Nuclear Coupling Schemes with a Surface Delta Interaction*†

IRA M. GREEN
TRW Space Technology Laboratories, Redondo Beach, California

AND

S. A. MOSZKOWSKI
University of California, Los Angeles, California

(Received 1 April 1965)

Recent work indicates that the major part of the residual interaction in nuclei, namely, both the quadrupole and pairing interactions, acts primarily when particles are near the nuclear surface. Since the one-particle radial wave functions at the surface are essentially state-independent, a surface interaction implies that all radial integrals are approximately the same. Thus for a surface *delta* function all radial integrals are assumed to be identical. For mixed two-particle configurations, e.g. $(s,d)^2$, the surface delta-function interaction gives a first excited $2+$ state at a lower energy than a conventional delta function acting throughout the nuclear volume. For the $(s,d)^4_{T=2}$ configuration one obtains essentially a vibrational spectrum in both cases. However, for the $(s,d)^4_{T=0}$ configuration involving both neutrons and protons, the conventional delta function and surface delta function give quite different spectra. The former leads to a spectrum similar to the two-particle case, whereas for a surface delta-function interaction, the lowest states are $0+$, $2+$, and $4+$ with a near-rotational spacing. It appears then that it is possible to obtain a rotational spectrum even with a short-range interaction, provided we have mixed configurations *and* both neutrons and protons participating.

INTRODUCTION

THE pairing- and quadrupole-interaction model has been quite successful in accounting for many systematic features of nuclear levels.[1] These interactions represent the *residual* interactions left over when we take into account the effect of the average one-particle potential of the nucleons. The quadrupole interaction is well known to act mainly when nucleons are at the nuclear surface and the pairing interaction usually has been treated as a volume effect. Recent calculations suggest that the pairing energy would be extremely small ($\leq$100 keV) in nuclear matter[2] and that the empirical pairing energies ($\sim$1 to 2 MeV) are due primarily to interactions at the nuclear surface.[3] It is thus plausible to suppose that most, if not all, of the residual interactions which describe the deviation of the nuclear Hamiltonian from the independent-particle model, act at the nuclear surface. In other words, the nucleons move independently inside the nuclear interior and collide only when they are in the surface region.[4] Such a surface-interaction model has been used in some studies of nuclear reactions, and it appears to be consistent with empirical evidence.[5] Of course, there are also correlations in nuclear matter due to the short-range repulsion in the nucleon-nucleon interaction. However, these correlations are nearly state-independent and can be neglected at low energies.

Let us examine some of the consequences of the assumption that residual two-particle interactions take place at the nuclear surface only. First of all, the one-particle radial wave functions all have approximately the same amplitude at the surface. If they were *exactly* the same there, then the different radial integrals, say F^k, which appear in the well-known Slater expansion would be equal. (This is also obtained if the two-body interaction is only a function of the *angle* between the coordinate vectors of the two particles, measured from the center of the nucleus.) Of course, the different radial wave functions differ greatly in the nuclear interior. Consequently, for a conventional two-body interaction acting through the nuclear volume, the F^k are significantly state-dependent.

For mathematical simplicity we will consider here only a contact or delta-function interaction. This implies that all quantities $F^k/(2k+1)$ and $G^k/(2k+1)$ are the same. While this is clearly a very idealized model, it illustrates the differences between the surface and volume interactions, and we will discuss these differences for some simple configurations. We shall see that the surface delta-function interaction combines some of the desirable features of both pairing and quadrupole interactions. On the other hand, for surface interaction we cannot use the Talmi expansion,[6] which requires

* Part of this work was done by one of us (I. M. G.) in partial fulfillment of the Ph.D. requirements at the University of California.

† Work supported in part by the National Science Foundation.

[1] L. S. Kisslinger and R. A. Sorenson, Rev. Mod. Phys. **35**, 853 (1963).

[2] K. A. Brueckner, T. Soda, P. W. Anderson, and P. Morel, Phys. Rev. **118**, 1442 (1960); V. J. Emery and A. M. Sessler, *ibid.* **119**, 248 (1960).

[3] R. C. Kennedy, L. Wilets, and E. Henley, Phys. Rev. Letters **12**, 36 (1964); S. Nagata and H. Bando, Phys. Letters **11**, 155 (1964).

