Dynamical Groups and Mass Formula

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The homogeneous Lorentz group and the 4+1 de Sitter group are interpreted as the dynamical groups of a nonrelativistic and a relativistic "rotator," respectively. In an irreducible representation of the latter group we obtain for certain states the mass formula $m^2=m_0^2+\lambda^2 j(j+1)$. The contraction of the dynamical groups [Euclidean group in three dimensions and Poincaré group, respectively] destroys the energy or mass spectrum and can be associated with the limit $\hbar\rightarrow 0$. The model of an elementary particle as a de Sitter "rotator" is discussed.

I. THE CONCEPT OF DYNAMICAL GROUPS

By a dynamical group we mean a group (in general a noncompact one) which gives the actual energy or mass spectrum of a quantum-mechanical composite system. The idea is to reduce a theory in flat space with an interaction (e.g., Hamiltonian) to a group of motions in curved space, i.e., to geometry, in a way similar to that done in general relativity. The representation of the dynamical group will give us the quantized states of the system. That this idea also works in the domain of quantum theory is shown explicitly in this paper in the case of two simple examples describing the quantized energy states of a rotator and the quantized mass states of a relativistic "rotator." The latter has a strong bearing on the problem of the actual mass spectrum of strongly interacting particles. Our aim is to give a group-theoretical formulation of the so-called broken symmetries. The unitary symmetries used for the classification of strongly interacting particles (SU_3 or SU_6) are approximate and clearly their experimental success must be attributed to the existence of symmetry-breaking terms.[1,2] These terms are introduced in a phenomenological way and are treated as a perturbation in the mass operator. The possibility that the symmetry breaking can be explained within the framework of a larger dynamical group containing the group of degeneracy has been considered before.[3–5] We explicitly show in this paper how a mass spectrum may be obtained under the assumption of a dynamical group. The attractive feature of the model is thus its simplicity with no further assumptions being necessary than the group.[6]

In order to see how the dynamical group is introduced we start from a mass formula for mesons of the form[3,4,7]

$$m^2=m_0^2+\lambda^2[2J(J+1)-I(I+1)+Y^2/4], \quad (1.1)$$

where J, I, and Y represent the spin, isospin, and hypercharge of the particles and m is the measured mass. The scale factor λ can be found from the empirical masses to be

$$\lambda\cong 14\times 10^4(\text{MeV})^2. \quad (1.1a)$$

For further reference we note that this factor can be related to a length l_0 by the relation[3]

$$\lambda=\hbar^2/e^2 l_0, \quad (1.1b)$$

with the value $l_0\approx 10^{-12}$ cm.

Because m^2 is the value of the mass operator $P_\mu P^\mu$, Eq. (1.1) indicates clearly a coupling between $P_\mu P^\mu$ and the other operators J^2, I^2, Y^2. In the limit of exact degeneracy $P_\mu P^\mu$ is an invariant and P_μ's commute among themselves. The assumptions about the existence of symmetry-breaking terms really amounts to the noncommuting P_μ's and the fact that $P_\mu P^\mu$ is no longer an invariant. (In SU_3 and SU_6, $P_\mu P^\mu$ is assumed to transform as a tensor operator of the group.)

For quantized systems the energy E or m^2 are not invariants but take definite values as a function of quantum numbers.

In general, however, the energy or mass spectrum (m^2) is not simply linearly related to the quantum numbers as in Eq. (1.1), e.g., H atom. But the simplicity of Eq. (1.1) suggests that it can be derived from the second-order Casimir operator of a group. In this paper we shall concentrate on the first two terms of (1.1), i.e.,

$$P_\mu P^\mu+\lambda^2 J(J+1)=\rho^2 \quad (1.2)$$

and shall interpret the left-hand side of (1.2) as terms entering into the second-order Casimir operator of a group, ρ^2 being related to the value of this Casimir operator in an irreducible representation of the group. The group containing $P_\mu P^\mu$ and J^2 (and more generally I^2, Y^2, $\cdots$) must be semi-simple. This is because the invariant operator of an invariant subgroup of a group is also an invariant operator of the group itself,[8] and if,

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[1] For symmetry-breaking terms in SU_3 see M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964).

[2] For symmetry-breaking terms in SU_6 see F. Gürsey and L. Radicati, Phys. Rev. Letters **13**, 173 (1964). M. M. B. Bég and V. Singh, *ibid.* **13**, 418 (1964).

[3] A. O. Barut, University of Colorado report, 1963 (unpublished).

