PHYSICAL REVIEW VOLUME 139, NUMBER 4B 23 AUGUST 1965

Five-Dimensional Quasispin. Exact Solutions of a Pairing Hamiltonian in the *J-T* Scheme*

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(Received 8 April 1965)

The matrix elements of a charge-independent pairing Hamiltonian spanning several single-particle states have been expressed in terms of the matrix elements of the infinitesimal operators of R_5, the rotation group in a five-dimensional space. General algebraic expressions for these matrix elements have been calculated for states with reduced isospin $t=0, \frac{1}{2}$, and 1, in a scheme in which both nucleon number N and isospin T are good quantum numbers, making it possible to find exact solutions to the charge-independent pairing Hamiltonian for states with individual level seniorities $v \leqslant 2$. Exact solutions are compared with perturbation-theory formulas for some simple models. The results indicate that perturbation theory may be used as a guide to an understanding of the charge-independent pairing interaction in its dependence on N and T. For relatively strong pairing and fixed T, the dependence on nucleon number N is similar to that for configurations of identical nucleons; while for fixed N, the T dependence is given mainly by a term of simple $T(T+1)$ form.

1. INTRODUCTION

THE usefulness of a generalized (five-dimensional) quasispin formalism has recently been pointed out by several authors[1–5] in connection with the classification of states of a neutron-proton configuration j^N. This quasispin formalism based on a rotation group in five dimensions R_5 is particularly suited to the study of a charge-independent pairing interaction spanning several single-particle states and may be employed to find exact solutions to this problem in a J-T scheme. An interaction acting only on pair states coupled to angular momentum $J=0$ and isospin $T=1$, even if charge-independent, may be an extremely poor approximation for a neutron-proton configuration, since the neutron-proton interaction in the $T=0$ state cannot be ignored and may in fact be more important than the $J=0$, $T=1$ or "pairing" interaction.[6] Nevertheless, model studies leading to exact solutions of a charge-independent pairing Hamiltonian may be of interest in view of recent attempts to generalize the Bardeen-Cooper-Schrieffer, Bogolyubov, Valatin formalism to properly include the effects of correlations between unlike nucleons.[6–8]

The three-dimensional quasispin formalism, applicable to configurations of identical nucleons, has been adapted to the study of nuclei by Kerman[9] and employed by Kerman, Lawson, and Macfarlane[10] to obtain exact solutions of a pairing Hamiltonian spanning a single-particle energy spectrum of several levels. The method involves the diagonalization of relatively large matrices. However, these matrices are very simple in form. Also Flowers *et al.*[11] have pointed out that such matrices can be kept to manageable size since the strength of the nuclear pairing interaction relative to the single-particle energy separations is relatively small, particularly in the lighter nuclei, so that the matrices can be cut off at excitations corresponding to a small number of pairs. The three-dimensional quasispin formalism has also been employed to extract the N dependence of nuclear matrix elements in the seniority scheme.[12] (N=nucleon number.)

The generalization of the quasispin formalism to the five-dimensional case makes it possible to take up analogous studies for configurations involving both protons and neutrons. Pairing in a single degenerate level (pure configurttion j^N) has been discussed by Flowers and Szpikowski[2] and by Parikh.[3] Some of the simpler R_5 Wigner coefficients needed to extract the N-T dependence of nuclear matrix elements in the seniority scheme have been calculated in Ref. 4. Recently Ginocchio[5] has calculated Wigner coefficients involving the four-dimensional (spinor) representation of R_5 to extract the N-T dependence of the fractional parentage coefficients in the seniority scheme.

It is the purpose of the present paper to set up the machinery for finding exact solutions to a charge-independent pairing Hamiltonian spanning several single-particle states. The matrices of such a Hamiltonian can be expressed solely in terms of the matrix

* The major part of this work was carried out at the University of Sussex, England. Machine computations and the later phases of the work were supported by the U. S. Office of Naval Research under Navy Theoretical Physics Contract No. Nonr 1224(15).
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[6] See, for example, the discussion given by A. M. Lane, *Nuclear Theory. Pairing Correlations and Collective Motion* (W. A. Benjamin, Inc., New York, 1964), Chap. 5.
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[8] The present investigation was motivated in part by the recent work of J. P. Elliott and D. A. Lea on pairing correlations in both the J-T and L-S-T schemes (private communication).
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[11] B. H. Flowers and J. M. Irvine, Proc. Phys. Soc. (London) **83**, 335 (1964). B. H. Flowers, J. M. Irvine, I. P. Johnstone, Proc. Phys. Soc. (London) **84**, 111 (1964).
[12] R. D. Lawson and M. H. Macfarlane, Nucl. Phys. **66**, 80 (1965).

elements of the infinitesimal operators of R_5. It has not been possible to give completely general algebraic expressions for these matrix elements in a scheme in which nucleon number N and isospin T are simultaneously good quantum numbers. However, such expressions can be derived for specific values of seniority v and reduced isospin t. Algebraic expressions for the cases $t=0, \frac{1}{2}, 1$, (arbitrary v and j), are given in Sec. 3. Since only small values of the seniority quantum number v will be of physical interest in problems in which a pairing interaction is the dominant interaction, and since small v implies small t, these expressions should be sufficient for most problems of actual interest. Perturbation formulas for both the weak-pairing and strong-pairing limit are derived in Sec. 4. Exact solutions for a simple system made up of four Nilsson-like single-particle states are discussed in Sec. 5. This system is simple enough to be easily soluble, and it was hoped that it would be complex enough to show the essential features of the pairing interaction. Comparison shows that the perturbation formulas are in good agreement with the exact results over a surprisingly large range of the pairing and single-particle energy parameters, so that perturbation formulas may in many cases be sufficient to give some insight to the action of a charge-independent pairing force.

2. REVIEW OF THE QUASISPIN FORMALISM

In order to establish the notation, a brief review of the quasispin formalism will be given.

A. Three-Dimensional Quasispin

For configurations of identical fermions (*either* neutrons *or* protons) the pairing Hamiltonian can be expressed in terms of the quasispin operators[9,13]

$$S_{j+}=\sum_{m>0}(-1)^{j-m}a_{jm}{}^\dagger a_{j-m}{}^\dagger\,,\quad S_{j-}=\sum_{m>0}(-1)^{j-m}a_{j-m}a_{jm}$$
$$S_{j0}=\tfrac{1}{2}[N_{j\,\mathrm{op}}-(j+\tfrac{1}{2})],\quad\text{where}\quad N_{j\,\mathrm{op}}=\sum_m a_{jm}{}^\dagger a_{jm}\,. \qquad (1)$$

These operators satisfy the "angular-momentum" commutation relations

$$[S_{j+},S_{j'-}]=2\delta_{jj'}S_{j0}\,,\quad [S_{j0},S_{j'\pm}]=\pm\delta_{jj'}S_{j\pm}\,.$$

The eigenvalues of the "angular-momentum" operators $\mathbf{S}_j{}^2$, S_{j0} are given in terms of quasispin quantum numbers S_j, M_{Sj}. These are related to the seniority v and nucleon number N for level j through

$$S_j=\tfrac{1}{2}(j+\tfrac{1}{2}-v_j)\quad\text{and}\quad M_{Sj}=\tfrac{1}{2}[N_j-(j+\tfrac{1}{2})]\,. \qquad (2)$$

The highest weight, largest possible value of M_{Sj}, which determines the quantum number S_j follows from the largest possible nucleon number in a state with seniority v_j, $(N_j)_{\max}=2j+1-v_j$.

In terms of the quasispin operators, a pairing plus single particle Hamiltonian takes the form

$$\mathcal{H}=\sum_j 2\epsilon_j S_{j0}+\sum_j \epsilon_j(j+\tfrac{1}{2})-\sum_{jj'}G_{jj'}S_{j+}S_{j'-}\,. \qquad (3)$$

The coefficients $G_{jj'}$ are usually taken independent of $j(j')$, but this leads to no particular simplification in the quasispin formalism. In the scheme $|S_1M_{S_1}S_2M_{S_2}\cdots S_iM_{S_i}\rangle$ the single-particle Hamiltonian is diagonal. (To avoid complicated subscripts only the labels 1, 2, $\cdots i$ are used for the quasispin quantum numbers associated with levels j_1, j_2, $\cdots j_i$.) The off-diagonal matrix elements of the pairing term are given by the simple matrix elements of the infinitesimal operators $S_{j\pm}$ of the group R_3. Alternately, in the scheme $|(S_1S_2)S_{12}S_3\cdots SM_S\rangle$ the pairing term is diagonal if the pairing strength is independent of $j(j')$. The single-particle term is off-diagonal, and its matrix elements are most easily expressed in terms of Racah coefficients. This scheme is of particular advantage only for a perturbation treatment in the strong-pairing limit.

B. Five-Dimensional Quasispin

Generalization to isospin-$\frac{1}{2}$ fermions leads to the ten quasispin operators[1–3]

$$A_j{}^\dagger(M_T)=\tfrac{1}{2}\sum_{m_t}\sum_m\langle\tfrac{1}{2}m_t\tfrac{1}{2}m_t'|1M_T\rangle(-1)^{j-m}a_{jmm_t}{}^\dagger a_{j-mm_t'}{}^\dagger\,,$$
$$A_j(M_T)=\tfrac{1}{2}\sum_{m_t}\sum_m\langle\tfrac{1}{2}m_t\tfrac{1}{2}m_t'|1M_T\rangle(-1)^{j-m}a_{j-mm_t'}a_{jmm_t}\,,$$
$$[\tfrac{1}{2}N_{j\,\mathrm{op}}-(j+\tfrac{1}{2})],\ \text{where}\ N_{j\,\mathrm{op}}=\sum_m\sum_{m_t}a_{jmm_t}{}^\dagger a_{jmm_t}\,, \qquad (4)$$
$$T_{j\pm}=\sum_m a_{jm\pm\frac{1}{2}}{}^\dagger a_{jm\mp\frac{1}{2}}\,,\quad T_{j0}=\tfrac{1}{2}\sum_m(a_{jm\frac{1}{2}}{}^\dagger a_{jm\frac{1}{2}}-a_{jm-\frac{1}{2}}{}^\dagger a_{jm-\frac{1}{2}})\,.$$

The fermion-creation (annihilation) operators now include the isospin quantum number $m_t=+\frac{1}{2}(-\frac{1}{2})$ for neutron (proton) states. The set of ten operators of Eq. (4) form the infinitesimal operators which generate the group R_5. The explicit connection with the generators of R_5 is shown in Table I. The irreducible representations of R_5 can be labeled by (ω_1,ω_2), the highest weights, $(H_{1\,\mathrm{eigen}}, H_{2\,\mathrm{eigen}})$ of the representation. In a state with seniority v, reduced isospin t, the largest eigenvalue of H_1 is $\frac{1}{2}N_{\max}-(j+\frac{1}{2})$, with $N_{\max}=4j+2-v$.

[13] The fermion creation (annihilation) operators are expressed in terms of the single-particle quantum numbers j and m (Condon-Shortley phase convention). It should be pointed out, however, that j serves mainly to specify the degeneracy of the single-particle state. In calculations with Nilsson states, for example, set $j=\frac{1}{2}$ and interpret $(-1)^{j-m}a_{j,-m}{}^\dagger$ as the creation operator for the time-reversed partner to the state $a_{jm}{}^\dagger|0\rangle$.

TABLE I. The five-dimensional quasispin operators and the infinitesimal operators of R_5.

(a) Quasispin operators	(b) 5-dimensional angular-momentum operators	(c) Infinitesimal operators in standard form for B_2 (R_5)	(d) Infinitesimal operators in standard form for C_2 (Sp_4)
$\frac{1}{2}N_{\rm op}-(j+\frac{1}{2})$	J_{12}	H_1	$S_0{}^n+S_0{}^p$
T_0	J_{34}	H_2	$S_0{}^n-S_0{}^p$
$A^\dagger(1)$	$\frac{1}{2}[(J_{14}+J_{23})+i(J_{24}+J_{31})]$	E_{11}	$S_+{}^n(F_{10})$
$A(1)$	$\frac{1}{2}[(J_{14}+J_{23})-i(J_{24}+J_{31})]$	E_{-1-1}	$S_-{}^n(F_{-10})$
$A^\dagger(-1)$	$\frac{1}{2}[(J_{14}-J_{23})+i(J_{24}-J_{31})]$	$-E_{1-1}$	$S_{+p}(F_{01})$
$A(-1)$	$\frac{1}{2}[(J_{14}-J_{23})-i(J_{24}-J_{31})]$	$-E_{-11}$	$S_{-p}(F_{0-1})$
$A^\dagger(0)$	$(J_{52}+iJ_{15})/\sqrt{2}$	E_{10}	$F_{\frac{1}{2}\frac{1}{2}}$
$A(0)$	$(J_{52}-iJ_{15})/\sqrt{2}$	E_{-10}	$F_{-\frac{1}{2}-\frac{1}{2}}$
T_+	$(J_{45}+iJ_{53})$	$\sqrt{2}E_{01}$	$\sqrt{2}F_{\frac{1}{2}-\frac{1}{2}}$
T_-	$(J_{45}-iJ_{53})$	$\sqrt{2}E_{0-1}$	$\sqrt{2}F_{-\frac{1}{2}\frac{1}{2}}$

[a] The operators as defined in Eqs. (4).

[b] The 5-dimensional angular momentum operators satisfy the same commutation relations as the operators $J_{mn}=-i(X_m\partial/\partial X_n-X_n\partial/\partial X_m)$, m, $n=1, \cdots 5$; but no restriction to 5-dimensional "orbital" angular momentum is implied. The vector $\mathbf{T}$ has been chosen to span the 3, 4, 5 subspace.

[c] The infinitesimal operators in standard form for root diagrams of Cartan's symmetry B_2 (the group R_5). The operators E_{ab} step up (down) the quantum numbers $\frac{1}{2}N-(j+\frac{1}{2})$ and M_T by a and b units, respectively. The operators E_{ab} are the same as those defined in Ref. 3.

[d] The infinitesimal operators in standard form for root diagrams of Cartan's symmetry C_2 (the symplectic group Sp_4). These operators are natural for the group chain $R_5\supset R_4\equiv R_3\times R_3$, where the neutron and proton quasispin operators, $\mathbf{S}^n$, $\mathbf{S}^p$, are the infinitesimal operators for the two commuting R_3 groups. The "angular-momentum" operators $\mathbf{S}^n$, $\mathbf{S}^p$ were denoted by $\mathbf{J}$ and Λ, respectively, in Ref. 4. (The notation has been changed here since J is at present reserved for the angular momentum quantum number in physical space.) The operators F_{ab} are those defined in Ref. 4. They step up (down) the eigenvalues of $S_0{}^n$, $S_0{}^p$ by a and b units, respectively.

The state with $4j+2-v$ nucleons (v holes) has unique isospin t. The largest eigenvalue of $H_2\equiv T_0$ in this state is thus t. This leads to the identification of the R_5 quantum numbers (ω_1, ω_2) for level j:

$$\omega_1=(j+\tfrac{1}{2}-\tfrac{1}{2}v_j)\,, \quad \omega_2=t_j\,. \tag{5}$$

From the relations $H_1=S_0{}^n+S_0{}^p$, $H_2=S_0{}^n-S_0{}^p$ (Table I) the quantum numbers ω_1,ω_2 can also be expressed in terms of the neutron and proton quasispin quantum numbers of the highest weight state, $\omega_1=S_{\max}{}^n+S_{\max}{}^p$, $\omega_2=S_{\max}{}^n-S_{\max}{}^p$. (In Ref. 4 these were denoted by $J_m+\Lambda_m$, $J_m-\Lambda_m$, respectively.) The quantum numbers $S_{\max}{}^n$, $S_{\max}{}^p$ are most natural for the symplectic group in four dimensions. In the present discussion standard notation for R_5 will be used throughout. Note that $S_{\max}{}^n=\frac{1}{2}(j+\frac{1}{2}-\frac{1}{2}v+t)$, $S_{\max}{}^p=\frac{1}{2}(j+\frac{1}{2}-\frac{1}{2}v-t)$. A complete labeling of the basis states of the irreducible representation (ω_1,ω_2) is furnished by the neutron and proton quasispin quantum numbers $S^nS^pM_{S^n}M_{S^p}$. (In Ref. 4 these were denoted by $J\Lambda M_JM_\Lambda$, the quantum numbers of the so-called mathematically natural scheme.) The operators H_1 and $H_2=T_0$ are diagonal in this scheme so that N and M_T are good quantum numbers, but $\mathbf{T}^2$ is not diagonal in this scheme. Alternately, in the scheme $|S^nS^pTM_T\rangle$, built from the former through the vector coupling $\mathbf{T}=\mathbf{S}^n-\mathbf{S}^p$, T and M_T are now good quantum numbers, but the number operator is no longer diagonal. The states of physical interest are those in which $\mathbf{T}^2$, T_0 and $N_{\rm op}$ are simultaneously diagonal. For the complete specification of the states (ω_1,ω_2) a fourth operator is needed. An operator which commutes with $\mathbf{T}^2$, T_0, $N_{\rm op}$, and the two Casimir invariants for R_5 must be constructed. The simplest such operator involves products of four of the infinitesimal operators of R_5. In terms of the infinitesimal operators in the form J_{ij}, (Table I), such an operator would be

$$0=\sum_{\alpha,\beta}\sum_{ij} J_{\alpha i}J_{\beta j}J_{\beta i}J_{\alpha j}\,, \tag{6}$$

where the sums are restricted to i, $j=1, 2$; $\alpha, \beta=3, 4, 5$. This operator, unlike the operators $\mathbf{S}^{n^2}$, $\mathbf{S}^{p^2}$, for example, is not simply related to one of the mathematically natural subgroups of R_5. Its algebraic properties are very complicated, so that it is very cumbersome to work with this operator. In practice an alternate labeling scheme, suggested by Racah,[14] has been used. This is based on the fact that a state of seniority v and total isospin T can be constructed by vector coupling the reduced isospin t of the v nucleons not coupled in pairs to $J=0$, with the resultant isospin T_p of the p pairs of nucleons coupled to $J=0$, $T=1$, to obtain the total isospin T; $(\mathbf{T}=\mathbf{t}+\mathbf{T}_p)$. Since the pair creation operators, $A^\dagger$ of Eq. (4), are commuting isospin-1 operators, the values of T_p for the p pairs are restricted to p, $(p-2)$, $\cdots$with $p\leq j+\frac{1}{2}-\frac{1}{2}v$. Since the label T_p cannot be associated with the eigenvalue of a Hermitean operator (commuting with $\mathbf{T}^2$, T_0, $N_{\rm op}$), a labeling scheme based on T_p does not lead to an orthogonal set of basis states. In the $|(\omega_1,\omega_2)T_pNTM_T\rangle$ labeling scheme states with the same values of N, T, M_T, but different values of T_p, are not in general orthogonal to each other. In practice, however, this difficulty is easily overcome in almost all irreducible representations of actual interest in nuclear spectroscopy. Almost all of these fall essentially into two categories. The first category includes the irreducible representations $(\omega_1\,0)$, $(\omega_1\,\frac{1}{2})$, and $(t\,t)$; that is those with reduced isospin $t=0$, $\frac{1}{2}$, and with $t=\omega_1$. In these irreducible representations a given isospin T occurs at most once for each nucleon number (eigenvalue of H_1). The quantum number T_p is therefore redundant in this case. The second category is illustrated in Table II by the irreducible representation $(\omega_1\,1)$, that is one with $t=1$, and includes also the irreducible representations $(\omega_1\,\frac{3}{2})$ and $(t+1, t)$. For irreducible representations in this category a particular value of T occurs at most twice for each nucleon number (eigenvalue of H_1). The two independent states, such as those with $H_1=\omega_1-2$, $T=1$ in the irreducible representation $(\omega_1\,1)$, for example, can be chosen as properly orthogonalized linear combinations of states with $T_p=0$ and 2, or in general with $T_p=T+1$ and $T-1$. Since such linear combinations can be chosen in an infinite number of ways, it seems that a labeling scheme based on the quantum number T_p cannot serve to uniquely specify such states. In such cases, however, the self-adjoint

[14] G. Racah, *Proceedings of the Rehovoth Conference on Nuclear Structure*, edited by H. J. Lipkin (North-Holland Publishing Company, Amsterdam, 1958).