[4] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **27**, 16 (1953) introduced a quadrupole coupling of nucleons via the surface. J. P. Elliott, Proc. Roy. Soc. (London) **A245**, 128 (1958) introduced the quadrupole-quadrupole interaction $r_1^2 r_2^2 P_2(\cos\theta_{12})$ which is also peaked at the nuclear surface. The general surface-interaction model was discussed by S. A. Moszkowski, Phys. Rev. **110**, 403 (1958).

[5] See, for example, the review by P. E. Hodgson, in *Selected Topics in Nuclear Spectroscopy*, edited by B. J. Verhaar (North-Holland Publishing Company, Amsterdam, 1964), in particular pp. 291–292.

[6] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), pp. 238–244.

separation of wave functions into center-of-mass and relative coordinates.

TWO-PARTICLE CONFIGURATIONS

Consider a pair of particles in states of orbital angular momentum l_1 and l_2, coupling to a resultant orbital angular momentum L. The interaction energy for even L due to a spin-independent delta function[7] is given by

$$U(l_1l_2)_L=((2l_1+1)(2l_2+1)/(2L+1)) \times C^2(l_1l_2L,\,000)F^0. \quad (1)$$

The interaction energy vanishes in all spatially antisymmetric states. For a surface delta function, all of the F^0 are the same. Likewise it is readily shown for this interaction that every off-diagonal matrix element is just the geometric mean of the corresponding diagonal matrix elements.

$$U[(l_1l_2)_L \rightarrow (l_1'l_2')_L]=[U(l_1l_2)_L U(l_1'l_2')_L]^{1/2} \quad (2)$$

As an example, the interaction energy matrix (in units of F^0) for the states of an $(s,d)^2$ configuration is given in Table I. We are assuming that the single-particle

TABLE I. Energy matrix for two particles in s,d shell interacting via surface delta-function potential.

$L=0$			$L=2$			$L=4$	
	d^2	s^2		d^2	$2^{-\frac{1}{2}}(ds+sd)$		d^2
d^2	5	$\sqrt{5}$	d^2	10/7	$(20/7)^{\frac{1}{2}}$	d^2	10/7
s^2	$\sqrt{5}$	1	$2^{-\frac{1}{2}}(ds+sd)$	$(20/7)^{\frac{1}{2}}$	2		

$U=0$ for $(d^2)_{L=1}$, $(d^2)_{L=3}$, and $2^{-\frac{1}{2}}(ds-sd)_{L=2}$.

s and d states are degenerate. The energy eigenvalues are $6F^0$ and 0 for $L=0$, $(24/7)F^0$ and 0 for $L=2$, and $(10/7)F^0$ for $L=4$. For identical particles, the Pauli principle requires $S=0$ for spatially symmetric states of two particles. Thus all these levels have $J=L$. It is interesting that for each L value there is only a *single* level with nonvanishing interaction energy. This follows quite generally from Eq. (2) in the case of a surface delta interaction for an *arbitrary* mixed configuration, as long as the different single-particle states are all degenerate. We find

$$E=\sum_{l_1}\sum_{l_2}U(l_1l_2)_L \quad (3)$$

where the sum is to be taken over all *pairs* of degenerate orbits. In this sense a surface delta-function interaction acts like a pairing interaction.[8] Consider any matrix in which each off-diagonal element is the geometric mean of the diagonal elements in its row and column. Regardless of the size of the matrix, there will be only one nonzero eigenvalue which equals the trace of the matrix.

[7] Reference 6, p. 219.

[8] For a review of the pairing interaction see, for example, A. Lane, *Nuclear Theory* (W. A. Benjamin, Inc., New York, 1964), Chap. 1.

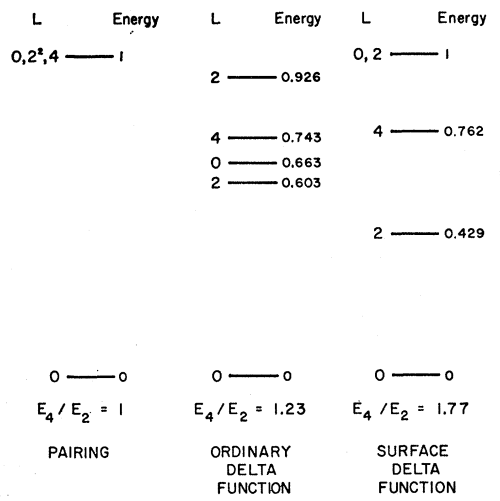

FIG. 1. Energy levels of $(s,d)^2$ configuration for pairing, ordinary delta function, and surface delta function interactions. Only spatially symmetric states are shown and the scale is chosen so that the ground state is at 0 and zero-energy shift is at unity.