[4] A. O. Barut, *Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1964).

[5] A. O. Barut, Phys. Rev. **135**, B839 (1964).

[6] The dynamical groups used here are among the groups of motion of Riemannian spaces. For a classification of such groups of motion see A. Rączka (to be published). In the usual mass formulas it is not clear whether one should use m^2, m, or $1/m^2$, etc. The present model definitely gives m^2 from the Casimir operator.

[7] Similar mass formulas involving $J(J+1)$ terms have now been obtained from broken SU_6 symmetry. See Ref. 2.

[8] A. Böhm, University of Marburg Report, 1964 (unpublished).

TABLE I. Passage from a dynamical group to the kinematical group.

System	Mathematical representation
Composite quantum system with mass spectrum	Irreducible representation of the dynamical group
↓	↓ group contraction
System without internal structure	Representation of the kinematical symmetry group

for example, P_μ or **J** (or **I**) would generate an invariant subgroup, they could not enter into the Eq. (1.2). Thus, as far as the mass-spectrum problem is concerned, P_μ's do not form an invariant subgroup. This is not surprising because in the problem of mass states of a quantum-mechanical composite system we do not have translational invariance. For example, there is no translational invariance in the H-atom problem. There is however, a translational invariance if we consider the motion of an H atom as a whole.

The passage from a dynamical group giving the mass spectrum of the system to what one may call the "kinematical group" of the system describing single mass states will be done by the process of group contraction.[9] The contracted kinematical groups are indeed classical groups, and it is interesting that the contraction can be associated in a natural way with the limit $\hbar \to 0$. We obtain thus the situation shown in Table I.

In Sec. II we discuss first a nonrelativistic model which contains all the ingredients of the idea of a dynamical group and its contraction, but is mathematically much simpler. From a realistic model of a particle we shall require that the contraction contains the Poincaré group. Such a model is discussed in Sec. III together with its implications for the structure of strongly interacting particles.

II. DYNAMICAL GROUP OF THE NON-RELATIVISTIC ROTATOR

We shall now interpret the homogeneous Lorentz group L as the dynamical group of a system. This model, which is described by the irreducible representations of L, will be called, for reasons which will be seen later, a rotator.

(1) The well-known commutation relations of L:

$$[M_{\mu\nu},M_{\sigma\rho}]=-g_{\nu\sigma}M_{\mu\rho}-g_{\mu\rho}M_{\nu\sigma}+g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma}$$
$$g_{00}=1;\quad g_{ii}=-1,\quad i=1,2,3;\quad g_{\nu\mu}=0,\quad \nu\neq\mu, \tag{2.1}$$

may be written in the following form:

$$\begin{aligned}
&[H_+,H_3]=-H_+,\quad [H_-,H_3]=H_-,\quad [H_+,H_-]=2H_3,\\
&[H_+,F_+]=[H_-,F_-]=[H_3,F_3]=0,\\
&[F_+,F_3]=H_+,\quad [F_-,F_3]=-H_-,\quad [F_+,F_-]=-2H_3,\\
&[H_+,F_3]=-F_+,\quad [H_-,F_3]=F_-,\\
&[H_+,F_-]=-[H_-,F_+]=2F_3,\\
&[F_+,H_3]=-F_+,\quad [F_-,H_3]+F_-,
\end{aligned} \tag{2.2}$$

where we have introduced the notations[10]

$$H_+=H_1+iH_2,\quad H_-=H_1-iH_2,$$
$$F_+=F_1+iF_2,\quad F_-=F_1-iF_2,$$
$$M_{\mu\nu}=-i\begin{pmatrix}0 & F_1 & F_2 & F_3\\ & 0 & H_3 & -H_2\\ & & 0 & H_1\\ & & & 0\end{pmatrix}. \tag{2.3}$$

The irreducible representations of these commutation relations in Hilbert space can be written as follows[10]:

$$\begin{aligned}
H_+f_\nu{}^k&=[(k+\nu+1)(k-\nu)]^{1/2}f_{\nu+1}{}^k,\\
H_-f_\nu{}^k&=[(k+\nu)(k-\nu+1)]^{1/2}f_{\nu-1}{}^k,\\
H_3f_\nu{}^k&=\nu f_\nu{}^k,\\
F_+f_\nu{}^k&=[(k-\nu)(k-\nu-1)]^{1/2}C_kf_{\nu+1}{}^{k-1}-[(k-\nu)(k+\nu+1)]^{1/2}A_kf_{\nu+1}{}^k+[(k+\nu+1)(k+\nu+2)]^{1/2}C_{k+1}f_{\nu+1}{}^{k+1},\\
F_-f_\nu{}^k&=-[(k+\nu)(k+\nu-1)]^{1/2}C_kf_{\nu-1}{}^{k-1}-[(k+\nu)(k-\nu+1)]^{1/2}A_kf_{\nu-1}{}^k-[(k-\nu+1)(k-\nu+2)]^{1/2}C_{k+1}f_{\nu-1}{}^{k+1},\\
F_3f_\nu{}^k&=[(k-\nu)(k+\nu)]^{1/2}C_kf_\nu{}^{k-1}-\nu A_kf_\nu{}^k-[(k+\nu+1)(k-\nu+1)]^{1/2}C_{k+1}f_\nu{}^{k+1},
\end{aligned}$$

where

$$A_k=ik_0c/k(k+1),$$
$$C_k=i/k[(k^2-k_0{}^2)(k^2-c^2)/(4k^2-1)]^{1/2} \tag{2.4}$$
$$\nu=-k,\,-k+1,\cdots,k;\quad k=k_0,\,k_0+1,\,k_0+2,\cdots.$$

Here $f_\nu{}^k$ are elements of the irreducible representation (Hilbert) space $\mathcal{H}(k_0,c)$ where (k_0,c)—k_0 integral or half-integral non-negative, c arbitrary complex number —characterize the irreducible representation. These representations are unitary representations of L for $C=ia$ and $(k_0,a)\in X$ where X is the following set:

$$X=\{(k_0,a)\,|\,k_0=\tfrac{1}{2},1,\tfrac{3}{2},\cdots,-\infty<a<+\infty\}$$
$$\cup\{(k_0,a)\,|\,k_0=0,\,0\leq ia\leq 1\}. \tag{2.4a}$$

[9] E. Inönü and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **39**, 510 (1953); **40**, 119 (1954).

[10] M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, Inc., New York, 1964) and references given there.

(The irreducible unitary representations belonging to the first set of X are called representations of the main series; those belonging to the second set are called representations of the supplementary series.[10])

The irreducible representation space of L is the direct sum of irreducible representation space of the rotation group

$$\mathcal{H}(k_0,a)=\sum_{j=k_0}^{\infty}\oplus\mathfrak{M}_j. \tag{2.5}$$

The invariant operators of L,

$$Q=\tfrac{1}{2}M_{\mu\nu}M^{\mu\nu},\quad \bar{Q}=\tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma}, \tag{2.6}$$

have in an irreducible representation $\mathcal{H}(k_0,a)$ the following values:

$$\begin{aligned}&\bar{Q}f_{j_3}{}^j(k_0,a)=k_0af_{j_3}{}^j(k_0,a)\\&Qf_{j_3}{}^j(k_0,a)=(1+a^2-k_0{}^2)f_{j_3}{}^j(k_0,a)\equiv\alpha^2f_{j_3}{}^j(k_0,a).\end{aligned} \tag{2.7}$$

(2) To perform the contraction of L with respect to its rotation subgroup[9] we introduce the following transformation of the Lie algebra

$$P_i=\lambda F_i,\quad J_i=H_i. \tag{2.8}$$

Then the commutation relations (2.1) become

$$\begin{aligned}[][P_i,P_j]&=i\lambda^2\epsilon_{ijk}J^k\\ [P_i,J_k]&=i\epsilon_{ik}{}^lJ_l\\ [J_i,J_k]&=i\epsilon_{ik}{}^lJ_l,\end{aligned} \tag{2.9}$$

and the Casimir operator (2.6) can be written in the form

$$\lambda^2Q=P_\mu P^\mu-\lambda^2\mathbf{J}^2. \tag{2.6'}$$

For $\lambda\to 0$, Eq. (2.9) goes over to the commutation relations of the 3-dimensional Euclidean group E_3, and the P_i and J_i represent the momentum and angular-momentum operators.