TABLE II. Allowed values of H_1, T in the representation $(\omega_1,1)$.[a]

H_1	T_p	Possible values of T
ω_1	0	1
ω_1-1	1	$0\ 1\ 2$
ω_1-2	$0,\ 2$	$1^2\ 2\ 3$
ω_1-3	$1,\ 3$	$0\ 1\ 2^2\ 3\ 4$
ω_1-4	$0,\ 2,\ 4$	$1^2\ 2\ 3^2\ 4\ 5$
⋮	⋮	⋮
1	$\cdots,(\omega_1-3),(\omega_1-1)$	$\cdots(\omega_1-3)(\omega_1-2)^2(\omega_1-1)\omega_1$
0	$\cdots,(\omega_1-2),\ \omega_1$	$\cdots(\omega_1-3)^2(\omega_1-2)(\omega_1-1)^2\omega_1$
-1	$\cdots,(\omega_1-3),(\omega_1-1)$	$\cdots(\omega_1-3)(\omega_1-2)^2(\omega_1-1)\omega_1$
⋮	⋮	⋮
$-(\omega_1-4)$	$0,\ 2,\ 4$	$1^2\ 2\ 3^3\ 4\ 5$
$-(\omega_1-3)$	$1,\ 3$	$0\ 1\ 2^2\ 3\ 4$
$-(\omega_1-2)$	$0,\ 2$	$1^2\ 2\ 3$
$-(\omega_1-1)$	1	$0\ 1\ 2$
$-\omega_1$	0	1

[a] In general, in the representation (ω_1,t) the allowed values of H_1T are given by the possible vector couplings $\mathbf{T}=\mathbf{T}_p+\mathbf{t}$, where $T_p=n,\ n-2,\ \cdots$ for $H_1=\pm|\omega_1-n|$, $n=0, 1, 2, \cdots (\leq\omega_1)$; subject to the restrictions:
1. $T\leq\omega_1$
2. $T=\omega_1-m$ (with $m=0, 1, 2, \cdots$) occurs at most k times for a specific value of H_1, where $k=\min.(m+1,\ \omega_1-t+1)$. [These rules follow from the allowed values of S^n, S^p, Eq. (11) of Ref. 4.]

property of the rotation group in five dimensions[15] can be exploited to lead to a unique labeling of the double states. The distinction between the double states can be made unique by choosing them as built from those specific linear combinations of T_p for which the resultant states have the symmetry property + or −, respectively, under conjugation (the adjoint or contragredient operation which transforms particle into hole, H_1 into $-H_1$, M_T into $-M_T$). For such states then, the quantum number T_p is used merely as an auxiliary label, while the symmetry property under particle-hole conjugation serves to specify the states. For particle $j\leq\frac{9}{2}$ all irreducible representations of R_5 fall into one or the other of the two categories discussed here. For particle $j=\frac{11}{2}$, there is only one case in one irreducible representation of R_5, the case of the three independent states with $H_1=0$, $T=2$ in the representation (42), for which the auxiliary label T_p, supplemented by the symmetry property under particle-hole conjugation, is insufficient for a unique labeling of the states. This is, therefore, one of the few cases of interest in nuclear spectroscopy where an operator such as 0 of Eq. (6) is actually needed. The problem of the fourth quantum number, although of great general interest, is thus largely an academic problem since the auxiliary label T_p, supplemented by the symmetry requirement under conjugation, can be used to uniquely specify almost all states of practical interest.

3. MATRIX ELEMENTS OF THE PAIRING HAMILTONIAN

A charge-independent pairing interaction spanning several single-particle states leads to the model Hamiltonian

$$\mathcal{H}=\sum_j \epsilon_j N_j-\sum_{jj'} G_{jj'}[A_j^\dagger(1)A_{j'}(1)+A_j^\dagger(0)A_{j'}(0)+A_j^\dagger(-1)A_{j'}(-1)], \quad (7)$$

where the pairing term is expressed in terms of the five-dimensional quasispin operators, the $J=0$, $T=1$ pair-creation and annihilation operators. The coefficients $G_{jj'}$ are usually taken independent of $j(j')$. This Hamiltonian commutes with the total number operator $N=\sum_j N_j$, or $H_1=\sum_j H_{1j}$. It also commutes with the R_5 Casimir operators for the individual levels

$$C_j=(H_1)_j{}^2+(H_2)_j{}^2+\sum_{a,b}(E_{ab})_j(E_{-a-b})_j \quad (8)$$

(see Table I), and the companion quartic R_5 invariants.[16] The Hamiltonian is thus diagonal in the quantum numbers N and $(\omega_1,\omega_2)_j=(j+\frac{1}{2}-\frac{1}{2}v_j,\ t_j)$ where the subscript j refers to the individual single-particle states. In the uncoupled scheme[17]

$$|(\omega_1\omega_2)_1\kappa_1(H_1)_1T_1M_{T_1};(\omega_1\omega_2)_2\kappa_2(H_1)_2T_2M_{T_2};\cdots(\omega_1\omega_2)_i\kappa_i(H_1)_iT_iM_{T_i}\rangle, \quad (9)$$

the analog of the $|S_1M_{S_1}S_2M_{S_2}\cdots S_iM_{S_i}\rangle$ scheme of the three-dimensional case, the single particle part of the Hamiltonian (7) is diagonal, but the pairing term leads to matrix elements off-diagonal in $\kappa_iH_{1i}T_iM_{T_i}$. (The label κ_i is used as the "fourth quantum number." In all simple cases, it can either be replaced by the label T_{pi}, or be based on this auxiliary label supplemented by the symmetry requirement under particle-hole conjugation as discussed in Sec. 2.) Alternatively, in the coupled scheme

$$|\{[(\omega_1\omega_2)_1(\omega_1\omega_2)_2](\omega_1\omega_2)_{12}(\omega_1\omega_2)_3\}\times(\omega_1\omega_2)_{123}\cdots(\omega_1\omega_2)\kappa H_1TM_T\rangle, \quad (10)$$

the analog of the $|[(S_1S_2)S_{12}S_3]S_{123}\cdots SM_S\rangle$ scheme of the three-dimensional case, the pairing term is diagonal (provided the coefficient G is independent of j and j'), while the single-particle part of the Hamiltonian leads to matrix elements off-diagonal in $(\omega_1,\omega_2)_{12},\cdots(\omega_1,\omega_2)$. The $(\omega_1,\omega_2)_{12}$ are those irreducible representations which occur in the Kronecker product of $(\omega_1,\omega_2)_1$ with $(\omega_1,\omega_2)_2$. The final resultant representation (ω_1,ω_2) can be associated with over-all seniority and reduced isospin quantum numbers, v and t, $\omega_1=\sum(j+\frac{1}{2})-\frac{1}{2}v$, $\omega_2=t$; where these become good quantum numbers in the strong-pairing or degenerate-level limit in which the energy differences between single-particle states become negligible compared with the pairing strength parameter G.

Since the pairing Hamiltonian (7) is charge independent (a scalar in isospin space), the intermediate

[15] F. D. Murnaghan, *The Theory of Group Representations* (The Johns Hopkins Press, Baltimore, Maryland, 1938), p. 262.

[16] H. Micu, Nucl. Phys. **60**, 353 (1964).

[17] Again, only the labels 1, 2, $\cdots i$ are used for the R_5 quantum numbers associated with single parricle levels j_1, j_2, $\cdots j_i$.

scheme

$$|(\omega_1\omega_2)_1\kappa_1(H_1)_1T_1;\ (\omega_1\omega_2)_2\kappa_2(H_1)_2T_2;$$
$$\times\cdots(\omega_1\omega_2)_i\kappa_i(H_1)_iT_i;\ \cdots;\ T_{12}\cdots T_{ik}\cdots TM_T\rangle \quad (11)$$

is also useful. In this scheme the irreducible representations of R_5 are uncoupled as in (9); but the isospin quantum numbers T_i, associated with individual levels j_i, are coupled to resultant isospin T, making it possible to treat the dependence on the isospin quantum numbers by ordinary angular momentum coupling and recoupling techniques. It is this scheme which has been used in the calculations to be presented here. The diagonal matrix elements in the scheme (11) [or (9)] follow from the relation between the pairing terms and the R_5 Casimir operators for individual j's.

$$\sum_{M_T} A_j^\dagger(M_T)A_j(M_T)=\tfrac{1}{2}\{C_j-\mathbf{T}_j^2-(H_1)_j^2+3(H_1)_j\} \quad (12)$$

giving diagonal matrix elements

$$\epsilon_jN_j-\tfrac{1}{2}G\{\omega_{1j}(\omega_{1j}+3)+t_j(t_j+1)-T_j(T_j+1)$$
$$-\tfrac{1}{4}(N_j-2j-1)(N_j-2j-7)\}. \quad (13)$$

In the scheme (11) the off-diagonal matrix elements of the Hamiltonian (7) can be expressed in terms of reduced (R_5/R_3) Wigner coefficients and ordinary 6-j symbols involving the isospin quantum numbers

$$\langle(\omega_1\omega_2)_i(\omega_1\omega_2)_j,\ \kappa_i',\ H_{1i}+1,\ T_i';\ \kappa_j',\ H_{1j}-1,\ T_j';\ T_{ij}\cdots|\mathcal{H}|(\omega_1\omega_2)_i(\omega_1\omega_2)_j,\kappa_iH_{1i}T_i;\ \kappa_jH_{1j}T_j;\ T_{ij}\cdots\rangle$$
$$=-G[\omega_{1i}(\omega_{1i}+3)+t_i(t_i+1)]^{1/2}[\omega_{1j}(\omega_{1j}+3)+t_j(t_j+1)]^{1/2}$$
$$\times\langle(\omega_1\omega_2)_i\kappa_iH_{1i}T_i;\ (11)11\|(\omega_1\omega_2)_i\kappa_i',\ H_{1i}+1,\ T_i'\rangle_1\langle(\omega_1\omega_2)_j\kappa_jH_{1j}T_j;\ (11)-11\|(\omega_1\omega_2)_j\kappa_j',\ H_{1j}-1,\ T_j'\rangle_1$$
$$\times(-1)^{T_{ij}+1+T_i'+T_j'}[(2T_i'+1)(2T_j'+1)]^{1/2}\begin{Bmatrix} T_{ij} & T_i & T_j \\ 1 & T_j' & T_i' \end{Bmatrix}. \quad (14)$$

The matrix elements of the infinitesimal operators have been expressed in terms of R_5 Wigner coefficients which can be factored into products of reduced (R_5/R_3) Wigner coefficients, denoted by a double bar, and ordinary (R_3) Wigner coefficients which carry all of the dependence on the quantum number M_T,

$$\langle(\omega_1'\omega_2')\kappa'H_1'T'M_T';\ (\omega_1''\omega_2'')\kappa''H_1''T''M_T''|(\omega_1\omega_2)\kappa H_1TM_T\rangle$$
$$=\langle(\omega_1'\omega_2')\kappa'H_1'T';\ (\omega_1''\omega_2'')\kappa''H_1''T''\|(\omega_1\omega_2)\kappa H_1T\rangle\langle T'M_T'T''M_T''|TM_T\rangle. \quad (15)$$

This factoring makes it possible to carry out the M_T sums implied in the matrix element (14) leading to the ordinary 6-j symbol in the isospin quantum numbers. Further isospin recoupling transformations are, in general, needed to express some of the matrix elements of the Hamiltonian in the scheme (11) in terms of those of Eq. (14). In a $\cdots(T_iT_j)T_{ij}\cdots(T_kT_l)T_{kl}\cdots$ scheme, for example, matrix elements off-diagonal in $(H_1)_i$ and $(H_1)_k$ can be expressed in terms of Eq. (14) after a recoupling to the $\cdots(T_iT_k)T_{ik}\cdots(T_jT_l)T_{jl}\cdots$ scheme. Since the infinitesimal operators transform according to the ten-dimensional irreducible representation (11), the R_5 Wigner coefficients of interest here are those with $(\omega_1'',\omega_2'')=(11)$. The Kronecker product $(\omega_1,\omega_2)\times(11)$ contains the irreducible representation (ω_1,ω_2) twice. The matrix elements of an operator with R_5 tensor character (11) between states (ω_1',ω_2') and $(\omega_1,\omega_2)=(\omega_1',\omega_2')$ must thus, in general, be expressed in terms of two independent R_5 Wigner coefficients and two reduced matrix elements[4] to be denoted by subscripts 1 and 2. The two Wigner coefficients have been chosen[4] so that the reduced matrix elements of the infinitesimal operators are different from zero only in states 1. Therefore, only R_5 Wigner coefficients with subscript 1 are needed for the matrix elements of the pairing Hamiltonian. The nonzero matrix elements of the operator $A^\dagger(1)(=E_{11})$, for example, can be written

$$\langle(\omega_1\omega_2)\kappa H_1TM_T|A^\dagger(1)|(\omega_1'\omega_2')\kappa'H_1'T'M_T'\rangle=\delta_{\omega_1\omega_1'}\delta_{\omega_2\omega_2'}\delta_{H_1',H_1-1}\delta_{M_T',M_T-1}[\omega_1(\omega_1+3)+\omega_2(\omega_2+1)]^{1/2}$$
$$\times\langle(\omega_1\omega_2)\kappa',\ H_1-1,\ T';\ (11)H_1''=1\|(\omega_1\omega_2)\kappa H_1T\rangle_1\langle T'(M_T-1)11|TM_T\rangle. \quad (16)$$

It has not been possible to calculate algebraic expressions for such R_5 Wigner coefficients in the $|\kappa H_1TM_T\rangle$ scheme for arbitrary irreducible representations (ω_1,ω_2). Such expressions can, however, be given for representations with small values of the reduced isospin. Since the states of greatest physical interest in problems dominated by a pairing interaction are those with small values of the seniority quantum number v, and since small v implies small t, these representations are precisely the ones of greatest interest. General algebraic expressions for the R_5 Wigner coefficients which decompose the product $(\omega_1,\omega_2)\times(11)$ have been calculated in the $|S^nS^pM_{S^n}M_{S^p}\rangle$ scheme based on neutron and proton quasispins in Ref. 4. In Ref. 4, a technique has also been developed for the calculation of the transformation coefficients from the $|S^nS^pM_{S^n}M_{S^p}\rangle$ to the $|\kappa H_1TM_T\rangle$ scheme. In particular, explicit algebraic expressions are given for the transformation coefficients $\langle S^nS^pM_{S^n}M_{S^p}|T_pH_1TM_T\rangle$ for irreducible representations with $t=0$ and $t=\tfrac{1}{2}$. These have been extended here (Appendix 1) to include the case $t=1$ needed for seniority 2 states. The unitary transformation coefficients $\langle S^nS^pM_{S^n}M_{S^p}|\kappa H_1TM_T\rangle$ can then be used to transform the R_5 Wigner coefficients from the $|S^nS^pM_{S^n}M_{S^p}\rangle$ scheme to the $|\kappa H_1TM_T\rangle$ scheme of actual interest. Although this process can in principle be generalized to

TABLE III. The R_5 Wigner coefficients $\langle (j+\frac{1}{2},0)H_1'T';\ (11)H_1''T''\|(j+\frac{1}{2},0)H_1T\rangle$.

H_1'	T'	H_1''	T''	H_1	T	$\langle (j+\frac{1}{2},0)H_1'T';\ (11)H_1''T''\|(j+\frac{1}{2},0)H_1T\rangle$ [a,b]
H_1-1	$T-1$	1	1	H_1	T	$\left[\dfrac{T(j+\frac{5}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)}{2(2T+1)(j+\frac{1}{2})(j+\frac{7}{2})}\right]^{1/2}$
H_1-1	$T+1$	1	1	H_1	T	$-\left[\dfrac{(T+1)(j+\frac{7}{2}-H_1+T)(j+\frac{1}{2}+H_1-T)}{2(2T+1)(j+\frac{1}{2})(j+\frac{7}{2})}\right]^{1/2}$
H_1+1	$T-1$	-1	1	H_1	T	$\left[\dfrac{T(j+\frac{5}{2}+H_1-T)(j+\frac{3}{2}-H_1+T)}{2(2T+1)(j+\frac{1}{2})(j+\frac{7}{2})}\right]^{1/2}$
H_1+1	$T+1$	-1	1	H_1	T	$-\left[\dfrac{(T+1)(j+\frac{7}{2}+H_1+T)(j+\frac{1}{2}-H_1-T)}{2(2T+1)(j+\frac{1}{2})(j+\frac{7}{2})}\right]^{1/2}$
H_1	T'	0	1	H_1	T	$\delta_{TT'}\left[\dfrac{T(T+1)}{(j+\frac{1}{2})(j+\frac{7}{2})}\right]^{1/2}$
H_1	T	0	0	H_1	T	$\dfrac{H_1}{[(j+\frac{1}{2})(j+\frac{7}{2})]^{1/2}}$

[a] When not needed, as in the representations $(j+\frac{1}{2},0)$ and (11), the fourth quantum number κ is omitted.
[b] The Kronecker product $(j+\frac{1}{2},0)\times(11)$ contains the representation $(j+\frac{1}{2},0)$ only once.

irreducible representations with higher reduced isospin, the resultant algebraic expressions for $t\geqslant 1$ become so complicated that the method becomes very tedious. In practice, therefore, it has been possible to calculate general algebraic expressions for R_5 Wigner coefficients in the physically interesting scheme, based on nucleon number and isospin T, only for those coefficients involving irreducible representations with reduced isospins of 0, $\frac{1}{2}$, and 1. Coefficients involving irreducible representations with $\omega_1=t$ again become simple,[5] and the technique outlined here can be used to calculate all other coefficients numerically.