In fact, the $L=0$ energy matrix and ground-state wave function is the same as that for a pairing interaction. However, the surface delta function, unlike a pairing interaction, also acts in states with $L\neq 0$. A similar argument can be given for the case of jj coupling.[9] For two identical particles, i.e., $T=1$, Eqs. (2) and (3) still hold with l replaced by j and L replaced by J.

The $(s,d)^2$ energy-level scheme with a surface delta-function interaction differs significantly from that with a conventional delta function. For the latter let us use oscillator wave functions and assume the s and d levels belong to the $N=2$ shell. Figure 1 shows the calculated energy levels for these two interactions as well as a pairing interaction.

For the ordinary delta function, the $L=0$ state is well isolated from all the others. This is similar to the case of the pairing interaction. However, in going from the ordinary to the surface delta function, the $L=2$ state drops down markedly. The difference between the two delta-function level schemes can be traced in large measure to the larger off-diagonal matrix elements of the surface interaction.[9a] These in turn are due to the coherence of the radial wave functions when only the surface matters. If the interaction can take place in the nuclear interior as well, then the imperfect overlap of the different wave functions reduces the off-diagonal elements considerably.

[9] Ira M. Green, Ph.D. dissertation, University of California, Los Angeles, 1964 (unpublished).

[9a] *Note added in proof.* Suppose we normalize an ordinary delta interaction so that its diagonal d^2 matrix elements are the same as for the surface delta interaction shown in Table 1: Then its off-diagonal $L=0$ matrix element is only 1.331 (versus 2.236 for surface delta interaction) while its off-diagonal $L=2$ matrix element is 0.891 (versus 1.690). Similarly the $L=2$ diagonal matrix element for sd is smaller (1.190 versus 2). On the other hand, the ordinary delta interaction has a larger $L=0$ diagonal matrix element for s^2 (2.440 versus 1).

FOUR-PARTICLE CONFIGURATIONS

Consider now the energy levels for some four-particle configurations with surface delta-function interactions. First let us assume that we have identical particles only, i.e., states of maximum isospin. We find that the states are characterized by definite seniority.[9b] Thus all the states which appear for $(s,d)^2$ will likewise appear for $(s,d)^4$ at the same excitation energy. However, the ground-state energy for four particles is just twice as large as that for two particles. The excitation energies for two- and four-particle configurations are shown in Fig. 2. For $(s,d)^6$, which constitutes a half-filled shell, all the $(s,d)^4$ levels appear again as well as a new $0+$ level at 0.64 in our units. The energy spectrum is seen to resemble a quadrupole vibrational one at least qualitatively, i.e., for $(s,d)^6$, we have E_0'/E_2, E_2'/E_2 and E_4/E_2 equal to 1.5, 1.6, and 1.8, respectively. To the extent that the $2+$ state can be regarded as the one-phonon quadrupole state, we can say that $\hbar\omega_2$ remains constant as we fill the shell. This feature of the spectrum holds for any combination of degenerate single-particle orbits.[10] It is also satisfied, of course, for a pairing interaction. In the case of a conventional delta function, it holds exactly for a pure l^n or j^n configuration and only approximately for a mixed configuration such as $(s,d)^n$.

Now consider four-particle configurations involving both neutrons and protons, i.e., $T=0$ states. Our first example assumes only a single orbit l^n or j^n. In this case all particles must of course have the same radial wave function; thus there is no difference between ordinary and surface delta functions. Consider for example a degenerate $j=\frac{9}{2}$ shell. For the two-particle configuration, the energy of the first excited $2+$ state is slightly less than $\frac{1}{4}$ of the energy of the ground state. For the four-particle $T=2$ configuration, we get, as is well known,[11] a seniority-type spectrum, just as discussed above for the $(s,d)^n$ case with surface delta-function interaction. On the other hand, for $(\frac{9}{2})^4_{T=0}$, the excitation spectrum is only slightly different from the $(\frac{9}{2})^4_{T=0}$ case for the lowest states. These and other results are summarized in Fig. 3. The $T=0$ spectrum is not even approximately rotational. This result is consistent with the usual viewpoint[12] that nuclear deformations and rotational spectra are due to the long-range part of the interaction. Similar calculations made for other j values up to $j=11/2$ suggest that, in the limit as $j\to\infty$, the difference between the four-particle $T=2$ and $T=0$ spectra disappears. This is not surprising, since the Pauli principle (which inhibits correlations for $T=2$ but not for $T=0$) will have relatively less and less effect as j increases.

