_{\parallel}$ plane. Hence the only contributions come from those poles of f_0 within the contour. For example, if the zero-order distribution is of the form (36), the calculus of residues immediately yields (37) for F, with the real Doppler shift $-k_{\parallel}v_{\parallel}$ going over to an imaginary Doppler shift $ik_{\parallel}u$ through the substitution of ζ for ν.

Finally, we wish to show the equivalence of the representations (38) for $\Upsilon_\zeta(\beta)$. From the definition of Υ, Eq.

[49] G. N. Watson, Ref. 13, p. 177, Eq. (5).

[50] From the point of view of its mathematical properties, $\Upsilon_\nu(\beta)$ is somewhat to be preferred over $Q(\beta,\nu)$ and has been retained in the text for this reason. For our specific application, and in particular for the discussion of this Appendix, $Q(\beta,\nu)$ is sometimes simpler.

[51] By use of the representation (38e), Eq. (A7) can be shown to be identical to the result of Harris, Ref. 2, Eq. (46). (Caution: our ω is the negative of that of Harris.)

(38a) or (A3),

$$\Upsilon_\zeta(\beta) = -i\zeta \sum_{m=0}^{\infty} e^{2\pi i m\zeta} \int_0^{2\pi} dx\, e^{i\zeta x} J_0(2\beta \sin\tfrac{1}{2}x)$$
$$= -i\zeta[1-e^{2\pi i\zeta}]^{-1} e^{i\pi\zeta} \int_{-\pi}^{\pi} dx\, e^{i\zeta x} J_0(2\beta \cos\tfrac{1}{2}x)$$

from which (38b) follows. In turn, (38b) is directly integrable,[24] yielding (38c). Thus (A6) also follows immediately on use of the identity $J_{\mu\pm1}(z) = \mu z^{-1} J_\mu(z) \mp J_\mu'(z)$. Next, Eq. (38d) is obtainable from an expansion given by Watson[52]

$$J_\zeta(\beta) J_{-\zeta}(\beta) = \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}\beta)^{2m} (2m)!}{(m!)^2 \Gamma(\zeta+m+1)\Gamma(-\zeta+m+1)}$$

if one also makes use of the identity

$$\pi \csc(\pi x) = \Gamma(x)\Gamma(1-x).$$

Lastly, Eq. (38e) is quickly obtained from (38a) using the identity

$$J_0(2\beta \sin\tfrac{1}{2}x) = \sum_{k=-\infty}^{\infty} J_k^2(\beta) \cos(kx).$$

APPENDIX B. MINIMUM WAVE NUMBERS

In the application of the theory of instabilities of an infinite medium to systems of finite geometry, a controlling parameter is often the minimum wave number (maximum wavelength) which the disturbance is permitted to have in a given direction. Since the reasons for imposing a minimum wave number differ with the mechanisms of instability (e.g., the electron-ion instabilities of Sec. 5 and the double-distribution modes of Sec. 7), and since we have also studied modes applicable to finite systems in which wavelengths along the field are taken essentially infinite (cf. the discussion of Type-A ion-ion modes Sec. 6A) it is useful to attempt to discuss more fully the points of view we have used in this paper.

In the first place, if the principal motions in an interaction are "localizable" in the sense that they take place in two dimensions rather than three, then an "open-circuit" termination of the plasma in the third dimension (i.e., such that no *a priori* restrictions are placed on the amplitudes of the perturbation at the ends) does nothing to limit the modes found from the infinite-geometry dispersion relation. Such a circumstance could be expected to be a good approximation, for example, in the case of purely Type-A ion-ion interactions in which the temperature parallel to the magnetic field is small and in which the plasma is kept from contacting the end walls of the system by magnetic mirrors. We wish to emphasize, however, that both features—an inappreciable longitudinal motion due to zero-order velocities *and* the fact that longitudinal motion of the perturbation is unnecessary—are required for such a conclusion.