The R_5 Wigner coefficients needed for the matrix elements of the infinitesimal operators [see Eq. (16)] are given in Table III for irreducible representations $(j+\frac{1}{2},0)$, that is those with $t=0$, and in Table IV for representations $(j\,\frac{1}{2})$, that is those with $t=\frac{1}{2}$. In these representations the values of T_p are uniquely determined by the values of H_1 and T, so that the label $\kappa(\equiv T_p)$ is not needed to specify the states. Expressions for the Wigner coefficients, however, are functions of this fourth quantum number in the irreducible representation $(j\,\frac{1}{2})$. States with $T_p=T+\frac{1}{2}$ and $T_p=T-\frac{1}{2}$ lead to different algebraic expressions for the Wigner coefficients so that the fourth quantum number cannot be ignored. The quantum number T_p has been defined in Sec. 2 as the resultant isospin of the p pairs of nucleons coupled to $J=0$, $T=1$, with $p=\frac{1}{2}(N-v)$. In Ref. 4, the transformation coefficients from the $|S^nS^pM_{S^n}M_{S^p}\rangle$ to the $|\kappa H_1TM_T\rangle$ scheme have been derived by techniques involving step-down operators, starting with the state of highest weight or $4j+2-v$ nucleons. In this process a state of $N=2p-1$ nucleons and $t=\frac{1}{2}$ was constructed by operating on a state of $2p$ nucleons with an appropriate annihilation operator (stepping down H_1 by $\frac{1}{2}$ unit). The quantum numbers, T_p, associated with these $p=\frac{1}{2}(N+v)$ nucleon pairs differ from those associated with the $p=\frac{1}{2}(N-v)$ nucleon pairs for states with odd seniority (half-integral t). To avoid confusion between the two types of quantum numbers T_p, the fourth quantum number κ for states of the irreducible representation $(j\,\frac{1}{2})$ have been denoted by e and o. The labels e and o refer to states with $j+\frac{1}{2}-H_1-T$ equal to an even or odd integer, respectively, or alternatively with $\frac{1}{2}(N+v-1)+T$ equal to an even or odd integer. For states with integral t (even seniority), the quantum numbers T_p associated with $p=\frac{1}{2}(N+v)$ $J=0$-coupled pairs have the same set of values as those associated with the $p=\frac{1}{2}(N-v)$ pairs, and the label T_p will be used in such representations. The possible values of T_p are n, $n-2$, $\cdots$ for $H_1=\pm|\omega_1-n|$ (Table II).

In the irreducible representation $(\omega_1\,1)$, with reduced isospin $t=1$, the states with $T_p=T$ are single, while states with $T_p=T\pm 1$ are in general double (Table II). The transformation coefficients $\langle S^nS^pM_{S^n}M_{S^p}|T_pH_1TM_T\rangle$ for the representation $(\omega_1\,1)$ are given in Appendix 1. Since the states

$$|T_p=T+1,\,H_1TM_T\rangle \quad\text{and}\quad |T_p=T-1,\,H_1TM_T\rangle$$

are in general not orthogonal to each other, the label T_p must be replaced by the fourth quantum number κ, based on the symmetry requirement under particle-hole conjugation. In the representation $(\omega_1\,1)=(j-\frac{1}{2},1)$ the quantum number κ has been chosen in the following way:

1. $\kappa=0$ denotes states with $T_p=T$:

$$|(j-\tfrac{1}{2},1)\kappa=0,\,H_1TM_T\rangle = |(j-\tfrac{1}{2},1)T_p=T,\,H_1TM_T\rangle. \quad (17a)$$

2. For states with $\kappa=1$, or 2:

$$|(j-\tfrac{1}{2},1)\kappa H_1TM_T\rangle = x_\kappa|(j-\tfrac{1}{2},1)T_p=T-1,\,H_1TM_T\rangle + y_\kappa|(j-\tfrac{1}{2},1)T_p=T+1,\,H_1TM_T\rangle, \quad (17b)$$

TABLE IV. The R_5 Wigner coefficients $\langle (j\frac{1}{2})\kappa' H_1' T';\, (11)H_1'' T'' \| (j\frac{1}{2})\kappa H_1 T\rangle_1$.

κ'	H_1'	T'	H_1''	T''	κ	H_1	T	$\langle (j\frac{1}{2})\kappa' H_1' T';\, (11)H_1'' T'' \| (j\frac{1}{2})\kappa H_1 T\rangle_1$
					With $\kappa=e$.			
e	H_1-1	$T-1$	1	1	e	H_1	T	$\left[\frac{(2T-1)(j+\frac{5}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)}{8T[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
e	H_1-1	$T+1$	1	1	e	H_1	T	$-\left[\frac{(2T+3)(j+\frac{5}{2}-H_1+T)(j-\frac{1}{2}+H_1-T)}{8(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
e	H_1+1	$T-1$	-1	1	e	H_1	T	$\left[\frac{(2T-1)(j+\frac{1}{2}-H_1+T)(j+\frac{3}{2}+H_1-T)}{8T[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
e	H_1+1	$T+1$	-1	1	e	H_1	T	$-\left[\frac{(2T+3)(j+\frac{1}{2}-H_1-T)(j+\frac{7}{2}+H_1+T)}{8(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
o	H_1-1	T	1	1	e	H_1	T	$\left[\frac{(j+\frac{3}{2}+H_1+T)(j+\frac{5}{2}-H_1+T)}{8T(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
o	H_1+1	T	-1	1	e	H_1	T	$\left[\frac{(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}+H_1-T)}{8T(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
e	H_1	T'	0	1	e	H_1	T	$\delta_{TT'}\left[\frac{T(T+1)}{j(j+3)+\frac{3}{4}}\right]^{1/2}$
e	H_1	T	0	0	e	H_1	T	$\frac{H_1}{[j(j+3)+\frac{3}{4}]^{1/2}}$
					With $\kappa=o$.			
o	H_1-1	$T-1$	1	1	o	H_1	T	$\left[\frac{(2T-1)(j+\frac{3}{2}-H_1-T)(j+\frac{1}{2}+H_1+T)}{8T[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
o	H_1-1	$T+1$	1	1	o	H_1	T	$-\left[\frac{(2T+3)(j+\frac{7}{2}-H_1+T)(j+\frac{1}{2}+H_1-T)}{8(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
o	H_1+1	$T-1$	-1	1	o	H_1	T	$\left[\frac{(2T-1)(j+\frac{5}{2}+H_1-T)(j+\frac{3}{2}-H_1+T)}{8T[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
o	H_1+1	$T+1$	-1	1	o	H_1	T	$-\left[\frac{(2T+3)(j-\frac{1}{2}-H_1-T)(j+\frac{5}{2}+H_1+T)}{8(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
e	H_1-1	T	1	1	o	H_1	T	$\left[\frac{(j+\frac{3}{2}-H_1-T)(j+\frac{1}{2}+H_1-T)}{8T(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
e	H_1+1	T	-1	1	o	H_1	T	$\left[\frac{(j+\frac{3}{2}-H_1+T)(j+\frac{5}{2}+H_1+T)}{8T(T+1)[j(j+3)+\frac{3}{4}]}\right]^{1/2}$
o	H_1	T'	0	1	o	H_1	T	$\delta_{TT'}\left[\frac{T(T+1)}{j(j+3)+\frac{3}{4}}\right]^{1/2}$
o	H_1	T	0	0	o	H_1	T	$\frac{H_1}{[j(j+3)+\frac{3}{4}]^{1/2}}$

where the coefficients x_κ, y_κ are chosen such that states with $\kappa=2$ or 1, and $M_T=T$, have the symmetry property $+$ or $-$, respectively, under the adjoint, particle-hole or conjugation operation which transforms H_1 into $-H_1$, M_T into $-M_T$.

$$|(j-\tfrac{1}{2},1)\kappa H_1 T M_T\rangle^* = (-1)^{\kappa+T-M_T}|(j-\tfrac{1}{2},1)\kappa, -H_1, T, -M_T\rangle. \quad (17c)$$

Since the over-all phase under conjugation is arbitrary to within an ω_1- and t-dependent phase factor, only relative phases have any real significance. The above choice of phases is fixed by noting that the state of highest weight (a state with $\kappa=2$) has the symmetry property $+$ under the conjugation operation. The explicit expressions for the coefficients x_κ, y_κ are

$\kappa=2$:

$$x_2=-\frac{[2(T+1)]^{1/2}\{H_1^2[T(2j+2)+(j+\frac{1}{2})]+2H_1T^2(2j+2)-(j+\frac{3}{2}+T)(j+\frac{1}{2}-T)[T(2j+2)+(j+\frac{3}{2})]\}}{(2T+1)[(j+\frac{3}{2}+H_1+T)f(-H_1,T)g(H_1,T)]^{1/2}},$$

$$y_2=\frac{[T(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}-H_1+T)f(H_1,T)\varphi(H_1,T)]^{1/2}}{(2T+1)[(j+\frac{3}{2}+H_1+T)f(-H_1,T)g(H_1,T)]^{1/2}}. \tag{18a}$$

$\kappa=1$:

$$x_1=-\frac{[2T(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}-H_1+T)]^{1/2}\times\{H_1(j+\frac{1}{2})[T(2j+2)+(j+\frac{1}{2})]+(j+\frac{3}{2})(j+\frac{1}{2}-T)[T(2j+2)+(j+\frac{3}{2})]\}}{(2T+1)[(j+\frac{1}{2})(j+\frac{3}{2})(j+\frac{1}{2}+H_1-T)f(-H_1,T)g(H_1,T)]^{1/2}},$$

$$y_1=\frac{[(T+1)f(H_1,T)\varphi(H_1,T)]^{1/2}\{(j+\frac{3}{2})(j+\frac{1}{2}-T)-H_1(j+\frac{1}{2})\}}{(2T+1)[(j+\frac{1}{2})(j+\frac{3}{2})(j+\frac{1}{2}+H_1-T)f(-H_1,T)g(H_1,T)]^{1/2}},$$

where

$$\begin{aligned}f(H_1,T)&=2\{H_1[T(2j+2)+(j+\tfrac{1}{2})]+(j+\tfrac{1}{2}-T)[T(2j+2)+(j+\tfrac{3}{2})]\},\\ g(H_1,T)&=\{(j+\tfrac{3}{2})(j+\tfrac{1}{2}-T)(j+\tfrac{3}{2}+T)-(j+\tfrac{1}{2})H_1^2\},\\ \varphi(H_1,T)&=\{H_1[T(2j+2)+(j+\tfrac{3}{2})]+(j+\tfrac{3}{2}+T)[T(2j+2)+(j+\tfrac{1}{2})]\}.\end{aligned} \tag{18b}$$

TABLE V. The R_5 Wigner coefficients $\langle(j-\frac{1}{2},1)\kappa'H_1'T';(11)H_1''T''\|(j-\frac{1}{2},1)\kappa H_1T\rangle_1$ with $\kappa=0$.[a]

κ'	H_1'	T'	H_1''	T''	κ	H_1	T	$\langle(j-\frac{1}{2},1)\kappa'H_1'T';(11)H_1''T''\|(j-\frac{1}{2},1)\kappa H_1T\rangle_1$
0	H_1-1	$T-1$	1	1	0	H_1	T	$\left[\frac{(T-1)(T+1)(j+\frac{3}{2}-H_1-T)(j+\frac{1}{2}+H_1+T)}{2T(2T+1)(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
0	H_1-1	$T+1$	1	1	0	H_1	T	$-\left[\frac{T(T+2)(j+\frac{5}{2}-H_1+T)(j-\frac{1}{2}+H_1-T)}{2(T+1)(2T+1)(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
0	H_1+1	$T-1$	-1	1	0	H_1	T	$\left[\frac{(T-1)(T+1)(j+\frac{3}{2}+H_1-T)(j+\frac{1}{2}-H_1+T)}{2T(2T+1)(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
0	H_1+1	$T+1$	-1	1	0	H_1	T	$-\left[\frac{T(T+2)(j+\frac{5}{2}+H_1+T)(j-\frac{1}{2}-H_1-T)}{2(T+1)(2T+1)(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
2	H_1-1	T	1	1	0	H_1	T	$\left[\frac{2(j+\frac{3}{2}-H_1-T)(j+\frac{1}{2}+H_1+T)}{T(T+1)(j+\frac{3}{2})f(H_1-1,T)f(-H_1+1,T)g(H_1-1,T)}\right]^{1/2}\times\{[T(2j+2)+j+\frac{1}{2}][H_1^2-H_1(T+2)+(T+1)]-[T(2j+2)+(j+\frac{3}{2})](j+\frac{3}{2})(j+\frac{1}{2}-T)\}$
2	H_1+1	T	-1	1	0	H_1	T	$-\left[\frac{2(j+\frac{3}{2}+H_1-T)(j+\frac{1}{2}-H_1+T)}{T(T+1)(j+\frac{3}{2})f(H_1+1,T)f(-H_1-1,T)g(H_1+1,T)}\right]^{1/2}\times\{[T(2j+2)+j+\frac{1}{2}][H_1^2+H_1(T+2)+(T+1)]-[T(2j+2)+(j+\frac{3}{2})](j+\frac{3}{2})(j+\frac{1}{2}-T)\}$
1	H_1-1	T	1	1	0	H_1	T	$\left[\frac{2(j+\frac{5}{2}-H_1+T)(j-\frac{1}{2}+H_1-T)}{(j+\frac{1}{2})f(H_1-1,T)f(-H_1+1,T)g(H_1-1,T)}\right]^{1/2}\times\{T^2(2j+2)-T(2j+1)(j+\frac{3}{2})-(j+\frac{1}{2})^2+H_1[T(2j+2)-(j+\frac{1}{2})]\}$
1	H_1+1	T	-1	1	0	H_1	T	$\left[\frac{2(j+\frac{5}{2}+H_1+T)(j-\frac{1}{2}-H_1-T)}{(j+\frac{1}{2})f(H_1+1,T)f(-H_1-1,T)g(H_1+1,T)}\right]^{1/2}\times\{T^2(2j+2)-T(2j+1)(j+\frac{3}{2})-(j+\frac{1}{2})^2-H_1[T(2j+2)-(j+\frac{1}{2})]\}$
0	H_1	T'	0	1	0	H_1	T	$\delta_{TT'}\left[\frac{T(T+1)}{(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
0	H_1	T	0	0	0	H_1	T	$\frac{H_1}{[(j+\frac{1}{2})(j+\frac{3}{2})]^{1/2}}$

[a] $f(H_1,T)=2\{H_1[T(2j+2)+(j+\frac{1}{2})]+(j+\frac{1}{2}-T)[T(2j+2)+(j+\frac{3}{2})]\}$
$g(H_1,T)=\{(j+\frac{3}{2})(j+\frac{1}{2}-T)(j+\frac{3}{2}+T)-(j+\frac{1}{2})H_1^2\}$.

TABLE VI. The R_5 Wigner coefficients $\langle (j-\frac{1}{2},1)\kappa' H_1' T';\, (11) H_1'' T'' \| (j-\frac{1}{2},1)\kappa H_1 T\rangle_1$ with $\kappa=1$.[a]

κ'	H_1'	T'	H_1''	T''	κ	H_1	T	$\langle (j-\frac{1}{2},1)\kappa' H_1' T';\, (11) H_1'' T'' \| (j-\frac{1}{2},1)\kappa H_1 T\rangle_1$
1	H_1-1	$T-1$	1	1	1	H_1	T	$\dfrac{2F_2(H_1-1,T-1)[2T(j+\frac{3}{2}+H_1+T)(j+\frac{1}{2}-H_1-T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T-1)f(-H_1+1,T-1)g(H_1,T)g(H_1-1,T-1)]^{1/2}}$
2	H_1-1	$T-1$	1	1	1	H_1	T	$\dfrac{-2F_1(H_1-1,T-1)[2(T-1)(j+\frac{1}{2}+H_1-T)(j+\frac{5}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)(j-\frac{1}{2}+H_1+T)(j+\frac{3}{2}-H_1+T)(j+\frac{1}{2}-H_1-T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T-1)f(-H_1+1,T-1)g(H_1,T)g(H_1-1,T-1)]^{1/2}}$
2	H_1-1	$T+1$	1	1	1	H_1	T	$\dfrac{-2F_3(-H_1,T)[2(T+2)(j+\frac{3}{2}+H_1+T)(j+\frac{1}{2}-H_1-T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T+1)f(-H_1+1,T+1)g(H_1,T)g(H_1-1,T+1)]^{1/2}}$
1	H_1-1	$T+1$	1	1	1	H_1	T	$\dfrac{-2F_2(-H_1,T)[2(T+1)(j-\frac{3}{2}+H_1-T)(j+\frac{7}{2}-H_1+T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T+1)f(-H_1+1,T+1)g(H_1,T)g(H_1-1,T+1)]^{1/2}}$
2	H_1+1	$T+1$	-1	1	1	H_1	T	$\dfrac{2F_3(H_1,T)[2(T+2)(j+\frac{3}{2}-H_1+T)(j+\frac{1}{2}+H_1-T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T+1)f(-H_1-1,T+1)g(H_1,T)g(H_1+1,T+1)]^{1/2}}$
1	H_1+1	$T+1$	-1	1	1	H_1	T	$\dfrac{-2F_2(H_1,T)[2(T+1)(j-\frac{3}{2}-H_1-T)(j+\frac{7}{2}+H_1+T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T+1)f(-H_1-1,T+1)g(H_1,T)g(H_1+1,T+1)]^{1/2}}$
2	H_1+1	$T-1$	-1	1	1	H_1	T	$\dfrac{2F_1(-H_1-1,T-1)[2(T-1)(j+\frac{5}{2}+H_1-T)(j-\frac{1}{2}-H_1+T)(j+\frac{3}{2}+H_1+T)(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}-H_1+T)(j+\frac{1}{2}+H_1-T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T-1)f(-H_1-1,T-1)g(H_1,T)g(H_1+1,T-1)]^{1/2}}$
1	H_1+1	$T-1$	-1	1	1	H_1	T	$\dfrac{2F_2(-H_1-1,T-1)[2T(j+\frac{3}{2}-H_1+T)(j+\frac{1}{2}+H_1-T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T-1)f(-H_1-1,T-1)g(H_1,T)g(H_1+1,T-1)]^{1/2}}$
0	H_1+1	T	-1	1	1	H_1	T	$\dfrac{-[2(j+\frac{3}{2}-H_1+T)(j+\frac{1}{2}+H_1-T)]^{1/2}}{[(j+\frac{1}{2})f(H_1,T)f(-H_1,T)g(H_1,T)]^{1/2}}\{T^2(2j+2)-T(2j^2+2j-\frac{1}{2})-(j+\frac{1}{2})(j+\frac{3}{2})+H_1[T(2j+2)-(j+\frac{1}{2})]\}$
0	H_1-1	T	1	1	1	H_1	T	$\dfrac{-[2(j+\frac{3}{2}+H_1+T)(j+\frac{1}{2}-H_1-T)]^{1/2}}{[(j+\frac{1}{2})f(H_1,T)f(-H_1,T)g(H_1,T)]^{1/2}}\{T^2(2j+2)-T(2j^2+2j-\frac{1}{2})-(j+\frac{1}{2})(j+\frac{3}{2})-H_1[T(2j+2)-(j+\frac{1}{2})]\}$
1	H_1	T'	0	1	1	H_1	T	$\delta_{TT'}\left[\dfrac{T(T+1)}{(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
1	H_1	T	0	0	1	H_1	T	$\dfrac{H_1}{[(j+\frac{1}{2})(j+\frac{3}{2})]^{1/2}}$