To obtain the representations of the contracted group E_3, we choose a sequence [9] of irreducible unitary representations of L in such a way that $\lambda^2\alpha^2\to\epsilon^2$ as $\lambda\to 0$, whereby ϵ^2 can be chosen arbitrarily and characterizes that representation of E_3 to which the representation of L is contracted. Because of (2.7) this means $\lambda^2\alpha^2=\lambda^2(1+a^2-k_0{}^2)\to\epsilon^2$. We restrict ourselves to the case where k_0 remains constant during the contraction process. Since k_0 is the smallest spin occurring in an irreducible representation $\mathcal{H}(k_0,a)$ of L, this means that the irreducible representation spaces $\mathfrak{M}_j$ of the rotation group contained in $\mathcal{H}(k_0,a)$ are also contained in the contracted representation. Then $\lambda^2\alpha^2\to\epsilon^2$ as $\lambda\to 0$ means $\lambda^2a^2\to\epsilon^2$.

From (3.4) we obtain the contracted representation

$$J_3f_{j_3}{}^j=j_3f_{j_3}{}^j;\quad J_+f_{J_3}{}^J=[(j+j_3+1)(j-j_3)]^{1/2}f_{j_3+1}{}^j,\quad J_-f_{j_3}{}^j=[(j+j_3)(j-j_3+1)]^{1/2}f_{j_3-1}{}^j \tag{2.10a}$$

$$P_3f_{j_3}{}^j=\lambda F_3f_{j_3}{}^j=[(j-j_3)(j+j_3)]^{1/2}C_jf_{j_3}{}^{j-1}-j_3a_jf_{j_3}{}^j-[(j+j_3+1)(j-j_3+1)]^{1/2}C_{j+1}f_{j_3}{}^{j+1} \tag{2.10}$$

$$\begin{aligned}P_1f_3{}^j=\tfrac{1}{2}\lambda(F_++F_-)f_{j_3}{}^j=&\tfrac{1}{2}a_j\{[(j-j_3)(j-j_3-1)]^{1/2}f_{j_3+1}{}^{j-1}-[(j+j_3)(j+j_3-1)]^{1/2}f_{j_3-1}{}^{j-1}\}\\&-\tfrac{1}{2}a_j\{[(j-j_3)(j+j_3+1)]^{1/2}f_{j_3+1}{}^j+[(j+j_3)(j-j_3+1)]^{1/2}f_{j_3-1}{}^j\}\\&+\tfrac{1}{2}C_{j+1}\{[(j+j_3+1)(j+j_3+2)]^{1/2}f_{j_3+1}{}^{j+1}-[(j-j_3+1)(j-j_3+2)]^{1/2}f_{j_3-1}{}^{j+1}\}\end{aligned} \tag{2.10b}$$

$$\begin{aligned}P_2f_{j_3}{}^\alpha=\frac{\lambda}{2i}(F_+-F_-)f_{j_3}{}^\alpha=&\frac{C_j}{2i}\{[(j-j_3)(j-j_3-1)]^{1/2}f_{j_3+1}{}^{j+1}+[(j+j_3)(j+j_3-1)]^{1/2}f_{j_3-1}{}^{j-1}\}\\&-\frac{a_j}{2i}\{[(j-j_3)(j+j_3+1)]^{1/2}f_{j_3+1}{}^j-[(j+j_3)(j-j_3+1)]^{1/2}f_{j_3-1}{}^j\}\\&+\frac{C_{j+1}}{2i}\{[(j+j_3+1)(j+j_3+2)]^{1/2}f_{j_3+1}{}^{j+1}+[(j-j_3+1)(j-j_3+2)]^{1/2}f_{j_3-1}{}^{j+1}\},\end{aligned}$$

where

$$a_j=\lim_{\lambda_2a'\to\epsilon_2}\lambda A_j=\lim\left[-\frac{\lambda ak_0}{j(j+1)}\right]=-\epsilon\frac{k_0}{j(j+1)}$$

and[11] (2.11)

$$C_j=\lim_{\lambda_2a'\to\epsilon_2}\lambda C_j=\epsilon\frac{i}{j}\left[\frac{j^2-k_0{}^2}{4j^2-1}\right]^{1/2}.$$

Those irreducible representations of L for which $\alpha^2=0$ (or some fixed value in the process of contraction) go under the contraction into the representations of the rotation group (2.10a), for in this case $a_j=c_j=0$ and

[11] For groups like E_3, in which the translations form an invariant subgroup, one usually has representations in which the translations are diagonal:

$$P_i|p_1,p_2,p_3,\xi\rangle=p_i|p_1p_2p_3,\xi\rangle,\quad P_i=p\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}$$
$$J_3|0,0,p,\xi\rangle=\xi|0,0,p,\xi\rangle.$$

