PHYSICAL REVIEW VOLUME 139, NUMBER 4B 23 AUGUST 1965

Quantum Effects on the Accuracy of Momentum Measurements*

CHING-HUNG WOO
The Institute for Advanced Study, Princeton, New Jersey
(Received 7 April 1965)

It is shown in two examples that to measure the momentum of a particle with great accuracy, allowing for large uncertainties in the position, it is still necessary to consider additional quantum effects. For the case of measurements by electromagnetic means, the uncertainties in the measurability of field strengths and of charge-current densities should be considered. Within the framework of the Bohr-Rosenfeld analysis, uncertainties in the field measurements do not limit the accuracy of momentum measurements. What happens when the Bohr-Rosenfeld approximations are not made is an open question.

THE concept of local fields is sometimes suspected of not being meaningful on the grounds that measurements relating to an exact localization in space-time are not possible. On the other hand, no serious doubt has been raised about the measurability of momentum with arbitrary accuracy. The reason behind such attitudes is, roughly speaking, that, by the uncertainty principle, any localization in space-time with great precision would be accompanied by large uncertainties in the energy-momentum involved, and additional particles can be created. On the other hand, for a momentum measurement with great accuracy, the uncertainty relation requires only large extensions in the space and time of measurement. This seems to be a limit where classical macroscopic descriptions can be used with better and better justification, and uncertainties due to quantum effects can be minimized. These intuitive feelings, however, have not been verified in detail with respect to either the measurability of momentum or the nonmeasurability of space-time localizations. In fact, in nonrelativistic quantum mechanics, one is used to the complementary nature of x and p, and to a certain degree of symmetry with respect to the limitations on the accuracies of position and momentum measurements. It may be interesting, therefore, to consider in simple idealized experiments how relativistic effects such as pair creation can actually change the situation.

In this note we give a brief account of some preliminary considerations on momentum measurements. In Sec. I it is shown that to obtain great accuracy in momentum measurements, one should also consider uncertainties in the measurability of field strengths and charge-current densities. In Sec. II it is shown that, within the approximations of Bohr and Rosenfeld,[1] the uncertainties in field measurements do not set a limit to the accuracy obtainable for momentum measurements. On the other hand, within the same approximation framework, measurements with respect to space-time localization are also meaningful. This will be discussed separately. The question as to what happens when these approximations are not made is completely open.

* The study was supported by the U. S. Air Force Office of Scientific Research Grant No. AF-AFOSR-42-64.

[1] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **12**, 8 (1933); also Phys. Rev. **78**, 794 (1950).

I. THE RELEVANCE OF QUANTUM EFFECTS ON MOMENTUM MEASUREMENTS

We will first define the problem more specifically. By a momentum measurement we shall mean the determination of the momentum of a single elementary particle, which is approaching a pure momentum state with a limitation imposed only by the uncertainty relation. We will not be concerned with the measurement of the average momentum of a macroscopic uniformly charged body. The following idealizations will be made: (1) The constants c and $\hbar$ (speed of light and Planck's constant) and the particle mass m will be assumed to be given parameters. For example, we assume that the determination of the velocity of a particle is equivalent to the determination of its momentum without inquiring into the possibility that one may not be able in principle to determine the value of m, as well as c and $\hbar$, with absolute accuracy. Naturally, our measurements refer only to stable particles. (2) We shall use electromagnetic interactions for measurements, and will at the same time neglect gravitational and weak interactions. We also neglect cosmological limitations so that, for instance, the region of measurement can be extended arbitrarily far in a flat space. (3) We will be interested in seeing whether the uncertainties in momentum measurement can be reduced indefinitely. We will not be directly concerned with absolute certainty in the determination of momentum. Such certainty may have meaning only as the limit of a sequence of approximations by consideration of a theorem of Araki and Yanase.[2]

(a) First we consider the following simple arrangement for the selection of a particular momentum for charged particles having a fixed mass. Static and homogeneous **E** and **H** fields are imposed inside a long rectangular tube with $\mathbf{E}\times\mathbf{H}$ parallel to the axis of the tube and **E** and **H** each parallel to two of the walls. The condensers and solenoids, say, which serve as the sources of the fields, may be assumed to be arbitrarily heavy and define the laboratory system. The region outside the tube is assumed to be field free (fringe effects turn out to be unimportant for later considerations). For particles that enter the tube, only those having a particular velocity relative to the laboratory system, parallel to the tube axis in direction and equal to E/H in magnitude,

[2] H. Arkai and M. M. Yanase, Phys. Rev. **120**, 622 (1960); also E. P. Wigner, Z. Physik **133**, 101 (1952).