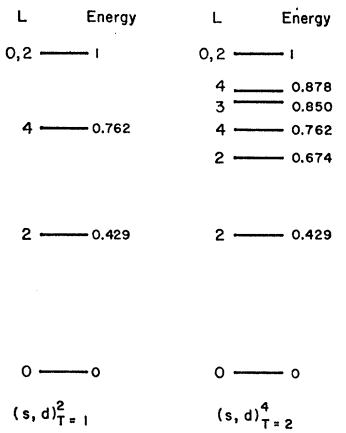

FIG. 2. Excitation energies of states in $(s,d)^2$ and $(s,d)^4$ configurations of identical particles with surface delta interaction. Energies are expressed in units of $6F^0$, the ground-state energy for $(s,d)^2$, and the scale is shifted so that the ground state is at 0. Only $S=0$ levels with excitation energies $\leq 6F^0$ are shown.

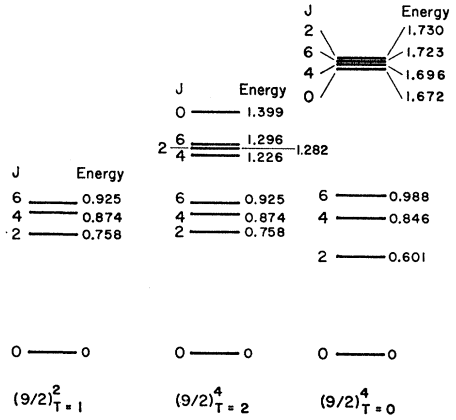

FIG. 3. Excitation spectrum for $(9/2)^2$ and $(9/2)^4$ configurations with delta interaction. Only the lowest two $J=0$, 2, 4, and 6 states are shown. Energies are expressed in units of the $(9/2)^2$ ground state energy and the scale is shifted so that the ground state is at 0.

Let us now study a four-particle configuration involving mixed orbits, e.g., $(s,d)^4$, *and* both neutrons and protons, i.e., $T=0$. In this case the ordinary and surface delta-function interactions give quite different spectra as is shown in Fig. 4. The ordinary delta-function interaction gives a spectrum similar to that for the two-particle case. On the other hand, for a surface delta function, the lowest $L=0$, 2, 4 states form a band which has nearly a rotational spacing of levels. Further calculations done by one of us (S. A. M.) using a two-dimensional analog of the (s,d) shell show explicitly that the ground-state wave function is very close to a projected determinant of deformed single-particle orbitals.[13] The results of these calculations also

[9b] For a discussion of seniority in mixed configurations, see, for example, R. D. Lawson and M. H. MacFarlane, Nucl. Phys. **66**, 80 (1965).

[10] This result is connected with the equality of the average particle-particle and particle-hole matrix elements for a delta-function potential. See for example, S. T. Belyaev, in *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963), in particular pp. 631–640.

[11] See, for example, A. de-Shalit and I. Talmi, Ref. 6, pp. 353–355.

[12] B. Mottelson in *Nuclear Spectroscopy* (Academic Press Inc., New York, 1962), pp. 44–99. O. Nathan and S. G. Nilsson, in *Alpha-, Beta- and Gamma-Ray Spectroscopy* edited by Kai Siegbahn (North-Holland Publishing Company, Amsterdam, 1965), Chap. 10.

[13] I. Kelson and C. A. Levinson, Phys. Rev. **134**, B269 (1964).