The transformation from the basis (2.10) to this basis $|p_1,p_2,p_3,\xi\rangle$ which lies outside the Hilbert space (Ref. 15), i.e.,

$$|p_1p_2p_3\xi\rangle=\sum_{j_3j}f_{j_3}{}^j(k_0\epsilon)\langle\epsilon k_0j_3j|p_1p_2p_3\xi\rangle$$

will be performed by the transition coefficients given explicitly

$$\langle\xi,p_3,p_2,p_1|j,j_3,k_0,\epsilon\rangle=(2j+1)\bar{D}_{j_3k_0}{}^j(\phi,\theta,-\varphi)\delta_{\xi k_0}\delta(\epsilon-|p|),$$

where $D_{j_3k_0}{}^j$ are the well-known rotation matrices [M. E. Rose *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1961)].

the momentum operators will be represented by the zero operator.

From (2.10) we see that the representation space obtained by contraction from $\mathcal{H}(k_0,a)$ which we denote by $\mathcal{H}(k_0,\epsilon)$ is again a direct sum of irreducible representation spaces of the rotation group

$$\mathcal{H}(k_0,\epsilon)=\sum_{j=k_0}^{\infty} \mathfrak{M}_j$$

and each basis vector $f_{j_3}{}^j(k_0,a)$ goes over into $f_{j_3}{}^j(k_0,\epsilon)$.[12]

(3) According to our general concept formulated in Sec. I we can now give the following physical interpretation. Our physical (model) system is described by the representation $\mathcal{H}(k_0,a)$ of the dynamical group L. By contraction this goes into the representation $\mathcal{H}(k_0,\epsilon)$ of the Euclidean group or into the representation $\mathcal{H}(k_0,0)$ of the rotation group. A system, which is defined by the representation of the rotation group, is called rotator; therefore the representation $\mathcal{H}(k_0,a)$ of the dynamical group L must give us the actual energy levels of the rotator. As we have seen in Sec. (2), the P_i correspond to the momentum operators and it is natural to interpret the expectation value p^2 of P_iP^i as the momentum square and hence $E=p^2/2\mu$ (μ a constant with dimension of mass) as the energy. The energy spectrum can then be calculated from the Casimir operator (2.6′) as follows:

$$2\mu E=p^2=(f_{j_3}{}^j,P_iP^if_{j_3}{}^j)=\lambda^2\alpha^2+\lambda^2(f_{j_3}{}^j,\mathbf{J}^2f_{j_3}{}^j)$$

or

$$E=(1/2\mu)\lambda^2\alpha^2+(\lambda^2/2\mu)j(j+1)\,, \qquad (2.12)$$

and for the special case $\alpha=0$

$$E=(\lambda^2/2\mu)j(j+1)\,. \qquad (2.12')$$

This is indeed the well-known energy spectrum of a rotator with $\lambda=\hbar/\rho_0$, where ρ_0 is a quantity with the dimension of a length. The first part in (2.12), $\lambda^2\alpha^2/2\mu$ $=$const, represents some translational energy and is without significance because E is only defined up to a constant.

Because $\lambda=\hbar/\rho_0$ the limiting process $\lambda\to 0$ from the dynamical group to the kinematical symmetry group can be taken to be the limit $\hbar\to 0$.

III. DE SITTER MODEL OF A RELATIVISTIC "ROTATOR"

The (4+1) de Sitter group ($NR_5{}^4$) as dynamical group furnishes us with a relativistic generalization of the rotator model.

(1) The commutation relations for the de Sitter group are

$$[L_{\alpha\beta},L_{\gamma\delta}]=-i_1(g_{\alpha\gamma}L_{\beta\delta}+g_{\beta\delta}L_{\alpha\gamma}-g_{\alpha\delta}L_{\beta\gamma}-g_{\beta\gamma}L_{\alpha\delta})$$
$$\alpha,\beta,\gamma,\delta=5,0,1,2,3 \qquad (3.1)$$
$$g_{55}=-1\,,\quad g_{00}=1\,,\quad g_{ii}=-1\,,\quad i=1,2,3\,,$$

and its invariant operators

$$Q=-\tfrac{1}{2}L_{\alpha,\beta}L^{\alpha\beta}$$
$$W=-w_\alpha w^\alpha\,,\quad w^\alpha=\tfrac{1}{8}\epsilon^{\alpha\beta\gamma\delta\rho}L_{\beta\gamma}L_{\delta\rho}\,. \qquad (3.2)$$