The electron-ion interactions are essentially nonlocalizable. For when the electron temperature is zero (Harris modes), it is necessary that perturbations in electron density be free to move distances of the order of the longitudinal wavelength in order to set up the plasma oscillations to which the ions couple, and this is possible only when the axial wavelength is less than the length of the machine. On the other hand, for the modes in which the electrons contribute a dissipation, it is necessary for a typical electron to be able to move about a wavelength in the longitudinal direction in order to pick up energy from the wave. If the particles are prevented from doing this by the extremeties of the system, again the instability as described by the analysis will not go. Thus, for the electron-ion modes a limitation to longitudinal wavelengths shorter than those of the order of the length of the system is a valid restriction.

We have also made use of finite-geometry limitations in our discussion of double distributions, and in this case one might be justified in viewing our procedure as something of a swindle. The rationale here lies with the requirement that the participating particles must be allowed to interact for a time sufficiently long that the instability can develop to appreciable proportions. For example, if the characteristic macroscopic length over which the parameters have an appreciable variation is d, then for an instability to be experimentally recognizable as such, $\mathrm{Im}(\omega/2\pi) > v_{\parallel i}/d$, where $v_{\parallel i}$ is a typical longitudinal velocity for the ions. But in the analysis of the double-distribution modes, $\mathrm{Im}(\omega)$ appears only in combination with $k_\parallel v_{\parallel i}$. Thus our lower bound on ω is equivalent to requiring $k_\parallel > 2\pi/d$, just the condition we have used.

APPENDIX C. ZERO-FREQUENCY MODES

After the present paper had been written, the authors were informed of the work of Dory, Guest, and Harris[53] who study the special case of instabilities in which the real part of the frequency is zero. Such instabilities have their physical origin in a process which is quite different from that described by our model of Sec. 2, and it is perhaps useful to describe the mechanical picture which we have formed in response to the work of Dory *et al.*[53] and to provide the slight extension necessary to take account of the passive background of electrons.

The instabilities under discussion occur when **k** is normal to the magnetic field. Thus they are also described by the dispersion equation governing Type-A ion-ion instabilities, Eq. (67), and, like the Type-A modes of Sec. 2A, only nonmonotonic transverse energy

[52] G. N. Watson, Ref. 13, p. 147.

[53] R. A. Dory, G. E. Guest, and E. G. Harris, Phys. Rev. Letters **14**, 131 (1965). We are indebted to these authors for making their results available to us before publication.

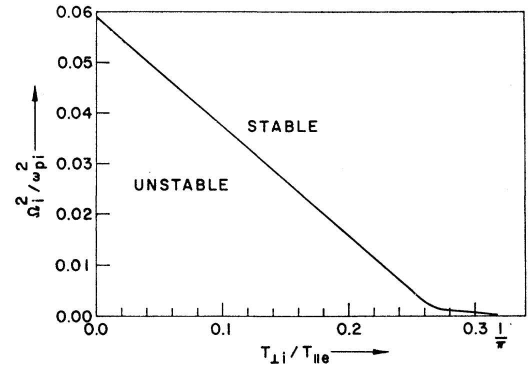

FIG. 9. Showing the instability boundary for zero-frequency ion-ion modes transverse to the magnetic field and with a delta-function distribution of transverse energy.

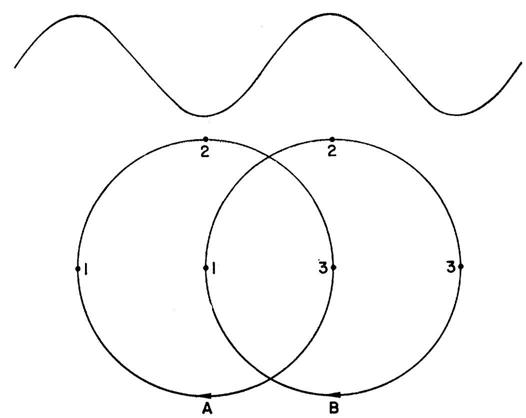

FIG. 10. Illustrating the mechanism of zero-frequency electrostatic instability. A and B are the unperturbed orbits of two particles whose guiding centers are displaced a half-wavelength along the electrostatic wave. The spatial dependence of the potential is also shown.