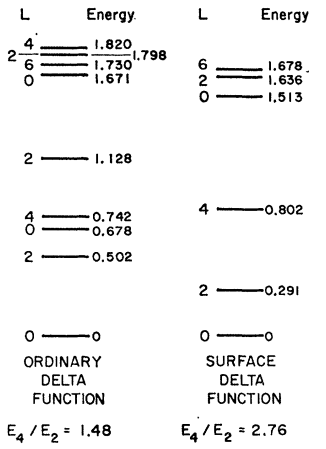

FIG. 4. Excitation energy of $(s,d)^4_{T=0}$ states for ordinary and surface delta-function interaction. Energies are expressed in units of the respective ground state energies for $(s,d)^2$ and the scale is shifted so that the ground state is at 0. Only $S=0$ levels with excitation energy ≤ 2 in our units are shown.

suggest that for a mixture of more single-particle orbits, e.g., $(s,d,g)^4$, a conventional delta-function interaction gives spectra similar to that for two particles. On the other hand, a surface delta function gives spectra which approach the rotational form as the number of degenerate orbits increases. These results can be understood on the basis of the following argument.

When we have mixed single-particle orbits available, it is possible to construct one-particle wave functions which are strongly localized in angle. Thus four particles coupling to $T=0$ can go into the same spatial state and thus be strongly correlated in angle. Using an attractive surface delta-function interaction [or the equivalent, namely a $\delta(\theta_{12})$ interaction], such "clusters" are strongly favored energetically. Of course, they cannot be static but must move around the nuclear surface; thus the lowest energy levels form a rotational band.

Why do we not find this type of spectrum for the other cases considered here? Because there is not too much overlap between the different radial wave-functions with a conventional delta function and there is somewhat less coherence between the matrix elements, there is less "angular correlation" than for a surface delta-function interaction. To obtain significant clustering for the other cases considered we need a finite-range interaction.

The situation is similar for a pure j^n configuration. In this case the particle orbits are not localized as well as they are for mixed configurations. Since the angular momentum is a good quantum number, the conjugate coordinate, namely the azimuthal angle, must be completely indeterminate. Thus, for $m=\pm j$, the particle density is localized close to the plane $\theta=\frac{1}{2}\pi$, but the azimuthal angle is indeterminate. It is thus not possible to form localized clusters when the l or j of each particle is a good quantum number.

CONCLUSION

As we have seen, it is possible to obtain rotational spectra even with short-range interactions. It is necessary, however, to have mixed configurations *and* both neutrons and protons participating. Indeed it is well known that low-lying rotational spectra occur only in nuclei with both neutrons *and* protons outside closed shells. The surface delta function seems to combine desirable features of both pairing and quadrupole interactions, the former through the interaction in two-particle $L=0$ states, the latter through the interaction in other two-particle states.

Of course, such an interaction is extremely idealized and cannot be expected to give good fits to specific nuclear levels. For example, it is well known that the effective particle-particle interactions in the oxygen isotopes must have a finite range.[14]

On the other hand, a delta-function interaction may be a somewhat better approximation to the particle hole interaction, e.g., in K^{40}. Indeed recent calculations[15] indicate that the low-energy levels of this nucleus can be fitted quite well with a surface delta-function interaction, and definitely better than with an ordinary delta-function interaction.

Arvieu and Veneroni[16] have pointed out that the off-diagonal matrix elements of the effective interactions in nuclei appear to be somewhat larger than expected for a conventional finite-range interaction. This is consistent with the assumption that the effective interactions are especially strong at the nuclear surface. The additional assumption of a zero range in this paper is of course made only for reasons of mathematical simplicity.

Of course, in light nuclei, most of the nucleus is below the nuclear matter density and the residual interactions presumably act over the entire nucleus. It is probably in heavy nuclei that the surface-interaction model may be approached most closely.

ACKNOWLEDGMENTS

We are very grateful to Dr. R. Arvieu, Dr. A. Goswami, Dr. D. Inglis, and Dr. C. Levinson for stimulating discussions.

[14] S. P. Pandya and I. M. Green, Nucl. Physics **57**, 658 (1964); T. Inoue, I. Sebe, H. Hagiwara, and A. Arima, *ibid.* **59**, 1 (1964).

[15] S. Szpikovski, Acta. Phys. Polon. **25**, 169 (1964) used a conventional delta function including some configuration mixing. For this nucleus the ordinary and surface delta function give nearly the same matrix elements (apart from a constant proportionality factor) except for one matrix element. This one is larger for the surface delta function by about a factor of five. It is precisely this matrix element that Szpikovski had to increase by just this factor in order to get a good fit.

[16] R. Arvieu (private communication).