[a] For $f(H_1,T)$, $g(H_1,T)$ see Table V or Eq. (18b).

$$F_1(H_1,T)=H_1[T^2(2j+2)^2+T(2j+2)(4j+3)+(j+\tfrac{1}{2})(5j+11/2)]+(j+\tfrac{1}{2}-T)[T(2j+2)+(j+\tfrac{3}{2})][T(2j+2)+(3j+5/2)].$$

$$\begin{aligned}F_2(H_1,T)=&(H_1^4+H_1^3)(j+\tfrac{1}{2})[T(2j+2)+(j+\tfrac{1}{2})][T(2j+2)+(3j+5/2)]\\&+H_1^2[T^4(2j+2)^2+T^3(2j+2)^2(2j^2+4j+9/2)+T^2(-8j^5-16j^4+14j^3+59j^2+53j+65/4)-T(j+\tfrac{1}{2})(j+\tfrac{3}{2})(16j^3+26j^2+6j-5/2)-(j+\tfrac{1}{2})(j+\tfrac{3}{2})(6j^3+12j^2+(13/2)j+1)]\\&+H_1[-T^4(2j+2)^2(j+\tfrac{3}{2})+T^3(2j+2)(j+\tfrac{1}{2})(4j^2+6j+3)+T^2(-4j^5-6j^4+14j^3+38j^2+(115/4)j+55/8)-T(8j^5+36j^4+59j^3+(85/2)j^2+(51/4)j+9/8)-(j+\tfrac{1}{2})^2(j+\tfrac{3}{2})^2(3j+5/2)]\\&+(j+\tfrac{3}{2})(j+\tfrac{3}{2}+T)(j+\tfrac{1}{2}-T)^2[T(2j+2)+j+\tfrac{3}{2}][-T^2(2j+2)+T(2j^2-2j-9/2)+(3j^2+2j-7/4)].\end{aligned}$$

$$\begin{aligned}F_3(H_1,T)=&H_1^3[T^2(2j+2)^2+T(2j+2)(4j+3)+(j+\tfrac{1}{2})(j+\tfrac{3}{2})]+H_1^2[T^3(2j+2)^2-T^2(2j+2)(2j^2-9j-9)-T(8j^3-11j^2-37j-71/4)-(3j^3+(5/2)j^2-(11/4)j-13/8)]\\&-H_1[T^4(2j+2)^2-T^3(2j+2)(4j^2+2j-3)+T^2(4j^4-4j^3-38j^2-40j-41/4)+T(8j^4+24j^3+9j^2-15j-33/4)+(j+\tfrac{1}{2})(j+\tfrac{3}{2})^3]\\&-[T(2j+2)+j+\tfrac{3}{2}][T^4(2j+2)-T^3(6j^2+6j+\tfrac{1}{2})+T^2(6j^3+3j^2-(17/2)j-29/4)+T(-2j^4+4j^3+16j^2+10j-11/8)-(j+\tfrac{1}{2})(j+\tfrac{3}{2})(3j^2+j-13/4)].\end{aligned}$$

$$\begin{aligned}F_4(H_1,T)=&(H_1^4+3H_1^3)(j+\tfrac{1}{2})[T(2j+2)+(j+\tfrac{1}{2})][T(2j+2)+(3j+5/2)]\\&+H_1^2[T^4(2j+2)^2+T^3(2j+2)^2(2j^2+2j+7/2)-T^2(8j^5+16j^4-10j^3-57j^2-59j-81/4)-T(16j^5+58j^4+60j^3+2j^2-25j-77/8)-(j+\tfrac{1}{2})(6j^4+21j^3+(39/2)j^2+(23/4)j+5/4)]\\&+H_1[T^4(2j+2)^2(j+5/2)+T^3(2j+2)(4j^3+12j^2+22j+29/2)-T^2(12j^5+34j^4+26j^3-26j^2-(221/4)j-205/8)-T(24j^5+96j^4+147j^3+(205/2)j^2+(111/4)j-5/8)\\&-(j+\tfrac{1}{2})(j+\tfrac{3}{2})(9j^3+(43/2)j^2+(71/4)j+53/8)]\\&+(j+\tfrac{3}{2})(j+\tfrac{1}{2}-T)[T(2j+2)+(j+\tfrac{3}{2})][T^4(2j+2)+T^3(-2j^2+8j+21/2)-T^2(2j^3+15j^2-\tfrac{1}{2}j-63/4)+T(2j^4+2j^3-22j^2-(37/2)j+51/8)+(3j^4+8j^3-(9/2)j^2-11j-9/16)].\end{aligned}$$

With these coefficients and the transformation coefficients of Appendix 1, R_5 Wigner coefficients, involving the irreducible representation $(j-\frac{1}{2},1)$ can be transformed from the neutron, proton quasispin scheme to the physically interesting scheme based on nucleon number and isospin T. The R_5 Wigner coefficients needed for the matrix elements of the infinitesimal operators in the representation $(j-\frac{1}{2},1)$ are given in Tables V, VI, and VII. The algebraic structure of these coefficients is very complicated in those cases in which $\kappa=1$, or 2 for two of the irreducible representations involved in the R_5 Wigner coefficient (Tables VI and VII). It does not seem worthwhile to continue with general algebraic techniques to even more complicated irreducible representations. Coefficients for representations with $t>1$, (and $\omega_1>t$), are best calculated numerically.

With the symmetry quantum numbers κ, the R_5 Wigner coefficients satisfy a symmetry property if the transformation $H_1\to -H_1$, $M_T\to -M_T$, is applied simultaneously to all three representations. In particular, the reduced (double barred) Wigner coefficients have the symmetry property

$$\begin{aligned}\langle(j-\tfrac{1}{2},1)\kappa H_1T;(11)\pm1,1\|(j-\tfrac{1}{2},1)\kappa'H_1'T'\rangle\\=(-1)^{\kappa-\kappa'}(-1)^{T+1-T'}\langle(j-\tfrac{1}{2},1)\kappa,-H_1,T;(11)\mp1,1\\\times\|(j-\tfrac{1}{2},1)\kappa',-H_1',T'\rangle,\quad(19)\end{aligned}$$

where the T-dependent phase factor balances the phase factor of the R_3 Wigner coefficient under the transformation $M_T\to -M_T$ for all three T's. Another symmetry property, involving interchange of the first and third representation, is also useful. In particular

$$\begin{aligned}\langle(j-\tfrac{1}{2},1)\kappa H_1T;(11)\pm11\|(j-\tfrac{1}{2},1)\kappa'H_1'T'\rangle\\=-\left[\frac{2T'+1}{2T+1}\right]^{1/2}\langle(j-\tfrac{1}{2},1)\kappa'H_1'T';(11)\mp11\\\times\|(j-\tfrac{1}{2},1)\kappa H_1T\rangle,\quad(20)\end{aligned}$$

where the phase factor follows from Eq. (33) of Ref. 4.

4. PERTURBATION THEORY

Since exact solutions of the charge-independent pairing Hamiltonian can be obtained only through the diagonalization of relatively large matrices, it may be of interest to develop perturbation-theory formulas, both in the weak and strong pairing limits.

Weak Pairing Limit

If the pairing strength G is small compared with the single-particle energies, the scheme of Eq. (11) furnishes a good zeroth approximation, and the effect of the off-diagonal matrix elements in the form of Eq. (14) can be treated as a perturbation. The lowest energy states will be those for which the single-particle levels are filled in

TABLE VII. The R_5 Wigner coefficients $\langle(j-\frac{1}{2},1)\kappa'H_1'T';(11)H_1''T''\|(j-\frac{1}{2},1)\kappa H_1T\rangle$, with $\kappa=2$.[a]

κ'	H_1'	T'	H_1''	T''	κ	H_1	T	$\langle(j-\frac{1}{2},1)\kappa'H_1'T';(11)H_1''T''\|(j-\frac{1}{2},1)\kappa H_1T\rangle_1$
2	H_1-1	$T-1$	1	1	2	H_1	T	$\dfrac{2F_4(H_1-1,T-1)[2(T-1)(T+1)(j-\frac{1}{2}+H_1+T)(j+\frac{5}{2}-H_1-T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})T(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T-1)f(-H_1+1,T-1)g(H_1,T)g(H_1-1,T-1)]^{1/2}}$
1	H_1-1	$T-1$	1	1	2	H_1	T	$\dfrac{-2F_3(H_1-1,T-1)[2(T+1)(j+\frac{1}{2}+H_1-T)(j+\frac{3}{2}-H_1+T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T-1)f(-H_1+1,T-1)g(H_1,T)g(H_1-1,T-1)]^{1/2}}$
2	H_1-1	$T+1$	1	1	2	H_1	T	$\dfrac{-2F_4(-H_1,T)[2T(T+2)(j+\frac{1}{2}+H_1-T)(j+\frac{3}{2}-H_1+T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})(T+1)(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T+1)f(-H_1+1,T+1)g(H_1,T)g(H_1-1,T+1)]^{1/2}}$
1	H_1-1	$T+1$	1	1	2	H_1	T	$\dfrac{-2F_1(-H_1,T)[2T(j+\frac{1}{2}-H_1-T)(j-\frac{3}{2}+H_1-T)(j+\frac{7}{2}-H_1+T)(j+\frac{3}{2}+H_1+T)(j+\frac{1}{2}+H_1-T)(j+\frac{3}{2}-H_1+T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1-1,T+1)f(-H_1+1,T+1)g(H_1,T)g(H_1-1,T+1)]^{1/2}}$
2	H_1+1	$T+1$	-1	1	2	H_1	T	$\dfrac{-2F_4(H_1,T)[2T(T+2)(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})(T+1)(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T+1)f(-H_1-1,T+1)g(H_1,T)g(H_1+1,T+1)]^{1/2}}$
1	H_1+1	$T+1$	-1	1	2	H_1	T	$\dfrac{2F_1(H_1,T)[2T(j+\frac{1}{2}+H_1-T)(j-\frac{3}{2}-H_1-T)(j+\frac{7}{2}+H_1+T)(j+\frac{3}{2}-H_1+T)(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T+1)f(-H_1-1,T+1)g(H_1,T)g(H_1+1,T+1)]^{1/2}}$
2	H_1+1	$T-1$	-1	1	2	H_1	T	$\dfrac{2F_4(-H_1-1,T-1)[2(T-1)(T+1)(j+\frac{5}{2}+H_1-T)(j-\frac{1}{2}-H_1+T)]^{1/2}}{[(j+\frac{1}{2})(j+\frac{3}{2})T(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T-1)f(-H_1-1,T-1)g(H_1,T)g(H_1+1,T-1)]^{1/2}}$
1	H_1+1	$T-1$	-1	1	2	H_1	T	$\dfrac{2F_3(-H_1-1,T-1)[2(T+1)(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)]^{1/2}}{[(2T+1)f(H_1,T)f(-H_1,T)f(H_1+1,T-1)f(-H_1-1,T-1)g(H_1,T)g(H_1+1,T-1)]^{1/2}}$
0	H_1-1	T	1	1	2	H_1	T	$\dfrac{[2(j+\frac{1}{2}+H_1-T)(j+\frac{3}{2}-H_1+T)]^{1/2}}{[(j+\frac{3}{2})T(T+1)f(H_1,T)f(-H_1,T)g(H_1,T)]^{1/2}}\{[H_1^2+H_1T][T(2j+2)+(j+\frac{1}{2})]-(j+\frac{3}{2})(j+\frac{1}{2}-T)[T(2j+2)+(j+\frac{3}{2})]\}$
0	H_1+1	T	-1	1	2	H_1	T	$\dfrac{-[2(j+\frac{1}{2}-H_1-T)(j+\frac{3}{2}+H_1+T)]^{1/2}}{[(j+\frac{3}{2})T(T+1)f(H_1,T)f(-H_1,T)g(H_1,T)]^{1/2}}\{[H_1^2-H_1T][T(2j+2)+(j+\frac{1}{2})]-(j+\frac{3}{2})(j+\frac{1}{2}-T)[T(2j+2)+(j+\frac{3}{2})]\}$
2	H_1	T'	0	1	2	H_1	T	$\delta_{TT'}\left[\dfrac{T(T+1)}{(j+\frac{1}{2})(j+\frac{3}{2})}\right]^{1/2}$
2	H_1	T	0	0	2	H_1	T	$\dfrac{H_1}{[(j+\frac{1}{2})(j+\frac{3}{2})]^{1/2}}$

[a] For $f(H_1,T)$, $g(H_1,T)$ see Table V or Eq. (18b). For $F_i(H_1,T)$ see Table VI.

order with the nucleons in the partly filled levels coupled to $v_j=0$ or $v_j=1$ for even or odd nuclei. Figure 1 illustrates the single-particle level structure for the two types of nuclei of importance. In case 1 (light nuclei), in which neutrons and protons are filling the same level α, the second-order perturbation terms, of order G^2/ϵ, merely lead to a renormalization of the single-particle energy and pairing strength parameters. For both the case $v_\alpha=0$ $(t_\alpha=0)$, and $v_\alpha=1$ $(t_\alpha=\frac{1}{2})$ the second-order corrections to the energy can be given by

$$\Delta E^{(2)}=-\sum_{p,i}\frac{3G^2(2j_p+1)(2j_i+1)}{8(\epsilon_p-\epsilon_i)}+\frac{3G^2}{8}(N_\alpha-2j_\alpha-1)\sum_i\frac{(2j_i+1)}{\epsilon_\alpha-\epsilon_i}-\frac{G^2}{8}\left\{\sum_i\frac{(2j_i+1)}{\epsilon_\alpha-\epsilon_i}+\sum_p\frac{(2j_p+1)}{\epsilon_p-\epsilon_\alpha}\right\}$$
$$\times\{(j_\alpha+\tfrac{1}{2}-\tfrac{1}{2}v_\alpha)(j_\alpha+\tfrac{7}{2}-\tfrac{1}{2}v_\alpha)+t_\alpha(t_\alpha+1)-T_\alpha(T_\alpha+1)-\tfrac{1}{4}(N_\alpha-2j_\alpha-1)(N_\alpha-2j_\alpha-7)\}\,. \quad (21a)$$

Comparison with Eq. (13) shows that, except for a constant term, the second-order terms merely lead to a renormalization of the single-particle energy and pairing strength parameters.

$$\epsilon_\alpha\rightarrow\epsilon_\alpha+\tfrac{3}{8}G^2\textstyle\sum_i(2j_i+1)/(\epsilon_\alpha-\epsilon_i) \quad (21b)$$
$$G\rightarrow G+\tfrac{1}{4}G^2\{\textstyle\sum_i(2j_i+1)/(\epsilon_\alpha-\epsilon_i)+\sum_p(2j_p+1)/(\epsilon_p-\epsilon_\alpha)\}\,.$$

In case 2 (heavier nuclei) it will be assumed that level α is partly filled with protons, but levels α and β are completely filled with neutrons, while level γ is partly filled with neutrons. The perturbation formulas are somewhat more complicated. For even-even nuclei, with both $v_\alpha=0$ and $v_\gamma=0$, the second-order corrections to the energy are

$$\Delta E^{(2)}=-\tfrac{3}{8}G^2\sum_{p,i}\frac{(2j_p+1)(2j_i+1)}{\epsilon_p-\epsilon_i}-\tfrac{1}{8}G^2\sum_{\beta,i}\frac{(2j_\beta+1)(2j_i+1)}{\epsilon_\beta-\epsilon_i}$$
$$-\tfrac{1}{8}G^2T_\alpha(2j_\alpha+3-2T_\alpha)\sum_i\frac{(2j_i+1)}{\epsilon_\alpha-\epsilon_i}-\tfrac{1}{8}G^2\{3(2j_\alpha+1)+(2j_\alpha-3)T_\alpha-2T_\alpha^2\}\sum_p\frac{(2j_p+1)}{\epsilon_p-\epsilon_\alpha}$$
$$-\tfrac{1}{8}G^2\{3(2j_\gamma+1)+(2j_\gamma-3)T_\gamma-2T_\gamma^2\}\sum_i\frac{(2j_i+1)}{\epsilon_\gamma-\epsilon_i}-\tfrac{1}{8}G^2T_\gamma(2j_\gamma+3-2T_\gamma)\sum_p\frac{(2j_p+1)}{\epsilon_p-\epsilon_\gamma}$$
$$-\tfrac{1}{8}G^2(T_\gamma+1)(2j_\gamma+1-2T_\gamma)\sum_\beta\frac{(2j_\beta+1)}{\epsilon_\gamma-\epsilon_\beta}-\tfrac{1}{8}G^2(T_\alpha+1)(2j_\alpha+1-2T_\alpha)\sum_\beta\frac{(2j_\beta+1)}{\epsilon_\beta-\epsilon_\alpha}$$
$$-\frac{G^2}{8(\epsilon_\gamma-\epsilon_\alpha)}\{(2j_\alpha+1)(2j_\gamma+1)(T_\alpha+T_\gamma+3)-2(2j_\alpha+1)T_\gamma(T_\gamma+2)-2(2j_\gamma+1)T_\alpha(T_\alpha+2)+4T_\alpha T_\gamma\}$$
$$-\tfrac{1}{8}G^2\sum_{p,\beta}\frac{(2j_\beta+1)(2j_p+1)}{\epsilon_p-\epsilon_\beta}, \quad (22)$$

where $T_\gamma=(1/2)N_\gamma$, with N_γ=neutron number for level γ; $T_\alpha=j_\alpha+\frac{1}{2}-(\frac{1}{2})Z_\alpha$, with Z_α=proton number for level α. For odd nuclei, the results are best expressed in terms of the energy differences between odd and even nuclei. For an odd number of neutrons in state γ $(v_\gamma=1)$

$$E_{N_\gamma+1,v_\gamma=1}-E_{N_\gamma,v_\gamma=0}=\epsilon_\gamma+GN_\gamma+\tfrac{1}{8}G^2\left\{(N_\gamma+3)\sum_i\frac{(2j_i+1)}{\epsilon_\gamma-\epsilon_i}+N_\gamma\sum_p\frac{(2j_p+1)}{\epsilon_p-\epsilon_\gamma}\right.$$
$$\left.+(N_\gamma+2)\sum_\beta\frac{(2j_\beta+1)}{\epsilon_\gamma-\epsilon_\beta}+\frac{1}{\epsilon_\gamma-\epsilon_\alpha}[(2j_\alpha+1)(N_\gamma+2)+Z_\alpha]\right\}. \quad (23a)$$