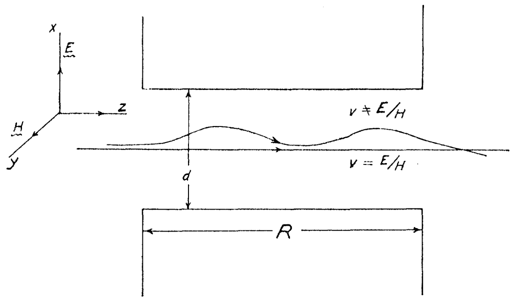

FIG. 1. Schematic diagram of the arrangement for momentum measurements by using crossed fields.

will follow a straight line. These will emerge from the other end of the tube. Particles with different velocities will perform trochoidal motions, and those that strike the wall are assumed to be completely absorbed. In this arrangement, therefore, any particles that emerge from the other end will be accepted as having the desired momentum. This is of course an extremely crude method, simplified so as to illustrate the essence of the problem. In this example, particles that have velocities slightly different from $(\mathbf{E}\times\mathbf{H})/H^2$ can also emerge from the other end, as illustrated in Fig. 1. With $R>2\pi p/eB$, and the distances between the walls both equal to d, the resolutions are given by:

$$\Delta p_z \approx Bed + 1/R\,, \tag{1}$$

$$\Delta p_x \approx Bed + 1/d\,, \tag{2}$$

$$\Delta p_y \approx 1/d + (pd^2)/(2R^2)\,. \tag{3}$$

Additional uncertainties in the momentum due to radiation will be considered later, but it is already clear from Eq. (2) that in order to attain great accuracy, one must let both d and R become large, and at the same time let the field strengths become arbitrarily weak. The question naturally arises as to whether it is possible to have an arbitrarily weak field whose action on a charged body obeys the classical calculations, and whether uncertainties in the *measurability* of field strengths will cause *deviations from classical expectations* as the field gets weaker and weaker. The words italicized in the last sentence serve to emphasize the reasons why this question is at all relevant. One may feel, for example, that as the applied fields get weaker, the behavior of a charged particle approaches that of a free particle, and no uncertainties should arise. However, a null field clearly cannot be used to discriminate different momenta. It is the ratio of field effects to the deviations of field effects from classical expectations that is at issue. Furthermore, while in dealing with relatively strong applied fields one may neglect a weak stray field, coming, say, from the outer space, in dealing with weaker and weaker applied fields one must make sure that no such unwanted fields are present. The question of the measurability of the field strengths is therefore relevant, and disturbances due to measurement should be considered. Before we examine this problem, however, let us consider another standard measurement procedure.

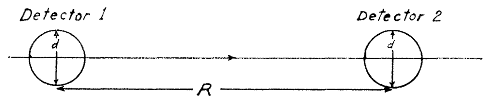

FIG. 2. Schematic diagram for velocity measurement by the time of flight.

(b) Let us now consider a velocity measurement by the time-of-flight method. Two detectors, 1 and 2, are placed a distance R apart as shown in Fig. 2. The particle passes detector 1 at time t_1, and detector 2 at time t_2. A velocity relative to the detectors, which can be assumed to be very heavy, is defined by the ratio

$$\mathbf{v}=\mathbf{R}/(t_2-t_1)\,. \tag{4}$$

One grants that the detector may have a finite size, represented by a linear dimension d, and that there is a corresponding uncertainty in the time when the particle passes through a detector. The resulting uncertainty in $\mathbf{v}$ will be denoted $\Delta\mathbf{v}$. On the other hand, as R becomes larger and larger, while d is kept fixed, $\Delta\mathbf{v}$ will become smaller and smaller, and $\mathbf{v}$ becomes better defined. It would seem that such an arrangement circumvents the consideration of quantum effects.