The contraction of the de Sitter group with respect to the homogeneous Lorentz group leads to the Poincaré group. We define

$$P_\mu=\lambda L_{5\mu}\,,\quad M_{\mu\nu}=L_{\mu\nu};\quad \mu,\nu=0,1,2,3\,. \qquad (3.3)$$

Then (3.1) reads

$$[P_\mu,P_\nu]=i\lambda^2M_{\mu\nu}$$
$$[M_{\mu\nu},P_\rho]=i(-g_{\mu\rho}P_\nu+g_{\nu\rho}P_\mu) \qquad (3.1')$$
$$[M_{\mu\nu},M_{\rho\sigma}]=-i(g_{\mu\rho}M_{\nu\sigma}+g_{\nu\sigma}M_{\mu\rho}-g_{\nu\rho}M_{\mu\sigma}-g_{\mu\sigma}M_{\nu\rho})\,.$$

The invariant operators can be written as

$$\lambda^2Q=P_\mu P^\mu+\lambda^2\mathbf{N}^2-\lambda^2\mathbf{M}^2$$
$$\lambda^2W=-\lambda^2(\mathbf{MN})^2-w_\mu w^\mu\,. \qquad (3.4)$$

For $\lambda\to 0$, (3.1′) gives the commutation relations of the Poincaré group, P_μ gives the momentum operators, $\mathbf{M}=(M_{23},M_{31},M_{12})$ the angular momentum, and $\mathbf{N}=(M_{01},M_{02},M_{03})$ the energy of the center of mass.

(2) The Hermitian representation of the commutation relations (3.1) of the (4+1) de Sitter group in the Hilbert space—in which the maximal compact subalgebra (D_2) is diagonal—were given by Thomas[13] and Newton.[14] Newton gives four classes of irreducible representations, of which class I and class II can be contracted to physical ($P_\mu P^\mu=m_0{}^2>0$) representations of the Poincaré group. The Hilbert-space basis of Thomas and Newton is not suitable for our purpose; we need a basis of the irreducible representation space, in which the subgroup L, with respect to which we contract, is diagonal. Such a basis will be given by the vectors defined in (2.4)

$$f_{j_3}{}^j(k_0,a)=|\,j,j_3,a,k_0;\,e,\beta\rangle\,, \qquad (3.5)$$

where a and k_0 are now variable and vary over a subset X' of X [see (2.4a)] and e and β characterize the irreducible representation of $NR_5{}^4$. The irreducible representation space $\mathcal{H}(e,\beta)$ is then a continuous direct sum[15] of Hilbert spaces $\mathcal{H}(k_0,a)$ of which each is an irreducible representation space of the Lorentz group L[16]:

$$\mathcal{H}(e,\beta)=\int_{X'}\oplus\mathcal{H}(k_0,a)d\mu(k_0,a)\,. \qquad (3.6)$$

[12] The reason for this is that we have chosen the representation conveniently in such a way that the subgroup with respect to which the contraction is performed is diagonally represented.

[13] L. H. Thomas, Ann. Math. 42, 113 (1941).

[14] T. D. Newton, Ann. Math. 3, 730 (1950).

[15] J. M. Gelfand and N. Ja Vilenkin, *Obobshchennye Funktsii* (Moscow, 1961) (translation in preparation in Academic Press Inc., New York), Vol. 4. A. Bohm, International Center for Theoretical Physics Report No. ICTP-9, 1964 (unpublished).

[16] We note that the basis vectors (3.5) are eigenvectors of unbounded operators with continuous spectrum and no longer elements of the Hilbert space but its rigging (Ref. 15).

A complete set of commuting operators of NR_5^4 is formed by

$$\mathbf{M}^2, M_3 = M_{12}, \quad \mathbf{M}^2 - \mathbf{N}^2, \mathbf{M}\cdot\mathbf{N}, L_{5\mu}L^{5\mu} = 1/\lambda^2 P_\mu P^\mu, W. \tag{3.7}$$

This set has the following eigenvalues on the basis vectors (3.5):

$$\begin{aligned} &\mathbf{M}^2 | jj_3ak_0e\beta\rangle = j(j+1)|\rangle; && M_3|\rangle = j_3|\rangle \\ &(\mathbf{M}^2-\mathbf{N}^2)|\rangle = (k_0^2-a^2-1)|\rangle, && \mathbf{M}\cdot\mathbf{N}|\rangle = k_0 a|\rangle \\ &L_{5\mu}L_5{}^\mu|\rangle = e^2|\rangle, && W|\rangle = \beta|\rangle. \end{aligned} \tag{3.8}$$