For an odd number of protons in state α $(v_\alpha=1)$

$$E_{Z_\alpha+1,v_\alpha=1}-E_{Z_\alpha,v_\alpha=0}=\epsilon_\alpha+G(Z_\alpha-\tfrac{1}{4})+\tfrac{1}{8}G^2\left\{(Z_\alpha+2)\sum_i\frac{(2j_i+1)}{\epsilon_\alpha-\epsilon_i}+Z_\alpha\sum_\beta\frac{(2j_\beta+1)}{\epsilon_\beta-\epsilon_\alpha}\right.$$
$$\left.+(Z_\alpha-1)\sum_p\frac{(2j_p+1)}{\epsilon_p-\epsilon_\alpha}+\frac{1}{\epsilon_\gamma-\epsilon_\alpha}[(2j_\gamma+1)(Z_\alpha-1)+N_\gamma]\right\}. \quad (23b)$$

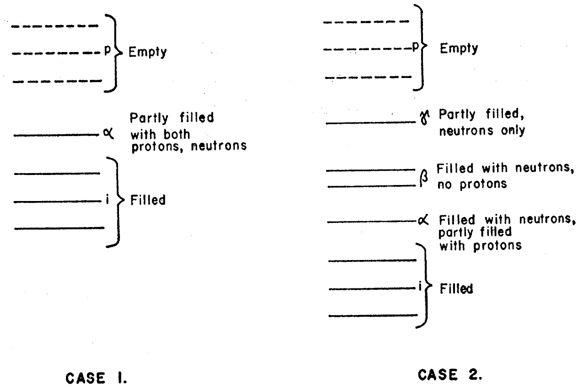

FIG. 1. Single-particle level schemes for typical nuclei. For case 1, see Eq. (21). For case 2, see Eqs. (22), (23a), and (23b).

Strong-Pairing Limit

If the energy differences between single-particle states are negligible compared with the pairing strength parameter, the coupled scheme of Eq. (10) furnishes a good zeroth approximation. The pairing term is diagonal in the final resultant R_5 irreducible representation $(\omega_1,\omega_2)=(\Omega-\frac{1}{2}v,\,t)$ where $\Omega=\sum_i(j_i+\frac{1}{2})$; but the single-particle part of the Hamiltonian is off-diagonal in the over-all seniroity v and reduced isospin t. In the scheme (10), these off-diagonal matrix elements can be expressed easily only in terms of generalized R_5 Racah coefficients. These have not been calculated, so that an alternate technique will be used for the strong-pairing perturbation theory.

It will be sufficient to consider the case of Ω Nilsson or "$j=\frac{1}{2}$-like" single-particle states with energies ϵ_1, ϵ_2, $\cdots\epsilon_i$, $\cdots\epsilon_\Omega$. By letting these become degenerate in groups of $j_i+\frac{1}{2}$ any arbitrary single-particle spectrum can be constructed. The greatest interest is in the state of over-all seniority zero, R_5 representation $(\Omega 0)$. Since this can be obtained from the Ω single-particle levels with individual R_5 representaions of (10) in only one way, the possibility of degeneracies of the ϵ_i plays no essential role. [Similarly, states with over-all seniority one or R_5 representation $(\Omega-\frac{1}{2},\frac{1}{2})$ can be obtained by coupling $\Omega-1$ states of representation (10) with a single state of representation $(\frac{1}{2},\frac{1}{2})$ in only one way.]

The perturbation technique to be employed makes use of the fact that the zeroth order or pairing part of the Hamiltonian

$$-G\sum_{i=1}^{\Omega}\sum_{i'=1}^{\Omega}\sum_{M_T}A_i^\dagger(M_T)A_{i'}(M_T) \tag{24}$$

is symmetric in the Ω level indices i. The zeroth order wave function can thus be classified not only according to irreducible representations of R_5, but also, according to irreducible representations of the symmetric group of order Ω. The latter are characterized as usual by the partition numbers of Ω objects, $[f_1f_2\cdots f_k]$ with $\sum f_k=\Omega$, or by Young tableaux with f_k nodes in the kth row. Although the Ω objects are here not particles but individual energy levels with R_5 transformation properties (10), the decomposition of the irreducible representations $[f_1f_2\cdots]$ into irreducible representations of R_5 is identical with that met in d-shell spectroscopy.[18] Totally symmetric states $[\Omega]$, for example, contain the R_5 representations $(\Omega 0)$, $(\Omega-2,0)\cdots(10)$ or (00); while states of $[\Omega-1,1]$ symmetry contain R_5 representations $(\Omega-1,1)$, $(\Omega-3,1)\cdots(21)$ or (11) and $(\Omega-2,0)$, $(\Omega-4,0)\cdots(10)$ or (00). The representation $(\Omega 0)$ is found only among the totally symmetric states $[\Omega]$.

The single-particle part of the Hamiltonian is off-diagonal not only in the over-all R_5 irreducible representation $(\Omega-\frac{1}{2},t)$, but also in the representations of the symmetric group. The single-particle part of the Hamiltonian can be written

$$\sum_{i=1}^{\Omega}(\epsilon_iN_i)=\frac{(\sum_{i=1}^{\Omega}\epsilon_i)(\sum_{i=1}^{\Omega}N_i)}{\Omega}+\sum_{k=1}^{\Omega-1}\frac{[(\sum_{i=1}^{\Omega-k}\epsilon_i)-(\Omega-k)\epsilon_{\Omega+1-k}][(\sum_{i=1}^{\Omega-k}N_i)-(\Omega-k)N_{\Omega+1-k}]}{(\Omega-k)(\Omega+1-k)}, \tag{25a}$$

where the first term, proportional to $\sum_i N_i$, is totally symmetric in the level indices and thus transforms according to the representation $[\Omega]$, while the $\Omega-1$ terms in the k sum transform according to the $(\Omega-1)$-dimensional representation $[\Omega-1,1]$. In particular, the terms

$$[(\sum_{i=1}^{\Omega-k}N_i)-(\Omega-k)N_{\Omega+1-k}]=2[(\sum_{i=1}^{\Omega-k}H_{1i})-(\Omega-k)H_{1(\Omega+1-k)}] \tag{25b}$$

with $k=1,\,2,\,\cdots\Omega-1$ have been constructed to have transformation properties which can be described by the Yamanouchi symbols[19,20] $\{2111\cdots1\}$, $\{1211\cdots1\}$, $\cdots$, $\{1111\cdots121\}$, respectively. Since the total number of nucleons is a good quantum number, matrix elements of the totally symmetric part, $[\Omega]$, are trivial. Matrix

[18] H. A. Jahn, Proc. Roy. Soc. (London) **A201**, 516 (1950).

[19] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1962), p. 221.

[20] Yamanouchi symbols are enclosed in curly brackets, $\{\rho_\Omega,\rho_{\Omega-1},\cdots,\rho_1\}$. Square brackets are used for the representation of the symmetric group, while parentheses are retained for representations of R_5.

elements of terms with symmetry $[\Omega-1,1]$ can be expressed in terms of a Wigner-Eckart theorem, for example,

$$\begin{aligned}\langle\alpha'[f_\Omega']\{\rho_\Omega'\cdots\rho_1'\}|T_{\{11\cdots121\cdots\}}{}^{[\Omega-1,1]}|\alpha[f_\Omega]\{\rho_\Omega\cdots\rho_1\}\rangle\\ =\langle\alpha'[f_\Omega']\|T^{[\Omega-1,1]}\|\alpha[f_\Omega]\rangle\\ \times\langle[f_\Omega]\{\rho_\Omega\cdots\rho_1\};[\Omega-1,1]\{11\cdots121\cdots1\}\\ \times|[f_\Omega']\{\rho_\Omega'\cdots\rho_1'\}\rangle,\quad(26)\end{aligned}$$

where the reduced (double-barred) matrix element is independent of the Yamanouchi symbols but carries the dependence on $[f]$ and all other quantum numbers, such as the R_5 quantum numbers, which are here denoted collectively by α. The dependence on the Yamanouchi symbols is given by the last coefficient, a Wigner coefficient for the inner product of the symmetric group.[21] To determine the values of the reduced matrix elements, it is sufficient to consider a single term of $[\Omega-1,1]$ symmetry. It is most convenient to choose the terms of $\{2111\cdots1\}$ symmetry, the term with $k=1$ in Eqs. (25). In normalized form, this can be written

$$\frac{2}{[(\Omega-1)\Omega]^{1/2}}\Big[\Big(\sum_{i=1}^{\Omega-1}H_{1i}\Big)-(\Omega-1)H_{1\Omega}\Big]=\frac{2}{[(\Omega-1)\Omega]^{1/2}}\Big[\sum_{i=1}^{\Omega}H_{1i}-\Omega H_{1\Omega}\Big],\quad(27)$$

where the totally symmetric part $2\sum_{i=1}^{\Omega}H_{1i}$ is diagonal, with trivial eigenvalue $N-2\Omega$, so that the only matrix elements which have to be calculated are those of $H_{1\Omega}$, essentially the number operator for the Ωth level. These are best calculated by expressing the Ω-level functions of definite permutation symmetry $[f_\Omega]\{\rho_\Omega,\cdots,\rho_1\}$ and R_5 irreducible representation $(\Omega-\frac{1}{2}v_\Omega,t_\Omega)$ through a fractional-parentage expansion involving functions for the $(\Omega-1)$-level system coupled through R_5 Wigner coefficients to the function for the Ωth level where the latter must have R_5 irreducible representation (10). In particular,

$$\begin{aligned}&|[f_\Omega]\{\rho_\Omega,\rho_{\Omega-1},\cdots\rho_1\}(\Omega-\tfrac{1}{2}v_\Omega,t_\Omega)\kappa H_1TM_T\rangle\\ &=\sum_{v_{\Omega-1},t_{\Omega-1}}\ \sum_{\substack{H_1''T''M_T''\\ \kappa'T'}}|[f_{\Omega-1}]\{\rho_{\Omega-1},\rho_{\Omega-2},\cdots\rho_1\}(\Omega-1-\tfrac{1}{2}v_{\Omega-1},t_{\Omega-1})\kappa'H_1'T'M_T'\rangle\\ &\times|[1]\{1\}(10)H_1''T''M_T''\rangle\langle(\Omega-1-\tfrac{1}{2}v_{\Omega-1},t_{\Omega-1})\kappa'H_1'T'M_T';(10)H_1''T''M_T''|(\Omega-\tfrac{1}{2}v_\Omega,t_\Omega)\kappa H_1TM_T\rangle\\ &\times\langle[f_{\Omega-1}]\{\rho_{\Omega-1},\rho_{\Omega-2},\cdots\rho_1\}(\Omega-1-\tfrac{1}{2}v_{\Omega-1},t_{\Omega-1})\|[f_\Omega]\{\rho_\Omega,\rho_{\Omega-1},\cdots\rho_1\}(\Omega-\tfrac{1}{2}v_\Omega,t_\Omega)\rangle,\quad(28)\end{aligned}$$

where the $\langle\ \|\ \rangle$ coefficients of the last line are fractional parentage coefficients which insure the $[f_\Omega]$ symmetry of the Ω-level function built from products of $(\Omega-1)$-level functions of $[f_{\Omega-1}]$ symmetry and one-level functions for the Ωth level. The $(\Omega-1)$-level functions with R_5 representation $(\Omega-1-\frac{1}{2}v_{\Omega-1},t_{\Omega-1})$ are coupled to the (10) function for the Ωth level through the R_5 Wigner coefficients of the second line. The operator $H_{1\Omega}$ simply multiplies the terms in the expansion by the number $H_1''=H_1-H_1'$.

Since the technique depends on knowledge of a new type of fractional parentage coefficient as well as the Wigner coefficients for the inner product of the symmetric group, for which general expressions are not known for arbitrary Ω, not very much progress seems to have been made. For the seniority zero state, of greatest interest, however, the only such coefficients actually needed have the trivial value unity, so that the technique becomes very simple. The seniority zero state, R_5 representation $(\Omega 0)$, belongs to the one-dimensional totally symmetric representation $[\Omega]$ of the symmetric group. Thus, only Wigner coefficients for the inner product $[\Omega]\times[\Omega-1,1]$ are needed, and these have the value unity:

$$\langle[\Omega]\{111\cdots1\};[\Omega-1,1]\{\rho_\Omega,\cdots\rho_1\}\times|[\Omega-1,1]\{\rho_\Omega,\cdots\rho_1\}\rangle=1\quad(29)$$

for all $\{\rho_\Omega,\cdots\rho_1\}$. The operator of Eq. (27) transforms according to the $[\Omega-1,1]$ representation of the symmetric group and the (11) representation of R_5. It can thus connect functions of $[\Omega]$ symmetry, $(\Omega 0)$ R_5 representation, only to functions of $[\Omega-1,1]$ symmetry, and $(\Omega-1,1)$ R_5 representation. [The Kronecker product $(\Omega 0)\times(11)$ contains the representations $(\Omega+1,1)$, $(\Omega,1)$, $(\Omega-1,1)$, and $(\Omega 0)$. Of these only the representation $(\Omega-1,1)$ is found in $[\Omega-1,1]$.] The fractional parentage expansion of Eq. (28) is thus needed only for the simple states $[\Omega](\Omega 0)$ and $[\Omega-1,1]$ $(\Omega-1,1)$. For both of these the fractional parentage expansion collapses to a single term since the parent states have a unique daughter. Thus,

$$\begin{aligned}&\langle[\Omega-1]\{11\cdots1\}(\Omega-1,0)\|[\Omega]\{111\cdots1\}(\Omega 0)\rangle=1,\\ &\langle[\Omega-1]\{11\cdots1\}(\Omega-1,0)\\ &\quad\times\|[\Omega-1,1]\{211\cdots1\}(\Omega-1,1)\rangle=1,\end{aligned}\quad(30)$$

and no complicated calculations are needed for either the coefficient of fractional parentage (c.f.p.) of Eq. (28)

[21] Ref. 19, Secs. 7-13, 7-14.

or the Wigner coefficients of Eq. (26). The reduced matrix elements for the single-particle part of the Hamiltonian, [Eq. (26)], follow from the matrix element

$$\langle[\Omega-1,1]\{211\cdots1\}(\Omega-1,1)\kappa H_1TM_T|\frac{2[(\sum_{i=1}^{\Omega}H_{1i}-\Omega H_{1\Omega}]}{[\Omega(\Omega-1]^{1/2}}|[\Omega]\{111\cdots1\}(\Omega0)H_1TM_T\rangle$$

$$=\frac{-2\Omega}{[\Omega(\Omega-1)]^{1/2}}\{\langle(\Omega-1,0)H_1-1,TM_T;(10)100|(\Omega0)H_1TM_T\rangle\langle(\Omega-1,0)H_1-1,TM_T;(10)100|(\Omega-1,1)\kappa H_1TM_T\rangle$$

$$-\langle(\Omega-1,0)H_1+1,TM_T;(10)-100|(\Omega0)H_1TM_T\rangle\langle(\Omega-1,0)H_1+1,TM_T;(10)-100|(\Omega-1,1)\kappa H_1TM_T\rangle\}. \quad (31)$$

This matrix element has been expressed solely in terms of R_5 Wigner coefficients involving the Kronecker product $(\Omega-1,0)\times(10)$. These have been calculated in the $|\kappa H_1TM_T\rangle$ scheme by the techniques discussed in Sec. 2. The results are shown in Table VIII, where the reduced (double-barred, R_5/R_3) coefficients are tabulated. [The isospin Wigner coefficients $\langle TM_T00|TM_T\rangle$ of Eq. (31) are all unity.] The only nonzero matrix elements in Eq. (31) are those with $\kappa=1$ and 2. With $\kappa=0$ the R_5 Wigner coefficients of Eq. (31) are zero. [The $\kappa=0$ states of $(\Omega-1,1)$ with $H_1=\Omega-n$ have even (odd) T for n odd (even), while the $H_1\pm1=\Omega-n\pm1$ states of $(\Omega-1,0)$ have odd (even) T for n odd (even).] Equation (31) combined with Eqs. (25), (26), and (29) gives the matrix elements for the single particle part of the Hamiltonian connecting the $(\Omega0)$ correlated ground state with the $(\Omega-1,1)$ excited state. The zeroth order energy difference between these states is $G\Omega$. The pairing energy can be expressed through terms of second order (ϵ^2/G) by

$$E=-\tfrac{1}{4}G[N(2\Omega+3-\tfrac{1}{2}N)-2T(T+1)]+(N/\Omega)\textstyle\sum_i\epsilon_i(j_i+\tfrac{1}{2})$$

$$\frac{-8f(\epsilon)\{[\Omega(\frac{1}{2}N-1-T)+\frac{1}{2}N]^2T(2\Omega+1-\frac{1}{2}N+T)+\Omega(\Omega+1)(T+1)(\frac{1}{2}N-T)(\frac{1}{2}N+T+1)(2\Omega-\frac{1}{2}N-T)\}}{G\Omega^2(\Omega-1)(2\Omega+1)[(2\Omega+1)(2T+1)(\frac{1}{2}N-T)+(2\Omega-\frac{1}{2}N-T)]}, \quad (32)$$

where

$$f(\epsilon)=[\sum_{i=1}^{\Omega}\epsilon_i^2-(1/\Omega)(\sum_{i=1}^{\Omega}\epsilon_i)^2],$$

which becomes

$$f(\epsilon)=[\sum_i(j_i+\tfrac{1}{2})\epsilon_i^2-(1/\Omega)(\sum_i(j_i+\tfrac{1}{2})\epsilon_i)^2]$$

for a set of degenerate levels j_i. The T values are restricted to T even for $\frac{1}{2}N$ even, T odd for $\frac{1}{2}N$ odd. The perturbation term, of order ϵ^2/G, has a seemingly very complicated N, T dependence; but, if plotted for fixed T as a function of N, its behavior is similar to that found in configurations of identical nucleons. The depression of the $v=0$ state is greatest for the half-filled system and approaches smaller values for the nearly filled or nearly empty system. For fixed N, the T-dependence is also relatively simple. For the half-filled system, $(N=2\Omega)$, the second-order ,ϵ^2/G, term of Eq. (32) collapses to the very simple form

$$\Delta E^{(2)}=\frac{-4f(\epsilon)[\Omega(\Omega+1)-T(T+1)]}{G\Omega(\Omega-1)(2\Omega+1)}. \quad (33)$$

In this case, therefore, the T dependence has exactly the $T(T+1)$ form, not only for the zeroth-order term, but for the second-order term as well. For N arbitrary, $(N\neq2\Omega)$, the T dependence deviates somewhat from the simple $T(T+1)$ form owing to the second-order term, but the deviations do not appear to be very large. In the strong pairing limit, therefore, the charge-independent nature of the pairing interaction makes itself felt largely through a $T(T+1)$-dependent term, a result which has recently also been predicted by Elliott and Lea[8] through a generalized Bardeen-Cooper-Schrieffer type of treatment of the charge-independent pairing interaction.

