Statistical Study of Linear Mass Relations for Elementary Particles*

W. J. FICKINGER AND R. M. STERNHEIMER
Brookhaven National Laboratory, Upton, New York
(Received 1 April 1965)

In an empirical study of linear relations among the masses of the known elementary particles, the expression $m=pA+qB$, where p and q are integers, has been compared, over a wide range of the constants A and B, with the set of the experimental mass values.

THERE has been some interest in the search for systematics in the group of numbers which represent the masses of the known fundamental particles. These studies have, in some cases, related particle masses to their established quantum numbers, while in other cases they have involved purely empirical examination of the masses of the strongly interacting particles in search of more general systematics.

This report is concerned with a study of the latter type, in which the masses of the following 28 particles[1] have been examined for additive relations: $ABC(290)$, $K(496)$, $\eta(549)$, $K^*(725)$, $\rho(750)$, $\omega(782)$, $K^*(888)$, $N(939)$, $X(960)$, $\phi(1019)$, $\Lambda(1115)$, $K^*(1175)$, $\Sigma(1193)$, $B(1220)$, $N^*(1238)$, $f(1255)$, $\Xi(1319)$, $Y^*(1385)$, $Y^*(1405)$, $N^*(1485)$, $N^*(1512)$, $Y^*(1520)$, $\Xi^*(1532)$, $Y^*(1660)$, $\Omega(1679)$, $N^*(1688)$, $Y^*(1815)$, $N^*(1922)$. Specifically, the form

$$m=pA+qB \qquad (1)$$

has been compared to the given mass values. The parameters p and q are integers and the values of A and B are constants, so that Eq. (1) represents each mass as the sum of integral numbers of two fundamental mass units, A and B. In the limit of small values of A and B, and correspondingly large integers, any mass can be fitted. Therefore, A and B were chosen large enough so that Eq. (1) represents a true constraint on the masses, and the range of integers used was taken wide enough so that (1) could cover the entire mass table. Equation (1) was evaluated for all values of A from 90 to 150 MeV and of B from 205 to 265 MeV in steps of 0.5 MeV, and the coefficients were constrained as follows: $-3\leq p\leq 17$ and $-2\leqq q\leqq 6$. For each of the $120\times 120=14{,}400$ (A,B) doublets, $21\times 9=189$ (p,q) lattice points were calculated and compared to the table of masses. A fit was defined as agreement between the calculated lattice point and an experimental mass to $\leqq 4$ MeV. When a particle fitted two lattice points it was counted as only one fit. It must be stressed that for many of the masses, 4 MeV is much less than the quoted experimental uncertainty $\Delta m_{\rm expt}$. If $\Delta m_{\rm expt}$ were used rather than 4 MeV, these particles would fit for all (A,B) and would contribute only background. It was felt that these masses could contribute some information to this study and that it was preferable to take $\Delta m=4$ MeV throughout rather than discard these mass values. The number of fits N_{AB} for each (A,B) doublet was plotted in the (A,B) plane. It was seen that, while there was considerable structure in this plot, with local maxima and minima, the whole pattern was dominated by clearly defined regions of low N_{AB} along certain lines radiating from the (A,B) origin. These lines are the loci of (A,B) values for which A/B equals a simple fraction. When A and B are so related, a value of m given by (1) may appear on several (p,q) lattice points. Consequently, the total mass interval covered by the regions 4 MeV to both sides of all lattice points is decreased, and the probability of fits is decreased proportionately.

To remove this source of "nonphysical" systematics, the function y_{AB} was plotted for each (A,B) doublet, rather than the number of fits N_{AB}, where y_{AB} is N_{AB} normalized to $N_{\rm ch}$, the number of fits expected from chance. $N_{\rm ch}$ is calculated for each (A,B) doublet from the ratio of the mass zone δM covered by the corresponding (p,q) lattice to the length ΔM of the zone covered by the mass table, $\sim$1900 MeV:

$$y_{AB}=N_{AB}/N_{\rm ch}\,; \quad N_{\rm ch}=28(\delta M/\Delta M)\,. \qquad (2)$$

The resulting plot of y_{AB} over the (A,B) plane showed greater uniformity than the N_{AB} plot, the radial zones of low N_{AB} having been normalized out. However, many local details persist. The great majority of these extrema are associated with the same radial lines described above. Along these lines of low N_{AB}, typically only 5 or 6 masses give fits, often each at several (p,q) lattice points, and the variations in y_{AB} along them represent statistical fluctuations in a small sample. The details of both of these "non-physical" effects were verified by repeating the entire calculation using a mass table of 28 random numbers, confined to a similar mass range. Both the radial depletions in N_{AB} and the fluctuations in y_{AB} along the radial lines were observed.

The results of the calculation for the physical masses are represented schematically in the left half of Fig. 1. The regions in the (A,B) plane for which $y_{AB}>1.5$ have been blackened. Most of these regions are associated

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] Wherever they are available, we have used the mass values given by A. H. Rosenfeld, *Proceedings of the International Conference on High-Energy Physics* (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 783. For the K meson, the nucleon, and the Σ and Ξ hyperons, we have used the average of the masses of the different charge states. The present mass values are the same as those used in R. M. Sternheimer, Phys. Rev. Letters **13**, 358 (1964).