It must be noted, however, that when $\mathbf{v}$ becomes very well defined, then the uncertainty in momentum relevant to the problem is given by $\Delta p=1/d$, irrespective of how large R is. This is because, when $1/d$ is larger than $|m\Delta\mathbf{v}|$, the $\mathbf{v}$ defined by Eq. (4) does not correspond to the velocity of the particle when it is between the two detectors. The particle has a distribution in momentum as a result of passing through the first detector, and it is only when it interacts with the second detector that a particular Fourier component is picked out. This can be seen by imagining that the two detectors are enclosed in a tube of diameter d with perfectly absorbing walls. If a particle passing through both detectors is taken to have velocity $\mathbf{v}$ defined by Eq. (4) while inside the tube, and if $|m\Delta\mathbf{v}|$ is less than $1/d$, the uncertainty principle is violated. Similarly, the velocity of the particle after it passes through the second detector also deviates from the $\mathbf{v}$ defined by Eq. (4) because of interactions with the second detector. So eventually the limit on the accuracy of momentum determination is set by the interactions with the detector, and the question is whether the uncertainty arising from this can be made arbitrarily small by extending the detector dimension d. It is clear that when d becomes larger and larger, the passage of a single charged particle corresponds to a smaller and smaller current density. One can ask whether uncertainties in the measurability of charge-current densities will cause difficulty in the measurement of an arbitrarily small current density without disturbing it. (It may seem absurd that, if a small cloud chamber can detect the passage of a charged particle with certainty, a large cloud chamber should have any difficulty doing it. It is to be noticed, however, that in ionizing even just one atom the particle receives a finite amount of unpredictable momentum transfer. In interacting with an atom, the test particle gets localized to a size of the atomic dimension, and the large extension in the chamber size is no longer relevant. Clearly, this is not the optimal arrangement.) We see therefore that in order to measure momentum with arbitrary accuracy, one must deal with arbitrarily weak field strengths and arbitrarily small charge-current densities. Consequently, uncertainties in the measurability of these quantities should be investigated. Since part of these uncertainties are due to virtual or real pair creation during the measurement process, one sees that the effects of particle creation should be considered not only for localization measurements, but for momentum measurements as well.

II. ACCURACIES OBTAINABLE IN THE BOHR-ROSENFELD FRAMEWORK

We will now consider the uncertainties in the measurement or preparation of electromagnetic field strengths. For this we will use the results of the work of Bohr and Rosenfeld. It is to be remembered that the Bohr and Rosenfeld analysis neglects completely the atomic structure of the test bodies that are used for field measurements; thus the considerations below are valid only within this approximation framework. On the other hand, the Bohr-Rosenfeld analysis does take into account the effect of pair creation during a measurement.

Within this framework, there are only two sources of uncertainty in the determination of average field strengths over a space-time region. First, there are field fluctuations. Second, there is the lack of commutativity for the average field strengths over two regions situated such that a signal can go from one region to the other. It has been shown by Corinaldesi[3] that for two space-time regions of linear dimension L ($T\approx L/c$), and displaced from each other by a distance of the same order, the magnitude of the commutator is either of the same order or smaller than the square of the fluctuation fields. For $T\approx L/c$ the latter is given by

$$(\Delta F_{\mu\nu})^2=\frac{1}{2(2\pi)^3V^2}\int (k_\mu{}^2+k_\nu{}^2-2g_{\mu\nu}k_\mu k_\nu)\delta(k^2)d^4k \times\int_V d^4x d^4x' \exp[ik(x-x')] \approx 1/L^4, \tag{5}$$

where V is a space-time region of dimension L. When the applied fields become weaker than this magnitude, the classical description of their effect on the particle trajectory becomes unreliable.