For small values of Ω, the strong-pairing perturbation techniques introduced in this section can quite easily be applied also to states with v, $t\neq0$. Some results for the case $\Omega=4$ are shown in Figs. 6, 7, and 8. A few of the details of this calculation are given in Appendix 2.

5. EXACT CALCULATIONS

To investigate the effects of the charge-independent nature of the pairing interaction further, a few simple model studies were undertaken, particularly in order to compare the exact solutions with the perturbation theory formulas. A simple system of four Nilsson or "$j=\frac{1}{2}$-like" single-particle states was chosen. This system is simple enough to be easily soluble. The biggest matrices which have to be diagonalized for the exact solutions are 24×24. Also, it was hoped that the system would be complex enough to show the essential features

TABLE VIII. The R_5 Wigner coefficients $\langle (j+\frac{1}{2},0)H_1'T';\,(10)H_1''T''\|(\omega_1\omega_2)\kappa H_1T\rangle$.[a,b]

H_1'	T'	H_1'' T''	$(\omega_1\omega_2)=(j+\frac{3}{2},0)$	$(\omega_1\omega_2)=(j-\frac{1}{2},0)$
H_1-1	T	1 0	$\left[\frac{(j+\frac{3}{2}+H_1-T)(j+\frac{5}{2}+H_1+T)}{(2j+3)(2j+4)}\right]^{1/2}$	$-\left[\frac{(j+\frac{3}{2}-H_1-T)(j+\frac{5}{2}-H_1+T)}{(2j+5)(2j+4)}\right]^{1/2}$
H_1+1	T	-1 0	$-\left[\frac{(j+\frac{3}{2}-H_1-T)(j+\frac{5}{2}-H_1+T)}{(2j+3)(2j+4)}\right]^{1/2}$	$\left[\frac{(j+\frac{3}{2}+H_1-T)(j+\frac{5}{2}+H_1+T)}{(2j+5)(2j+4)}\right]^{1/2}$
H_1	$T-1$	0 1	$\left[\frac{T(j+\frac{5}{2}+H_1+T)(j+\frac{5}{2}-H_1+T)}{(2T+1)(j+\frac{3}{2})(2j+4)}\right]^{1/2}$	$\left[\frac{T(j+\frac{3}{2}+H_1-T)(j+\frac{3}{2}-H_1-T)}{(2T+1)(j+\frac{5}{2})(2j+4)}\right]^{1/2}$
H_1	$T+1$	0 1	$-\left[\frac{(T+1)(j+\frac{3}{2}+H_1-T)(j+\frac{3}{2}-H_1-T)}{(2T+1)(j+\frac{3}{2})(2j+4)}\right]^{1/2}$	$-\left[\frac{(T+1)(j+\frac{5}{2}+H_1+T)(j+\frac{5}{2}-H_1+T)}{(2T+1)(j+\frac{5}{2})(2j+4)}\right]^{1/2}$
H_1'	T'	H_1'' T''	$(\omega_1\omega_2)=(j+\frac{1}{2},1),\ \kappa=1$	
H_1-1	T	1 0	$\frac{[2(j+\frac{3}{2}-H_1-T)(j+\frac{5}{2}+H_1+T)]^{1/2}}{[(j+\frac{5}{2})f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}\{(j+\frac{5}{2})(j+\frac{3}{2}-T)[T(2j+4)+(j+\frac{5}{2})]+[H_1T-H_1^2][T(2j+4)+(j+\frac{3}{2})]\}$	
H_1+1	T	-1 0	$\frac{[2(j+\frac{3}{2}+H_1-T)(j+\frac{5}{2}-H_1+T)]^{1/2}}{[(j+\frac{5}{2})f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}\{(j+\frac{5}{2})(j+\frac{3}{2}-T)[T(2j+4)+(j+\frac{5}{2})]-[H_1T+H_1^2][T(2j+4)+(j+\frac{3}{2})]\}$	
H_1	$T-1$	0 1	$\frac{-2H_1[T(2j+4)+(j+\frac{3}{2})][T(j+\frac{5}{2}+H_1+T)(j+\frac{5}{2}-H_1+T)(j+\frac{3}{2}+H_1-T)(j+\frac{3}{2}-H_1-T)]^{1/2}}{[(j+\frac{5}{2})(2T+1)f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}$	
H_1	$T+1$	0 1	$\frac{2H_1\{(j+\frac{3}{2}-T)[(j+\frac{5}{2})^2(2T+1)-T^2(2j+4)]-H_1^2[T(2j+4)+(j+\frac{3}{2})]\}[(T+1)]^{1/2}}{[(j+\frac{5}{2})(2T+1)f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}$	
H_1'	T'	H_1'' T''	$(\omega_1\omega_2)=(j+\frac{1}{2},1),\ \kappa=2$	
H_1-1	T	1 0	$\frac{[2T(T+1)(j+\frac{3}{2}+H_1-T)(j+\frac{5}{2}-H_1+T)]^{1/2}\{H_1[T(2j+4)-(j+\frac{3}{2})]-(j+\frac{3}{2}-T)[T(2j+4)+(j+\frac{5}{2})]\}}{[(j+\frac{3}{2})f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}$	
H_1+1	T	-1 0	$\frac{-[2T(T+1)(j+\frac{3}{2}-H_1-T)(j+\frac{5}{2}+H_1+T)]^{1/2}\{-H_1[T(2j+4)-(j+\frac{3}{2})]-(j+\frac{3}{2}-T)[T(2j+4)+(j+\frac{5}{2})]\}}{[(j+\frac{3}{2})f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}$	
H_1	$T-1$	0 1	$\frac{2\{(j+\frac{3}{2}-T)^2(j+\frac{5}{2}+T)[T(2j+4)+(j+\frac{5}{2})]-H_1^2[T^2(2j+4)+(j+\frac{3}{2})^2(2T+1)]\}[(T+1)]^{1/2}}{[(j+\frac{3}{2})(2T+1)f(H_1,T,j+1)f(-H_1,T_1,j+1)g(H_1,T,j+1)]^{1/2}}$	
H_1	$T+1$	0 1	$\frac{2(j+\frac{3}{2}-T)[T(2j+4)+(j+\frac{5}{2})][T(j+\frac{5}{2}+H_1+T)(j+\frac{5}{2}-H_1+T)(j+\frac{3}{2}+H_1-T)(j+\frac{3}{2}-H_1-T)]^{1/2}}{[(j+\frac{3}{2})(2T+1)f(H_1,T,j+1)f(-H_1,T,j+1)g(H_1,T,j+1)]^{1/2}}$	

[a] $f(H_1,T,j+1)=2\{H_1[T(2j+4)+(j+\frac{3}{2})]+(j+\frac{3}{2}-T)[T(2j+4)+(j+5/2)]\}$
$g(H_1,T,j+1)=\{(j+5/2)(j+\frac{3}{2}-T)(j+5/2+T)-(j+\frac{3}{2})H_1^2\}$.
[b] For $(\omega_1\omega_2)=(j+\frac{1}{2},1)$, $\kappa=0$, all *but* the coefficient $\langle(j+\frac{1}{2},0)H_1T;\,(10)01\|(j+\frac{1}{2},1)\kappa=0,H_1T\rangle=-1$ are zero.

of the pairing interaction. Figures 2, 3, 4, and 5 show the exact results for the case where the single particle spectrum is that of a $j=\frac{7}{2}$ level whose degeneracy has been removed by a quadrupole field. Figures 2, 3, and 4 show the spectra for eight nucleons, the half-filled system. Figure 2 shows the levels for states with individual level seniorities of zero, $v_j=0$ or $(\omega_1,\omega_2)_j=(10)$, for all four levels. These are the only states which contain the state of over-all seniority zero, $(\Omega 0)$. The lowest energy states for even T become the (40) states with over-all seniority $v=0$, $t=0$ in the limit of large pairing strength G. The next group of states must be compared with the so-called 2-quasiparticle states. Their energies are comparable to those for states with $\sum_j v_j=2$. One set of states of this type is shown in Fig. 3, those for which $v_j=1$ for both single-particle levels 2 and 3, while $v_j=0$ for $j=1$, and 4, the lowest and highest single-particle levels. There are five other sets of states of this type; those with $v_j=1$ for $j=2$ and 4, 3 and 4, 1 and 2, 1 and 3, 1 and 4, all with comparable energies, particularly in the large G limit, but none of which are shown in the figures. Another class of 2-quasiparticle states are those with $v_j=2$ for one level, $(\omega_1,\omega_2)_j=(00)$. Examples are shown in Fig. 4 for the cases $v_3=2$ and $v_2=2$. Finally, Fig. 5 shows the states with individual level seniorities of zero, all $v_j=0$, for the case of six nucleons. The lowest energy state for a given T is separated from the next group of states by an appreciable "energy gap" in general only for the states with $\sum_j v_j=0$ and with natural isospin, that is, even T for an even number of pairs, (Fig. 2), and odd T for an odd number of pairs, (Fig. 5). For eight nucleons, (Fig. 2), the lowest $T=0$ and 2 states, for example, lie appreciably below the next group of states. The lowest $T=1$ eight-nucleon state on the other hand forms a member of a triplet, particularly in the limit of large G. For the half-filled system, $N=8$, strong-pairing perturbation theory predicts a simple $T(T+1)$ dependence for the low-lying $v=0$ states of natural isospin. The

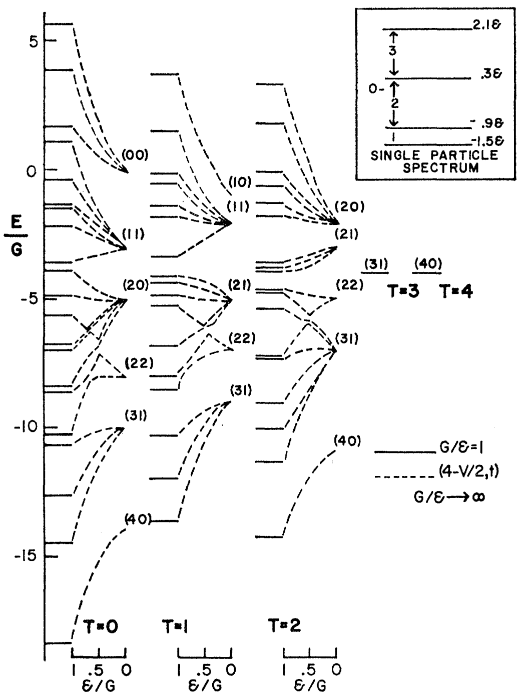

FIG. 2. Pairing-energy spectrum for eight nucleons distributed over a single-particle spectrum of four Nilsson or "$j=\frac{1}{2}$-like" levels ($\Omega=4$, $N=8$). Individual level seniorities are zero, $v_j=0$ or $(\omega_1,\omega_2)_j=(10)$, for all four levels. The single-particle Nilsson spectrum, shown in the insert, is that of a $j=\frac{7}{2}$ level whose m degeneracy has been removed by a quadrupole field. The solid lines show the spectrum for the case $\epsilon=G$. The dotted lines show the energies as a function of ϵ/G in the limit in which $G\rightarrow\infty$. Energies are plotted in units of G. In the strong-pairing limit ($\epsilon/G=0$), the states can be characterized by the degenerate level quantum numbers $(4-\frac{1}{2}v,\ t)$, the R_5 irreducible representations for a $j=\frac{7}{2}$ level.

ratio of the energy differences $[E(T=4)-E(T=0)]/[E(T=2)-E(T=0)]$ should thus be 10/3 in the limit of large G. For the value $\epsilon/G=0.5$ this ratio is found to be 3.36 for states with the single-particle spectrum shown in Fig. 2. Even for $\epsilon/G=1$ this ratio has the value 3.48, suggesting that the main dependence on isospin T may be given by the simple $T(T+1)$ form.

In order to see how far the perturbation-theory results might be used as a guide to the N and T dependence of the energy expressions, exact, and second-order per-

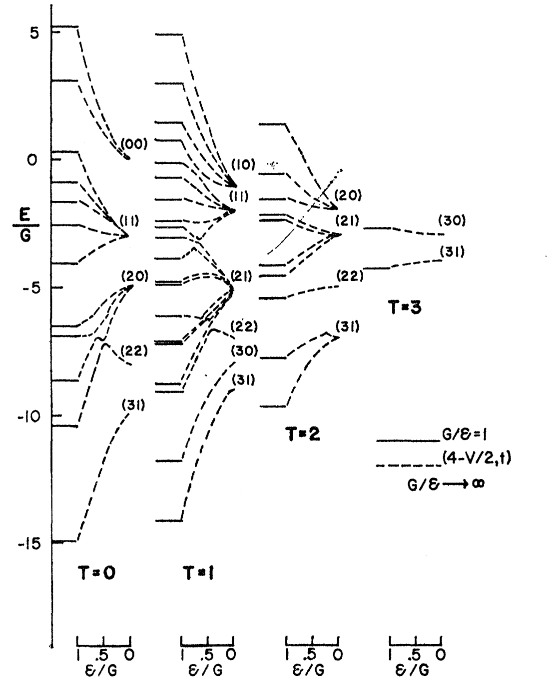

FIG. 3. Pairing-energy spectrum for $N=8$, $\Omega=4$, $\Sigma_j v_j=2$; eight nucleons distributed over the single-particle spectrum of four Nilsson levels shown in the insert to Fig. 2. The case shown is that with $v_j=1$, or $(\omega_1,\omega_2)_j=(\frac{1}{2}\frac{1}{2})$, for levels 2 and 3, while $v_j=0$, $(\omega_1,\omega_2)_j=(10)$, for levels 1 and 4, the lowest and highest single-particle levels.

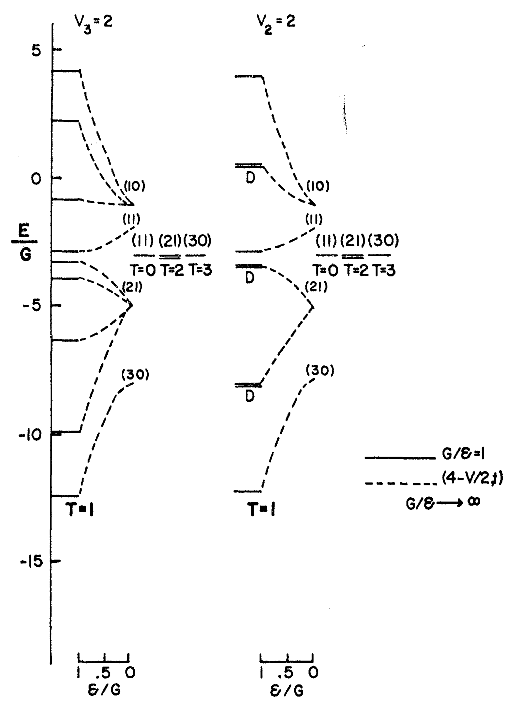

FIG. 4. Pairing-energy spectrum for $N=8$, $\Omega=4$, $\Sigma_j v_j=2$; eight nucleons distributed over the single-particle spectrum of four Nilsson levels shown in the insert to Fig. 2. The cases shown are those with $v_j=2$, $(\omega_1,\omega_2)_j=(00)$, for one level, while $v_j=0$ for the three remaining levels. Only the cases $v_3=2$ and $v_2=2$ are shown.

turbation theory results are compared in Figs. 6, 7, and 8 for the $T=0$, 1, and 2 levels, again, for the half-filled system of eight nucleons, this time for a single-particle spectrum of four equally spaced Nilsson or "$j=\frac{1}{2}$-like" states. With $N=8$ and four equally spaced levels, the system has a higher symmetry. Some of the states become doubly degenerate, and the strong-pairing perturbation calculations for states with over-all $v\neq 0$, $t\neq 0$ are somewhat simplified (see Appendix 2). The agreement between the exact solutions and the second-order perturbation-theory results, though not particularly good for some of the highly excited states, is quite striking for the lowest energy states with even T in both the weak and strong pairing limits, even for values of

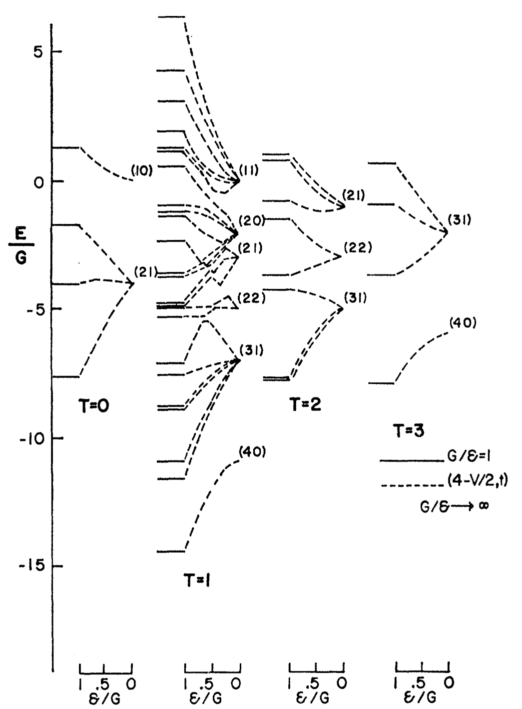

FIG. 5. Pairing-energy spectrum for $N=6$, $\Omega=4$, $\Sigma_j v_j=0$; the case of six nucleons distributed over the single-particle spectrum of four Nilsson levels shown in the insert to Fig. 2.