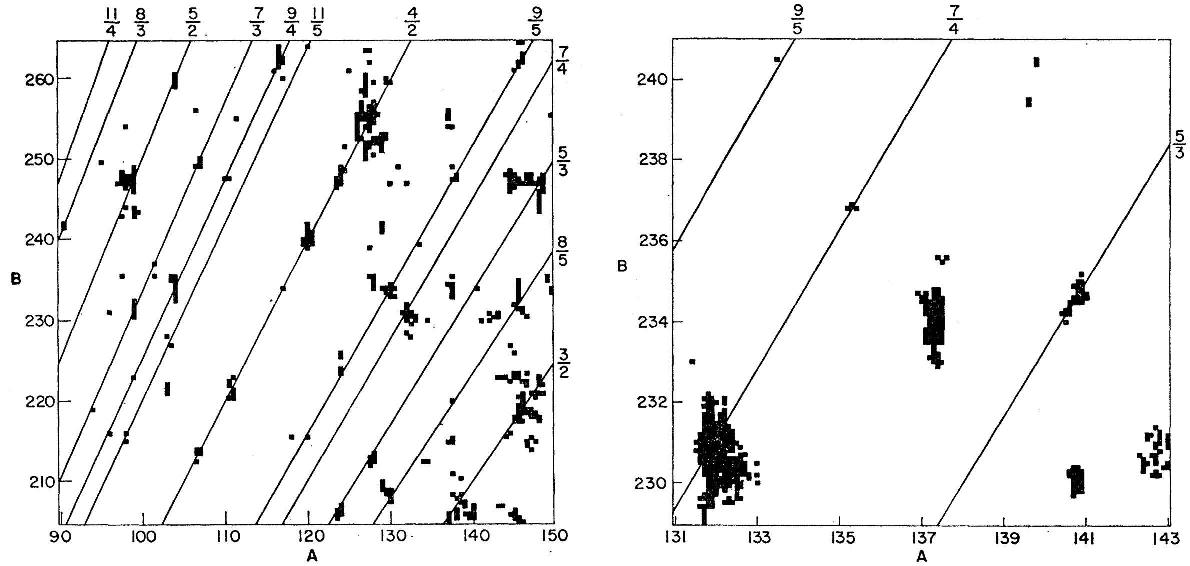

FIG. 1. Plots of the (A,B) plane on which the regions for which $y_{AB}>1.5$ have been blackened. The left-hand plot covers the entire (A,B) range studied; the right-hand plot covers the vicinity of the (m_π,κ) cluster. The values of A and B are in MeV. The straight lines correspond to $B/A=\text{constant}\equiv f$, where f is the fraction indicated next to each line.

with the lines $B/A=l/m$, l and m integral, $[(3/2)\leq(l/m)\leq(11/4),\ m\leq 5]$ and correspond to low values of N_{AB}. In addition, however, there are several regions between the radial lines which correspond to both $N_{AB}>19$ and $y_{AB}>1.5$. We may point out, for example, the regions centered on the following (A,B) doublets: (145,206), (142.5,230), (137.5,234), (137,255), (132,231), (129,242), (103,221). The third of these local maxima corresponds closely to the (A,B) values $(m_\pi,\kappa)=137.3$, 234.7) suggested in several previous publications.[2,3] This particular region in the (A,B) plane was examined in greater detail by calculating N_{AB} and y_{AB} on an (A,B) grid with intervals of 0.1 rather than 0.5 MeV. The results are shown in the right half of Fig. 1, where the regions for which $y_{AB}>1.5$ have been blackened. The (m_π,κ) maximum was found to be centered at $(137.3\pm0.15,\ 233.9\pm0.8)$ and with peak values of $N_{m_\pi\kappa}=22$ and $y_{m_\pi,\kappa}=1.76$. The details of the arrangement of the physical masses on this particular (p,q) grid have been discussed by Sternheimer.[2,3] If these maxima indicate systematics in the mass table, it is probable that they are to some degree reflections of one another.

The purposes of the present paper are as follows: (1) to report that this study has been made; (2) to describe the nature of the purely mathematical systematics inherent in the form of Eq. (1); (3) to point out that there are relatively few values of (A,B) which provide good fits of Eq. (1) to the experimental masses; and (4) to show that one of these good fits is in agreement with the two parameters suggested by Sternheimer,[2] one of which is equal to the average of the charged and neutral pion masses.

We would like to thank Dr. R. F. Peierls and Dr. D. K. Robinson for helpful discussions.

[2] R. M. Sternheimer, Phys. Rev. Letters **13**, 37, 358 (1964); Phys. Rev. **136**, B1364 (1964).

[3] The constant $\kappa\equiv\frac{1}{8}(m_p+m_n)$ has been introduced by T. Takabayasi and Y. Ohnuki, Progr. Theoret. Phys. (Kyoto) **30**, 272 (1963). See also T. Takabayasi, Nuovo Cimento **30**, 1500 (1963).