It may seem already obvious that in this framework the uncertainties in field measurements have negligible effects. One may argue that since with the Bohr-Rosenfeld arrangement the dominant uncertainty is due to field fluctuations, and since fluctuations give *predictable* renormalization effects, no deviation from a classical trajectory should occur if one uses renormalized quantities. This is not quite correct because, due to interactions with the measuring apparatus, real unpredictable deviations are introduced. Their magnitudes can be estimated only from a consideration of the size of the test bodies to be used for field measurements, which determines the value of L in Eq. (5).

Suppose one determines directly the field strength in the region described in Fig. 1 by an actual measurement of the Bohr-Rosenfeld type; one then encounters two difficulties:

(i) If a large uniformly charged test body which fills the whole region is used, it can only measure the average field strength over the whole region, and does not ensure that the field is homogeneous in space and static in time. If smaller test bodies are used to measure fields in subregions in order to determine the homogeneity of fields, the value of L to be used in Eq. (5) must be much smaller than d or R in Fig. 1.

(ii) If the field measurement is completed before the particle whose momentum is to be measured enters the region, the field intensity experienced by the particle is not the same as that previously measured. The field and its time derivative do not commute. On the other hand, if the measurement is carried out simultaneously during the passage of the particle, it can be shown that the particle trajectory will be strongly disturbed by the measurement. Even though the test body for the field measurement is immersed in an oppositely charged, fixed background so that the field due to the test body is neutralized, the reaction field due to field measurements[4] has a non-negligible effect on the particle whose momentum is under measurement.

It is clearly necessary, therefore, to make use of one's knowledge of the charge-current distributions which serve as sources to the fields in the measurement region. This knowledge must in turn come from measurements. An actual measurement of the charge-current density will also involve uncertainties due to fluctuations, which become infinite when the boundary of the source region becomes sharply defined.[5] Thus there will be some uncertainties in the positions of the charge-current distributions which serve as the sources. Without elaborating on the tedious details, one may state that it can be shown that by letting the region depicted in Fig. 1 become arbitrarily large and by choosing heavy masses for the charged bodies which form the sources, it is indeed possible to prepare static and homogeneous **E** and **H** fields, subject to an uncertainty in field strength due only to fluctuation effects of the order $|\Delta\mathbf{E}|\sim|\Delta\mathbf{H}| \lesssim 1/d^2$. This is of crucial importance because it is now a trivial matter to compare the rate of decrease of the applied fields with that of the fluctuation fields, as the

[3] E. Corinaldesi, Nuovo Cimento, Suppl. **10**, 83 (1953).

[4] An estimate of the magnitude of field actions of the test bodies can be found in Ref. 1.

[5] W. Heisenberg. Leipzig. Ber. **86**, 317 (1934).

required momentum resolution becomes better and better.

A look at Eqs. (1) through (3) shows that to make $\Delta p_i \to 0$, it is only necessary to let

$$B \lesssim 1/d^{1+\epsilon}, \quad R \geq d^{1+\Delta}, \quad 1 > \Delta > \epsilon \qquad (6)$$

with ϵ and Δ real and positive, and to let d become arbitrarily large.

Thus eventually

$$(\text{applied } B \approx 1/d^{1+\epsilon}) \gg (\text{fluctuation fields} \lesssim 1/d^2).$$

There is an additional uncertainty in the momentum of the particle due to radiation of photons during the passage, which is given by

$$\Delta p \sim (\alpha^2/m^2) B^2 R.$$

Clearly this also decreases steadily with increasing d. One can further convince oneself of the negligibility of fluctuation effects that are averaged over a whole region of linear dimension d by the following qualitative consideration, which was used by Welton[6] to obtain an estimate of the Lamb shift. For a charged particle of mass m moving with nonrelativistic velocity, the acceleration due to a fluctuating field $\mathbf{E}$ is

$$\ddot{\mathbf{x}} = (e/m)\mathbf{E},$$

so that for a given Fourier component

$$(\Delta x)_\omega^2 = \frac{e^2}{m^2} \frac{|\mathbf{E}|^2}{\omega^4}.$$

Since

$$\langle \mathbf{E} \rangle_{\text{av}}^2 \approx \omega^3 d\omega,$$

$$(\Delta x)^2 \approx \frac{e^2}{m^2} \ln \frac{\omega_{\max}}{\omega_{\min}}.$$