G/ϵ (ϵ/G) approaching unity. To gain some notion of the importance of the relative strengths of the pairing parameter G and the single-particle energy separation ϵ, the expansion coefficients of the ground-state $T=0$ wave function are plotted as a function of ϵ/G in Fig. 9. The expansion scheme is that of Eq. (11) based on the individual level occupation numbers N_j. In the weak-pairing limit the lowest $T=0$ state is that with levels 1 and 2 completely filled with four nucleons, ($N_1N_2N_3N_4=4400$). For $\epsilon/G=1$, the 4400 state still accounts for 45% of the wave function (relatively weak pairing); but by $\epsilon/G=0.5$, the strengths are shared more or less equally

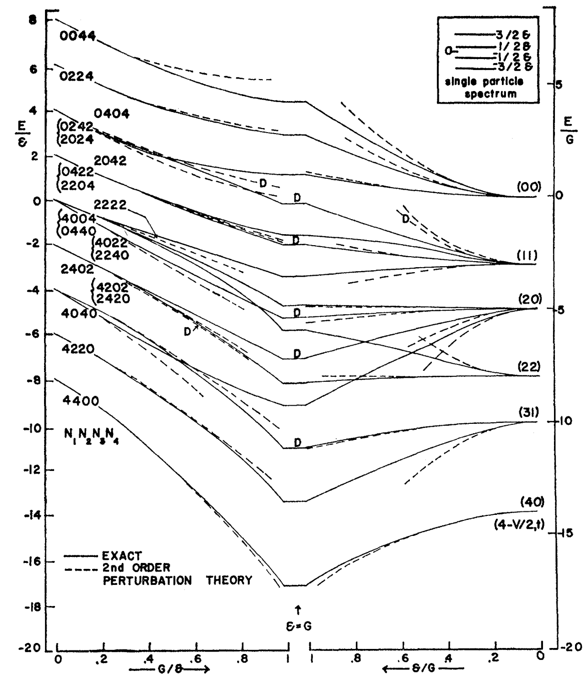

FIG. 6. Comparison between exact and perturbation theory results. The $T=0$ levels. The spectra are those for eight nucleons distributed over the single-particle spectrum of four equally spaced Nilsson-like levels, shown in the insert. The states are those with individual level seniorities of zero, all $v_j=0$. Some of the states, labeled D, are doubly degenerate as a result of the higher symmetry of the system of four equally spaced single-particle levels, with $N=8$. The energies for the weak-pairing limit, ($G/\epsilon<1$), are plotted as a function of G/ϵ on the left, with energies measured in units of ϵ. The energies for the strong-pairing limit, ($\epsilon/G<1$), are plotted as a function of ϵ/G on the right, with energies measured in units of G. The weak-pairing limiting quantum numbers shown are the occupation numbers of the four levels, $N_1N_2N_3N_4$. The strong-pairing limiting quantum numbers, the over-all seniority and reduced isospin for a degenerate level with $\Omega=4$, are indicated in terms of the R_5 representation labels $(4-\frac{1}{2}v,\, t)$.

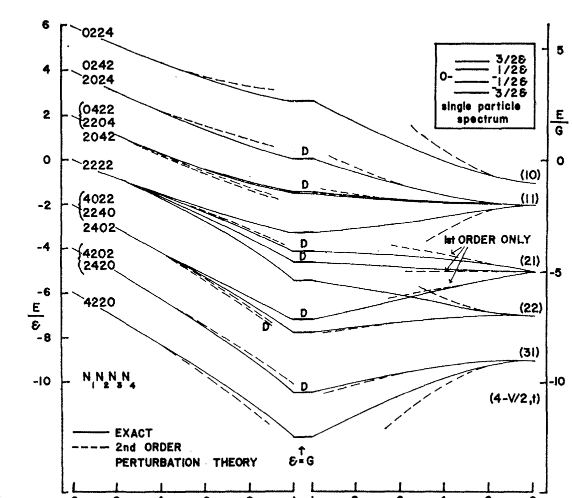

FIG. 7. Comparison between exact and perturbation-theory results. The $T=1$ levels. See caption for Fig. 6.

by all of the possible single-particle configurations, (strong pairing).

The good agreement between the perturbation theory and exact results for the correlated ground state may be due partly to the simplicity of the example chosen. However, the exact results do support the following conclusions about the correlated ground-state energies. For fixed T, the dependence on nucleon number N is similar to that for configurations of identical nucleons; while for fixed N, the T dependence is given mainly by a term of simple $T(T+1)$ form, a result which has recently also been predicted by Elliott and Lea.[8] Pairing in the J-T scheme is thus essentially quite simple. The J-T scheme sheds no light on the competition between pairing and "fouring" effects.[22] For this purpose studies in light nuclei, involving a generalized eight-dimensional quasi-spin[23] with its $T=0$ and $T=1$ interaction may be of greater interest.

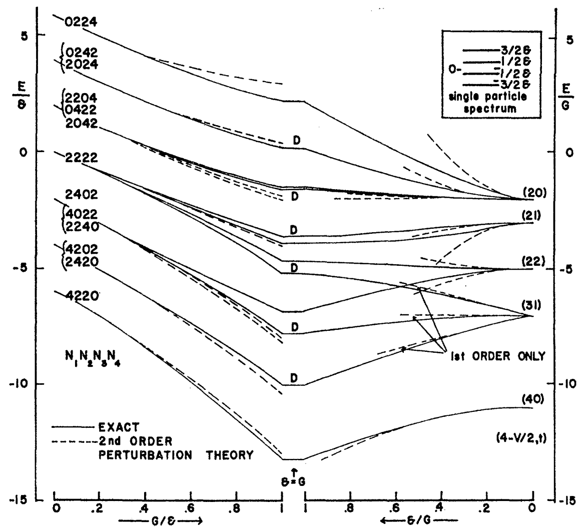

FIG. 8. Comparison between exact and perturbation-theory results. The $T=2$ levels. See caption for Fig. 6.

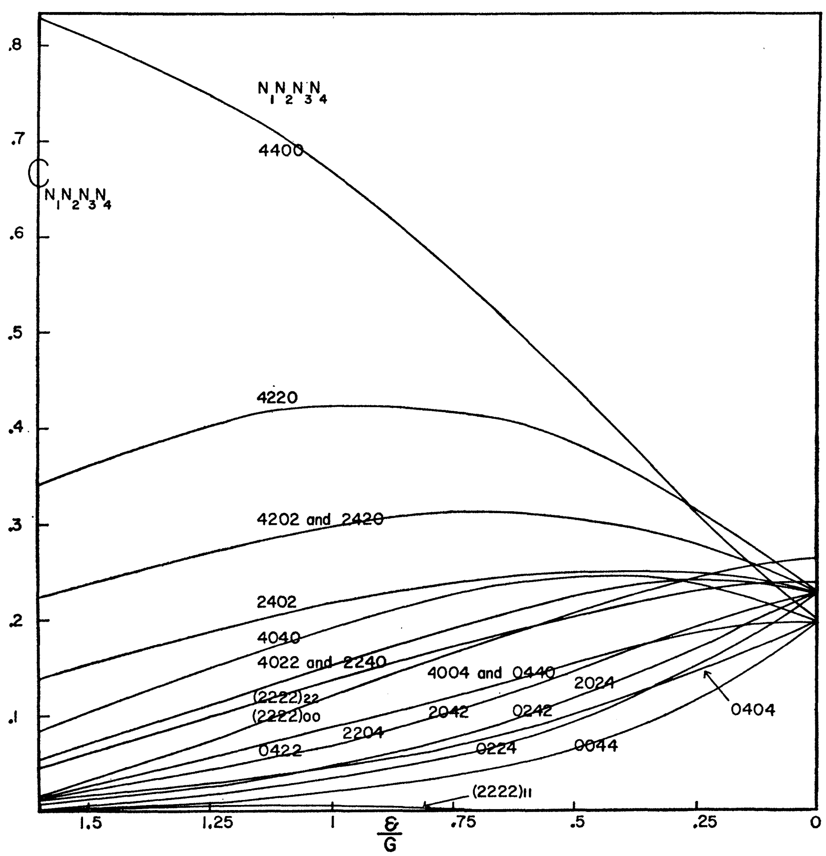

FIG. 9. Expansion coefficients for the $T=0$ ground state. The coefficients $C_{N_1N_2N_3N_4}$, plotted as a function of ϵ/G, are those for the lowest energy $T=0$ state of Fig. 6. The expansion is in terms of the scheme of Eq. (11). In place of the full set of individual level quantum numbers, $(\omega_1,\omega_2)_j=(10)$, κ_j, $H_{1j}=(1/2N_j-1)$, T_j, only the nucleon numbers N_j are indicated. In almost all cases, the quantum numbers $N_1N_2N_3N_4$ are sufficient to fully specify the states. When needed, the additional quantum numbers T_{12} and T_{34} are indicated by subscripts, $(N_1N_2N_3N_4)_{T_{12}T_{34}}$. In the limit $G\to 0$, $C_{4400}\to 1$.

ACKNOWLEDGMENTS

The major part of this work was carried out at the University of Sussex, England. The hospitality of the University of Sussex during a very pleasant summer is most gratefully acknowledged. The author would like especially to thank Dr. J. P. Elliott for many stimulating discussions and helpful comments. The author is also indebted to Dr. R. A. Leacock for programming and carrying out some of the machine computations at the University of Michigan Computing Center.

[22] B. H. Flowers, *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961* (Academic Press Inc., New York, 1961), p. 207.

[23] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) **84**, 673 (1964).

APPENDIX 1. TRANSFORMATION COEFFICIENTS FOR $t=1$ STATES

Specific algebraic expressions for the transformation coefficients from the $|S^nS^pM_{S^n}M_{S^p}\rangle$ to the $|T_pH_1TM_T\rangle$ scheme were calculated in Ref. 4 for irreducible representations with $t=0$ and $t=\frac{1}{2}$. (In Ref. 4 the quantum numbers J and Λ were used in place of S^n and S^p.) These calculations have now been extended to irreducible representations with $t=1$ by the general techniques outlined in Ref. 4. In particular, for the irreducible representation $(j-\frac{1}{2}, 1)$, and states with $M_T=T$, the transformation coefficients $\langle S^n(\mu)S^p(\mu)M_{S^n}(b)M_{S^p}(a,b) \times | T_p(a)H_1(a,b)T=M_T\rangle = C_\mu(a,b)$ are expressed through the parameters a, b, and μ, (all with integral values), where the parameters a and b are defined by

$$H_1=j-\tfrac{1}{2}-a-2b\,,\quad T_p=a\,.$$

The integer μ is a running index, $\mu=0, 1, 2, \cdots$

Case A. $(j-\frac{1}{2}, 1)$. $T_p=T$.

1. Coefficients with $S^n=\frac{1}{2}j-\frac{1}{4}-\mu$, $S^p=\frac{1}{2}j-\frac{1}{4}-\mu$.

$$C_\mu(a,b)=-(-1)^{b-\mu}\frac{[(2j+1)(j+\frac{3}{2})(j+\frac{1}{2}-2\mu)(2j+2-2b)]^{1/2}}{\mu!(j+\frac{1}{2}-\mu)!}\times\left[\frac{(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu)!(2j+1-2\mu)!(j-\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j-\frac{1}{2}-b-\mu)!}{(a+1)!(a-1)!(2a+2b+1)!(b-\mu)!(2j+3-2b)!(j-\frac{1}{2}-a-b-\mu)!(j-\frac{1}{2}-2\mu)(j+\frac{3}{2}-2\mu)}\right]^{1/2}.$$

2. Coefficients with $S^n=\frac{1}{2}j+\frac{3}{4}-\mu$, $S^p=\frac{1}{2}j-\frac{1}{4}-\mu$.

$$C_\mu(a,b)=-(-1)^{b-\mu}\frac{2[(2j+2-2b)]^{1/2}}{(\mu-1)!(j+\frac{1}{2}-\mu)!}\times\left[\frac{(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu-1)!(2j+2-2\mu)!(j-\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j+\frac{1}{2}-b-\mu)!}{(a+1)!(a-1)!(2a+2b+1)!(b+1-\mu)!(2j+3-2b)!(j-\frac{1}{2}-a-b-\mu)!(j+\frac{3}{2}-2\mu)}\right]^{1/2}.$$

3. Coefficients with $S^n=\frac{1}{2}j-\frac{1}{4}-\mu$, $S^p=\frac{1}{2}j+\frac{3}{4}-\mu$.

$$C_\mu(a,b)=(-1)^{b-\mu}\frac{2[(2j+2-2b)]^{1/2}}{(\mu-1)!(j+\frac{1}{2}-\mu)!}\times\left[\frac{(2a+1)!(a+b)!b!(a+b+1-\mu)!(2\mu-1)!(2j+2-2\mu)!(j-\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j-\frac{1}{2}-b-\mu)!}{(a+1)!(a-1)!(2a+2b+1)!(b-\mu)!(2j+3-2b)!(j+\frac{1}{2}-a-b-\mu)!(j+\frac{3}{2}-2\mu)}\right]^{1/2}.$$

Case B. $(j-\frac{1}{2}, 1)$ $T_p=T-1$.

1. Coefficients with $S^n=\frac{1}{2}j+\frac{1}{4}-\mu$, $S^p=\frac{1}{2}j+5/4-\mu$.

$$C_\mu(a,b)=(-1)^{b-\mu}\frac{(2\mu-1)[(2j+1)(2a+3)]^{1/2}}{a!(\mu-1)!(j+\frac{3}{2}-\mu)![(j+\frac{5}{2}-2\mu)]^{1/2}}\times\left[\frac{(2a+1)!(a+b)!b!(a+b+2-\mu)!(2\mu-2)!(2j+3-2\mu)!(j-\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j+\frac{1}{2}-b-\mu)!}{(2a+2b+1)!(b-\mu)!(2j+3-2b)!(j+\frac{1}{2}-a-b-\mu)![(2j+2)(2a+3)(j-\frac{1}{2}-a-b)+b]}\right]^{1/2}.$$

2. Coefficients with $S^n=\frac{1}{2}j+\frac{1}{4}-\mu$, $S^p=\frac{1}{2}j-\frac{3}{4}-\mu$.

$$C_\mu(a,b)=(-1)^{b-\mu}\frac{(2j+2-2\mu)[(2j+1)(2a+3)]^{1/2}}{a!\mu!(j+\frac{1}{2}-\mu)![j+\frac{1}{2}-2\mu]^{1/2}}\times\left[\frac{(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu)!(2j+1-2\mu)!(j-\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j+\frac{1}{2}-b-\mu)!}{(2a+2b+1)!(b-\mu)!(2j+3-2b)!(j-\frac{3}{2}-a-b-\mu)![(2j+2)(2a+3)(j-\frac{1}{2}-a-b)+b]}\right]^{1/2}.$$

3. Coefficients with $S^n=\frac{1}{2}j+\frac{1}{4}-\mu$, $S^p=\frac{1}{2}j+\frac{1}{4}-\mu$.

$$C_\mu(a,b)=-(-1)^{b-\mu}\frac{4[(j+\frac{3}{2})(2a+3)(j+\frac{3}{2}-2\mu)]^{1/2}}{a!(\mu-1)!(j+\frac{1}{2}-\mu)![(j+\frac{1}{2}-2\mu)(j+\frac{5}{2}-2\mu)]^{1/2}}$$
$$\times\left[\frac{(2a+1)!(a+b)!b!(a+b+1-\mu)!(2\mu-1)!(2j+2-2\mu)!(j-\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j+\frac{1}{2}-b-\mu)!}{(2a+2b+1)!(b-\mu)!(2j+3-2b)!(j-\frac{1}{2}-a-b-\mu)![(2j+2)(2a+3)(j-\frac{1}{2}-a-b)+b]}\right]^{1/2}.$$

Case C. $(j-\frac{1}{2},1)$ $T_p=T+1$.

1. Coefficients with $S^n=\frac{1}{2}j+5/4-\mu$, $S^p=\frac{1}{2}j+\frac{1}{4}-\mu$.

$$C_\mu(a,b)=-(-1)^{b-\mu}\frac{\{[1+(j+\frac{5}{2})b+(2j+2)a-ab]+\mu[(j-\frac{1}{2})-2b-(2j+1)a+2ab]\}}{a!(\mu-1)!(j+\frac{3}{2}-\mu)![(2a+1)(j+\frac{5}{2}-2\mu)]^{1/2}}$$
$$\times\left\{\frac{(2j+1)(2j+2-2b)(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu-2)!\times(2j+3-2\mu)!(j+\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j+\frac{1}{2}-b-\mu)!}{(2a+2b+1)!(b+2-\mu)!(2j+3-2b)!(j+\frac{1}{2}-a-b-\mu)!\times\{a[(2j^2+2j-\frac{1}{2})-b(2j+2)]-(j+\frac{1}{2})(j+\frac{1}{2}-b)\}}\right\}^{1/2}.$$

2. Coefficients with $S^n=\frac{1}{2}j-\frac{3}{4}-\mu$, $S^p=\frac{1}{2}j+\frac{1}{4}-\mu$.

$$C_\mu(a,b)=(-1)^{b-\mu}\frac{\{[a[(2j^2+2j-\frac{1}{2})-b(2j+2)]-(j+\frac{1}{2})(j+\frac{1}{2}-b)]+\mu[(j-\frac{1}{2})-2b-(2j+1)a+2ab]\}}{a!\mu!(j+\frac{1}{2}-\mu)![(2a+1)(j+\frac{1}{2}-2\mu)]^{1/2}}$$
$$\times\left\{\frac{(2j+1)(2j+2-2b)(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu)!\times(2j+1-2\mu)!(j+\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j-\frac{3}{2}-b-\mu)!}{(2a+2b+1)!(b-\mu)!(2j+3-2b)!(j+\frac{1}{2}-a-b-\mu)!\times\{a[(2j^2+2j-\frac{1}{2})-b(2j+2)]-(j+\frac{1}{2})(j+\frac{1}{2}-b)\}}\right\}^{1/2}.$$

3. Coefficients with $S^n=\frac{1}{2}j+\frac{1}{4}-\mu$, $S^p=\frac{1}{2}j+\frac{1}{4}-\mu$.

$$C_\mu(a,b)=(-1)^{b-\mu}\frac{2\{a(2j-1-2b)-(j+\frac{1}{2})\}[(2j+2-2b)(j+\frac{3}{2}-2\mu)(j+\frac{3}{2})]^{1/2}}{a!(\mu-1)!(j+\frac{1}{2}-\mu)![(2a+1)(j+\frac{1}{2}-2\mu)(j+\frac{5}{2}-2\mu)]^{1/2}}$$
$$\times\left\{\frac{(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu-1)!(2j+2-2\mu)!(j+\frac{1}{2}-a-b)!(j+\frac{3}{2}-b)!(j-\frac{1}{2}-b-\mu)!}{(2a+2b+1)!(b+1-\mu)!(2j+3-2b)!(j+\frac{1}{2}-a-b-\mu)!\times\{a[(2j^2+2j-\frac{1}{2})-b(2j+2)]-(j+\frac{1}{2})(j+\frac{1}{2}-b)\}}\right\}^{1/2}.$$

APPENDIX 2. STRONG-PAIRING PERTURBATION THEORY FOR $\Omega=4$, $N=8$

The strong-pairing perturbation technique outlined in Sec. 4 has been applied to calculate second-order perturbation formulas for all of the states $(\Omega-\frac{1}{2}v_\Omega, t_\Omega)$ in the special case $\Omega=4$; individual level seniorities of zero, that is $(\omega_1\omega_2)_j=(10)$, $j=1, \cdots 4$; and $H_1=0$ (the case of eight nucleons needed for Figs. 6, 7, and 8).