The eigenvalues of the second-order Casimiar operator Q on the space $\mathcal{H}(e,\beta)$ is given by

$$\alpha^2 = e^2 + a^2 + 1 - k_0^2.$$

The contraction of the irreducible representation of NR_5^4 into representations of the Poincaré group $\mathcal{P}$ is again performed via a sequence of representations $\alpha \to \infty$ such that

$$\lim \lambda^2\alpha^2 = \lim \lambda^2 e^2 = m_0^2, \tag{3.8'}$$

a value, which characterizes the irreducible representation of $\mathcal{P}$. In the contraction process a and k_0 are kept fixed and the contracted representation space contains therefore the same subset X' of representations of L as $\mathcal{H}(e,\beta)$. By the contraction process (3.8′) $\lambda^2\beta$ goes to $\sigma(\sigma+1)m_0^2$, $(\sigma = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots)$ and the contracted representations are characterized, as is well known, by $\mathcal{H}(m_0,\sigma)$. $\mathcal{H}(m_0,\sigma)$ is again a continuous direct sum of representation spaces $\mathcal{H}(k_0,a)$ of L:

$$\mathcal{H}(m_0,\sigma) = \int_{X'} \oplus \mathcal{H}(k_0,a) d\mu(k_0,a) \tag{3.9}$$

and each basis vector $|j,j_3,a,k_0;e,\beta\rangle$ goes into $|j,j_3,a,k_0;m_0\sigma\rangle$.[17]

(3) An "elementary particle" (EP) (without intrinsic degrees of freedom as isospin and hypercharge) is characterized by an irreducible representation of the Poincaré group as its (kinematical) symmetry group. A composite physical system, which is described by representation of the de Sitter group as dynamical group goes therefore by contraction into an EP. Its mass spectrum is described by the de Sitter group as the dynamical group in the following manner:

According to what we have said previously it appears natural to interpret $P_\mu P^\mu$ as the "mass operator" of the composite particle. If the physical system is in the state described by $|jj_3ak_0,e\beta\rangle$, the expectation value (=eigenvalue) of the mass operator is given according to (3.4) by

$$\begin{aligned} m^2 &= \langle \beta e; k_0 a j_3 j | P_\mu P^\mu | jj_3 ak_0; e\beta\rangle \\ &= \lambda^2\alpha^2 + \lambda^2\langle|\mathbf{M}^2|\rangle - \lambda^2\langle|\mathbf{N}^2|\rangle, \end{aligned} \tag{3.10}$$

and for those states for which

$$n^2 = \langle \beta e, k_0 a j_3 j | \mathbf{N}^2 | \rangle = 0 \tag{3.11}$$

we obtain the mass formula

$$m^2 = \lambda^2\alpha^2 + \lambda^2 j(j+1), \tag{3.12}$$

which is in agreement with the phenomenological mass formula (1.1). So we see, using (1.1b), that the limiting process $\lambda \to 0$ from the dynamical group to the kinematical symmetry group can be interpreted as the limit $\hbar \to 0$. We have started from a representation of the de Sitter group characterized by *one* α (or e) and *one* β. One could also have started from a reducible representation characterized by several β. This would be the case if the de Sitter group is embedded into a larger dynamical group, an irreducible representation of which then contains many irreducible representations of the de Sitter group.

(4) Some remarks are necessary with regard to the (3.11) and (3.12): The expectation values of $\mathbf{M}^2$ and $\mathbf{N}^2$ are the same before and after the contraction because

$$\begin{aligned} j(j+1) &= \langle \beta e k_0 a j_3 j | M^2 | jj_3 ak_0 e\beta\rangle \\ &= \langle \sigma m_0 k_0 a j_3 j | M^2 | jj_3 ak_0 m_0\sigma\rangle \end{aligned} \tag{3.13}$$

and similarly for N^2.

From the properties of the transition coefficients $\langle k_0 a j_3 j | p_i s_3\rangle$[18] one can see, using (4.18), (4.21), (4.24), (4.27), and (4.36) of Ref. 19, that for the rest states $|p=0,\ s_3;\ m,\ s\rangle$ there is a one-to-one correspondence between j and s: $j \leftrightarrow s$, because $\langle k_0 a j_3 j | p=0, s_3, ms\rangle \sim \delta_{sj}$ as one should expect. Thus only the states

$$|jj_3ak_0; m\sigma\rangle \quad \text{with} \quad j = s \tag{3.14}$$

contribute to the rest states. If we could show that for these states $n^2=0$, we would have proved that the mass formula (3.12) is valid for states which correspond to rest states after contraction, then we could call m^2 the "rest mass,[20]" what we actually want, as the phenomenological formula (1.1) holds for rest masses.