The upper cutoff $\omega_{\max}$ for the frequency is of the order of m by a consideration of retardation effects,[6] and the lower cutoff can be chosen to be $\omega_{\min} \sim 1/d$. Thus $(\Delta x)^2 \sim (\alpha/m^2) \ln(dm) \ll d^2$. One sees that because of the oscillatory nature of the field fluctuation, the deviation from classically computed trajectories is not serious. Within the framework of Bohr and Rosenfeld for the measurability of fields and charge-current densities, therefore, it is possible to achieve arbitrary accuracy in momentum resolution by the arrangement of Fig. 1.

Similar considerations apply to the second method using time of flight. In order for the detector to register the fact that a particle has passed through, the latter must be made to interact with something. To reduce the amount of momentum transfer during the interaction, one might, for instance, make the average interaction distance very large as d increases, as shown schematically in Fig. 3, where the particle interacts only with other particles in the shaded regions. It will not be satisfactory to use elastic scattering as the required interaction for this purpose because it will then be necessary to know the target momentum before and after the collision with arbitrary accuracy. This would be a "Zanzibar" in the sense of Cohen *et al.*[7] So some target particle must undergo a recognizable change as the result of the interaction. For simplicity one might take this to be a transition in a two-level system. As d becomes larger and larger, the energy between the two levels must become smaller and smaller, decreasing at least at the rate $\Delta E \sim 1/d$. It will also be necessary to have arbitrarily narrow natural line widths. This can be achieved, for instance, by Zeeman splitting a ground level, and letting the applied magnetic field B decrease as $1/d$ as d increases. Detection of a transition, which signals for the passage of a charged particle, can be effected by a Stern-Gerlach experiment. The main conclusion of the previous consideration is that the uncertainty in the determination of the field strength is of the order $\Delta B \sim 1/d^2$. Hence, here it is also possible to achieve arbitrary accuracy by letting $d \to \infty$.

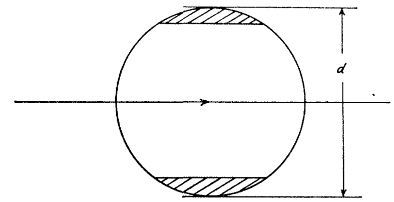

FIG. 3. Symbolic sketch of a detector, where only the shaded regions contain particles capable of being excited by a passing particle.

Thus, within the approximation framework of neglecting the structure of test bodies that are used to measure field strengths, it is possible to measure the momentum of a charged particle with arbitrary accuracy. But the considerations involved indicate a close connection between the measurability of momentum and such quantum effects as the zero-point oscillations of a field or a lattice.[8] It is an entirely open question how the removal of the Bohr-Rosenfeld approximation will affect the local field measurements as well as the momentum measurements.

ACKNOWLEDGMENTS

The author wishes to thank A. Dragt and J. B. Hartle for discussions. He is grateful to Professor E. P. Wigner for useful comments, to Professor J. A. Wheeler for a discussion of the Bohr-Rosenfeld analysis, and to Professor E. H. Wichmann for many relevant criticisms. He wishes to thank Professor J. R. Oppenheimer for the hospitality at The Institute for Advanced Study.

[6] T. Welton, Phys. Rev. **74**, 1157 (1948).

[7] E. H. Cohen, K. M. Crowe, and J. W. M. DuMond, *Fundamental Constants of Physics* (Interscience Publications, New York, 1957), preface.

[8] There is a third method of selecting a particular momentum, by diffraction-scattering off a lattice and selecting particles that come out at a fixed angle. For an indefinite increase in accuracy by this method, one must consider the zero-point oscillations of the lattice, which in the case of x-ray diffraction are known to contribute to the Debye-Waller factor. The author has not yet analyzed this case in detail.