distributions can give rise to instability. When $\mathrm{Re}(\omega)=0$, however, the imaginary part of the dispersion equation is *automatically* satisfied, and the condition for marginal instability becomes

$$(\Omega_i^2/\omega_{pi}^2)+(T_{\perp i}/T_{\parallel e})\lambda^{-1}=\langle\beta^{-1}\partial J_0^2(\beta)/\partial\beta\rangle, \quad \text{(C1)}$$

where $\lambda=k^2(\kappa T_{\perp i}/m_i\Omega_i^2)$.[18] For a delta-function distribution at energy $\kappa T_{\perp i}$, the angular brackets in (C1) fall away (with $\frac{1}{2}\beta^2=\lambda$) and the result is simply the limiting case of Eqs. (75) with $\nu=0$. [Eq. (74) again is automatically satisfied.] Except for the second term on the left, Eq. (C1) is equivalent to that considered by Dory, Guest, and Harris.[53]

When the distribution is a delta function, solutions can occur for sufficiently high density whenever β lies between a zero of J_0 and its succeeding extremum, i.e., for $2.40<\beta<3.83$, $5.52<\beta<7.02$, etc. These modes, therefore, can go unstable for somewhat smaller wave numbers than were possible for the oscillatory, or overstable, modes (the lower limit here being $\beta=2.40$ rather than $\beta=3.83$ as in the overstable case). In the present case, moreover, there is no real frequency shift (in contradistinction to the resonant modes of Sec. 6A), there being a continuum of wave numbers at the given frequency (zero) for which the right-hand side of (C1) is positive. Thus these modes are distinct from those we have already considered. They cannot be obtained as a limiting case of the oscillatory instabilities where the wave numbers appropriate to a given frequency are selected out of the continua by the requirement that the imaginary part of the dispersion relation be satisfied.

The right-hand side of Eq. (C1) was plotted versus λ^{-1} for the case of a delta function distribution of transverse energy, and the envelope of its intersection with the straight lines $a+b\lambda^{-1}$ determined graphically in order to determine the instability boundary. The results are shown in Fig. 9. As noted by Dory, Guest, and Harris,[53] no instability is possible for $\omega_{pi}^2<17.1\Omega_i^2$. If Fig. 9 is compared with Fig. 4, we see that the area of zero-frequency instability lies well within the area of overstability.

The physical origin of the zero-frequency instabilities is most easily understood from the diagram of Fig. 10. If we consider a monoenergetic distribution in transverse energy, the particle orbit diameters will be very nearly equal to integral multiples of the wavelength. For example, $\beta=k\rho_i$ lies between 2.40 and 3.83 for the fundamental modes, so that if l is the wavelength of the disturbance, $0.77l<2\rho_i<1.22l$. Consider therefore a particle whose unperturbed orbit is indicated by A in Fig. 10 and compare its contribution to the charge density with that of a particle in orbit B, displaced a half-wavelength along $\mathbf{k}$ with respect to A. Position 1 is on the crest of the electrostatic wave for particle A and at the bottom of the wave for particle B. At position 2 the condition is the reverse and at 3 the condition is again as at the start.

Because the potential energy of particle A is somewhat greater than that of particle B at positions 1, and since the total energy is the same for both, A will travel more slowly across the crest of the wave than B will travel through the trough, and will thus make a somewhat greater contribution to the density at the peak than will B at the bottom of the electrostatic well. Again, at positions 2, the particle at the peak will be going more slowly than the particle in the trough, and similarly at position 3. But then the contribution to the charge density is always such as to increase the height of the peaks and the depth of the wells, causing an even greater disparity in the relative velocities, and thus in the relative contributions to the local density. That such a mechanism should give rise to instabilities at zero frequency is therefore clear.