But as long as we do not know more about the representation of NR_5^4 we can only make some plausibility arguments with respect to this point. The equation

$$n^2 = j(j+1) + k_0 - a^2 - 1 = 0 \tag{3.15}$$

[17] This is a representation of the Poincaré group in which the homogeneous Lorentz group is diagonal (compare Ref. 11). The transformation of this basis to the well-known canonical basis Ref. 19 $|p_i,s_3;m,s\rangle$:

$$|p_i,s_3;m,s\rangle = \sum_{jj_3ak_0m,\sigma} |j,j_3ak_0m_0\sigma\rangle\langle\sigma m_0,k_0 a j_3 j | p_i,s_3,ms\rangle$$

will be performed by the transition coefficients

$$\langle \sigma m_0 a j_3 j | p_i,s_3;m,s\rangle = \delta_{s\sigma}\delta(m_0-m)\langle k_0 a j_3 j | p_i s_3\rangle,$$

where $\langle k_0 a j_3 j | p_i s_3\rangle$ are given by Joos [H. Joos, Fortschr. Physik **10**, 3 (1962), Secs. 4.2 and 4.3 (see also Chou Kuang-Chao and L. G. Zastavenko, Zh. Eksperim. i Teor. Fiz. **35**, 1417 (1959) [English transl.: Soviet Phys.—JETP **35**, 990 (1959)] and references given there)].

[18] Reference 16.

[19] H. Joos, Fortschr. Physik **10**, 3 (1962).

[20] Still the question, what this "rest mass" has to do with the experimentally measured mass, remains unanswered.

is possible for $(k_0,a)\epsilon X$ (III.4) as is easily seen (and $n^2 \to \infty$ could only be possible for $j \to \infty$). We suppose that those (k_0,a), for which (3.15) is fulfilled are also in $X' \subset X$ (III.2). The set X' of course depends on e and β; $X'(e,\beta)$, and we could hope that only those (k_0,a) fulfilling (15) and corresponding to rest states are in the set $X'(e\beta)$.

(5) The (4+1) de Sitter group is the group of motion in the de Sitter spherical world[21] with finite extension in space-like and infinite extension in time-like directions. Its curvature tensor is

$$R_{\mu\nu\lambda\rho} = -\lambda^2(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \tag{3.16}$$

and Einstein's law reads

$$G_{\nu\lambda} = 3\lambda^2 g_{\nu\lambda}, \tag{3.17}$$

where λ is the parameter introduced in (3.3). The radius of the de Sitter world is[21] $R=1/\lambda$, and if we use for λ the empirical value (1.1a) converting MeV into cm^{-1} we obtain for our de Sitter world a radius of $R \approx 10^{-13}$ cm.[22] Thus we can consider a strongly interacting particle as a de Sitter "spherical" world of 10^{-13} cm with finite space-like and infinite time-like extension,[23] a picture which is not too far from our usual image of a particle, which might indicate that our model is not too far from reality.[24]

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[21] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, New York, 1963), 10th ed., Chap. V

[22] A geometrical interpretation of the contraction process is difficult, as it does not simply mean $R \to \infty$ but $\alpha/R \to m_0$.

[23] If we had chosen the (3+2) de Sitter group [J. B. Ehrman, Proc. Cambridge Phil. Soc. **53**, 290 (1957)] instead of the (4+1), the strongly interacting particle would have been infinite in space but finite in time, in disagreement with our ideas of a particle, but then we would also have obtained $m^2 = \lambda^2\alpha^2 - \lambda^2 j(j+1)$ instead of (3.12), which is in obvious disagreement with experimental facts. It should be remarked that the use of the 3+2 deSitter group in the large [see, e.g., C. Fronsdal, Rev. Mod. Phys. **37**, 221 (1965)] is a completely different idea from the present one. There a single mass point is embedded in a large curved universe and has a discrete spectrum which in the limit $R \to \infty$ goes over into the continuum states of the mass point in flat space; it is a change of the kinematical group and has nothing to do with dynamics.

[24] For a discussion and interpretation of de Sitter rotator in which the compact subgroup R_4 is diagonalized see A. O. Barut, in Seminar on High Energy Physics and Elementary Particles, Trieste, 1965 (unpublished).