For small values of Ω it is actually not necesssy to calculate the c.f.p. of Eq. (28) since it is easy to express the strong-pairing eigenfunctions $\psi([f_\Omega]\{\rho_\Omega,\rho_{\Omega-1},\cdots\rho_1\}(\Omega-\frac{1}{2}v_\Omega, t_\Omega)\kappa H_1TM_T)$ in terms of linear combinations of the weak pairing eigenfunctions of Eq. (11).

For the $T=0$ states, the results are

$$\psi([4]\{1111\}(40)H_1=T=0)=(1/3\sqrt{7})((\sqrt{15})\psi_1([4]\{1111\})+2(\sqrt{10})\psi_2([4]\{1111\})+2\sqrt{2}\psi_3([4]\{1111\})),$$
$$\psi([4]\{1111\}(20)H_1=T=0)=(1/3\sqrt{5})(2(\sqrt{6})\psi_1([4]\{1111\})-\psi_2([4]\{1111\})-2(\sqrt{5})\psi_3([4]\{1111\})),$$
$$\psi([4]\{1111\}(00)H_1=T=0)=(1/\sqrt{35})(2\sqrt{2}\psi_1([4]\{1111\})-2\sqrt{3}\psi_2([4]\{1111\})+(\sqrt{15})\psi_3([4]\{1111\})),$$
$$\psi_a([31]\{2111\}(31)H_1=T=0)=\psi_3([31]\{2111\}),$$
$$\psi_b([31]\{2111\}(31)H_1=T=0)=(1/\sqrt{7})(\sqrt{3}\psi_1([31]\{2111\})+2\psi_2([31]\{2111\})),\qquad\text{(A2.1)}$$
$$\psi([31]\{2111\}(11)H_1=T=0)=(1/\sqrt{7})(2\psi_1([31]\{2111\})-\sqrt{3}\psi_2([31]\{2111\})),$$
$$\psi([211]\{3211\}(11)H_1=T=0)=\psi([211]\{3211\}),$$
$$\psi([22]\{2211\}(22)H_1=T=0)=(1/\sqrt{6})(\sqrt{3}\psi_1([22]\{2211\})-\sqrt{2}\psi_2([22]\{2211\})-\psi_3([22]\{2211\})),$$
$$\psi([22]\{2211\}(20)H_1=T=0)=(1/\sqrt{15})((\sqrt{6})\psi_1([22]\{2211\})+\psi_2([22]\{2211\})+2\sqrt{2}\psi_3([22]\{2211\})),$$
$$\psi([22]\{2211\}(00)H_1=T=0)=(1/\sqrt{10})(\psi_1([22]\{2211\})+(\sqrt{6})\psi_2([22]\{2211\})-\sqrt{3}\psi_3([22]\{2211\})),$$

where functions $\psi_k([f_\Omega]\{\rho_4\rho_3\rho_2\rho_1\})$ of definite permutation symmetry have been constructed from single functions $\psi(N_1N_2N_3N_4)$, such as $\psi(4400)$ or $\psi(4220)$, by the Young-Yamanouchi techniques.[19,24]

$$\psi_1([4]\{1111\})=(1/\sqrt{6})(\psi(4400)+\psi(4040)+\psi(4004)+\psi(0440)+\psi(0404)+\psi(0044)),$$
$$\psi_2([4]\{1111\})=(1/\sqrt{12})(\psi(4022)+\psi(0422)+\psi(4202)+\psi(0242)+\psi(4220)+\psi(0224)$$
$$+\psi(2402)+\psi(2042)+\psi(2420)+\psi(2024)+\psi(2240)+\psi(2204)),$$
$$\psi_3([4]\{1111\})=(\sqrt{5}/3)\psi(2222;T_{12}=T_{34}=0)+\tfrac{2}{3}\psi(2222;T_{12}=T_{34}=2),$$
$$\psi_1([31]\{2111\})=(1/\sqrt{6})(\psi(4400)+\psi(4040)-\psi(4004)+\psi(0440)-\psi(0404)-\psi(0044)),$$
$$\psi_2([31]\{2111\})=(1/\sqrt{6})(\psi(4220)-\psi(0224)+\psi(2420)-\psi(2024)+\psi(2240)-\psi(2204)),$$
$$\psi_3([31]\{2111\})=(1/\sqrt{12})(\psi(4022)+\psi(0422)+\psi(4202)+\psi(0242)-\psi(4220)-\psi(0224)\qquad\text{(A2.2)}$$
$$+\psi(2402)+\psi(2042)-\psi(2420)-\psi(2024)-\psi(2240)-\psi(2204)),$$
$$\psi([211]\{3211\})=\tfrac{1}{4}(\psi(4202)-\psi(0242)-\psi(4220)+\psi(0224)+\psi(2402)-\psi(2042)$$
$$-\psi(2420)+\psi(2024)+2\psi(2240)-2\psi(2204)),$$
$$\psi_1([22]\{2211\})=(1/2\sqrt{3})(2\psi(4400)-\psi(4040)-\psi(4004)-\psi(0440)-\psi(0404)+2\psi(0044)),$$
$$\psi_2([22]\{2211\})=(1/2\sqrt{6})(2\psi(4022)+2\psi(0422)-\psi(4202)-\psi(0242)-\psi(4220)-\psi(0224)$$
$$-\psi(2402)-\psi(2042)-\psi(2420)-\psi(2024)+2\psi(2240)+2\psi(2204)),$$
$$\psi_3([22]\{2211\})=\tfrac{2}{3}\psi(2222;T_{12}=T_{34}=0)-(\sqrt{5}/3)\psi(2222;T_{12}=T_{34}=2).$$

In general, the weak pairing scheme of Eq. (11) is fully specified only by the quantum numbers $(\omega_1,\omega_2)_j$, κ_j, N_j, T_j; $j=1,\cdots 4$; and the additional quantum numbers T_{12}, T_{34}, T, M_T in the four-level case. For the representation $(\omega_1,\omega_2)_j=(10)$, however, the individual level quantum numbers N_j are in most cases sufficient to fully specify the state. In the state with $N_1N_2N_3N_4=4220$, for example, the quantum numbers T_j, T_{12}, T_{34} follow unambiguously from the values of N_j: $T_1=0$, $T_2=1$, $T_3=1$, $T_4=0$, $T_{12}=1$, $T_{34}=1$. The quantum numbers κ_j are not needed for $(\omega_1,\omega_2)_j=(10)$. In the few cases where needed the quantum numbers T_{12}, T_{34} are identified explicitly in Eqs. (A2.2).

[24] D. E. Rutherford, *Substitutional Analysis* (Edinburgh University Press, Edinburgh, Scotland, 1948).

Similarly, for the $T=2$ states

$$\begin{aligned}
&\psi([4]\{1111\}(40)H_1=0,\, T=2)=(\sqrt{7}/3)\psi_2([4]\{1111\})+(\sqrt{2}/3)\psi_4([4]\{1111\}),\\
&\psi([4]\{1111\}(20)H_1=0,\, T=2)=(\sqrt{2}/3)\psi_2([4]\{1111\})-(\sqrt{7}/3)\psi_4([4]\}1111\}),\\
&\psi_a([31]\{2111\}(31)H_1=0,\, T=2)=\psi_2([31]\{2111\}),\\
&\psi_b([31]\{2111\}(31)H_1=0,\, T=2)=(\sqrt{3}/\sqrt{5})\psi_3([31]\{2111\})+(\sqrt{2}/\sqrt{5})\psi_4([31]\{2111\}),\\
&\psi([31]\{2111\}(20)H_1=0,\, T=2)=(\sqrt{2}/\sqrt{5})\psi_3([31]\{2111\})-(\sqrt{3}/\sqrt{5})\psi_4([31]\{2111\}),\\
&\psi([211]\{3211\}(21)H_1=0,\, T=2)=\psi([211]\{3211\}),\\
&\psi([22]\{2211\}(22)H_1=0,\, T=2)=(1/\sqrt{3})\psi_2([22]\{2211\})+(\sqrt{2}/\sqrt{3})\psi_4([22]\{2211\}),\\
&\psi([22]\{2211\}(20)H_1=0,\, T=2)=(\sqrt{2}/\sqrt{3})\psi_2([22]\{2211\})-(1/\sqrt{3})\psi_4([22]\{2211\}),
\end{aligned} \tag{A2.3}$$

where all but the functions $\psi_4([f_\Omega]\{\rho_4\rho_3\rho_2\rho_1\})$ are defined as in Eqs. (A2.2) with the exception that functions $\psi(N_1N_2N_3N_4)$ now imply a coupling to an over-all T of 2 rather than 0. The functions $\psi_4([f_\Omega]\{\rho_4\rho_3\rho_2\rho_1\})$ are defined by

$$\begin{aligned}
&\psi_4([4]\{1111\})=(\sqrt{7}/3\sqrt{2})[\psi(2222;\,T_{12}=0,\,T_{34}=2)+\psi(2222;\,T_{12}=2,\,T_{34}=0)]+(\sqrt{2}/3)\psi(2222;\,T_{12}=T_{34}=2),\\
&\psi_4([31]\{2111\})=(1/\sqrt{6})[\psi(2222;\,T_{12}=0,\,T_{34}=2)-\psi(2222;\,T_{12}=2,\,T_{34}=0)]\\
&\qquad\qquad -(2/\sqrt{6})\psi(2222;\,T_{12}=2,\,T_{34}=1),\\
&\psi_4([22]\{2211\})=(1/3)[\psi(2222;\,T_{12}=0,\,T_{34}=2)+\psi(2222;\,T_{12}=2,\,T_{34}=0)]-(\sqrt{7}/3)\psi(2222;\,T_{12}=T_{34}=2).
\end{aligned} \tag{A2.4}$$

Finally, for the $T=1$ states,

$$\begin{aligned}
&\psi([1^4]\{4321\}(10)H_1=0,\, T=1)=\psi([1^4]\{4321\}),\\
&\psi([22]\{2211\}(22)H_1=0,\, T=1)=\psi([22]\{2211\}),\\
&\psi_a([211]\{3211\}(21)H_1=0,\, T=1)=(1/\sqrt{3})\psi_1([211]\{3211\})+(\sqrt{2}/\sqrt{3})\psi_3([211]\{3211\}),\\
&\psi_b([211]\{3211\}(21)H_1=0,\, T=1)=\psi_2([211]\{3211\}),\\
&\psi([211]\{3211\}(11)H_1=0,\, T=1)=(\sqrt{2}/\sqrt{3})\psi_1([211]\{3211\})-(1/\sqrt{3})\psi_3([211]\{3211\}),\\
&\psi([31]\{2111\}(31)H_1=0,\, T=1)=(\sqrt{5}/\sqrt{7})\psi_1([31]\{2111\})+(\sqrt{2}/\sqrt{7})\psi_2([31]\{2111\}),\\
&\psi([31]\{2111\}(11)H_1=0,\, T=1)=(\sqrt{2}/\sqrt{7})\psi_1([31]\{2111\})-(\sqrt{5}/\sqrt{7})\psi_2([31]\{2111\}),
\end{aligned} \tag{A2.5}$$

where the functions $\psi_k([f_\Omega]\{\rho_4\rho_3\rho_2\rho_1\})$ of Eqs. (A2.5) are defined by

$$\begin{aligned}
&\psi([1^4]\{4321\})=(1/\sqrt{12})(\psi(4022)-\psi(0422)-\psi(4202)+\psi(0242)+\psi(4220)-\psi(0224)\\
&\qquad +\psi(2402)-\psi(2042)-\psi(2420)+\psi(2024)+\psi(2240)-\psi(2204)),\\
&\psi([22]\{2211\})=(1/2\sqrt{2})(\psi(4202)-\psi(0242)+\psi(4220)-\psi(0224)+\psi(2402)-\psi(2042)+\psi(2420)-\psi(2024)),\\
&\psi_1([211]\{3211\})=\tfrac{1}{4}(2\psi(4022)+2\psi(0422)-\psi(4202)-\psi(0242)+\psi(4220)+\psi(0224)\\
&\qquad -\psi(2402)-\psi(2042)+\psi(2420)+\psi(2024)),\\
&\psi_2([211]\{3211\})=(1/2\sqrt{2})(-\psi(4202)+\psi(0242)+\psi(4220)-\psi(0224)-\psi(2402)+\psi(2042)+\psi(2420)-\psi(2024)),\\
&\psi_3([211]\{3211\})=(1/\sqrt{6})[\psi(2222;\,T_{12}=0,\,T_{34}=1)+(\sqrt{5})\psi(2222;\,T_{12}=2,\,T_{34}=1)],\\
&\psi_1([31]\{2111\})=(1/\sqrt{6})(\psi(4022)+\psi(0422)+\psi(4202)+\psi(0242)+\psi(2402)+\psi(2042)),\\
&\psi_2([31]\{2111\})=(1/3\sqrt{2})[\sqrt{5}\psi(2222;\,T_{12}=0,\,T_{34}=1)+2\sqrt{3}\psi(2222;\,T_{12}=T_{34}=2)+\psi(2222;\,T_{12}=2,\,T_{34}=1)].
\end{aligned} \tag{A2.6}$$

From Eqs. (A2.1) through (A2.6), functions with the remaining Yamanouchi symmetries could be constructed by using the transformation properties of the Yamanouchi functions[25] under the permutations $P(12)$, $P(23)$, and $P(34)$; but these additional functions are not needed, since the functions of Eqs. (A2.1) through (A2.6) are sufficient to calculate the reduced matrix elements of Eq. (26). The Wigner coefficients for the inner product of the symmetric group have been calculated by the technique developed by Hamermesh.[26]

[25] Reference 19, pp. 224–5.

[26] Reference 19, Secs. 7–14.

Specifically, Wigner coefficients for the products [31]×[31], [211]×[31], and [22]×[31] are the only nontrivial ones needed for the four-level calculations. With these, all matrix elements of the single particle part of the Hamiltonian can be calculated.

In the zeroth approximation of the strong pairing limit, the only nondegenerate states are the states of [4] symmetry, R_5 representations (40), (20), (00), and the state of [1^4] symmetry, R_5 representation (10). The remaining states are at least two or three-fold degenerate, so that degenerate perturbation theory must be used. For the special case $\Omega=4$, $N=8$ ($H_1=0$), almost all matrix elements of the single-particle part of the Hamiltonian are off-diagonal in the R_5 quantum numbers $(4-\frac{1}{2}v, t)$. The exceptions occur in those representations in which a given value of T occurs more than once for $H_1=0$, specifically for states with $T=1$ in the representation (21), and for states with $T=2$ in the representation (31). These are, therefore, the only states with first-order corrections to the energy. Neglecting the higher order contributions of the off-diagonal matrix elements, the first-order corrections to the $T=2$ states of representation (31), for example, are eigenvalues of a 6×6 matrix which can be split into two 3×3 matrices which are identical except for an over-all sign.

$$\begin{matrix} \mp\sqrt{2}\alpha/\sqrt{5} & \pm\beta/\sqrt{10} & \pm\gamma/\sqrt{10} \\ \pm\beta/\sqrt{10} & \pm(\alpha-\sqrt{2}\beta)/\sqrt{10} & \pm\gamma/\sqrt{5} \\ \pm\gamma/\sqrt{10} & \pm\gamma/\sqrt{5} & \pm(\alpha+\sqrt{2}\beta)/\sqrt{10}, \end{matrix} \tag{A2.7}$$

where

$$\alpha=(1/\sqrt{3})(\epsilon_1+\epsilon_2+\epsilon_3-3\epsilon_4),$$
$$\beta=(\sqrt{2}/\sqrt{3})(\epsilon_1+\epsilon_2-2\epsilon_3), \quad \gamma=\sqrt{2}(\epsilon_1-\epsilon_2). \tag{A2.8}$$

In the case of four equally spaced single-particle levels, (Fig. 8), both the matrices (A2.7) lead to the simple eigenvalues, 0, $\pm(\sqrt{6})\epsilon$.

For the remaining states, the single-particle part of the Hamiltonian can lead only to second-order corrections to the energy. These can be calculated by bringing the first-order off-diagonal corrections into the n-fold matrices on the diagonal by standard perturbation techniques. For the 3-fold degenerate $T=0$ states of representation (31), for example, this leads to the following 3×3 matrix, correct to order ϵ^2/G

$$\begin{matrix} \frac{1}{G}\left\{\frac{\alpha^2}{15}-\frac{32(\beta^2+\gamma^2)}{105}\right\} & -\frac{13}{105G}\{\alpha\beta+\sqrt{2}(\beta^2-\gamma^2)\} & -\frac{13}{105G}\{\alpha\gamma-2\sqrt{2}\beta\gamma\} \\ -\frac{13}{105G}\{\alpha\beta+\sqrt{2}(\beta^2-\gamma^2)\} & -\frac{1}{G}\left\{\frac{32\alpha^2+6\beta^2+19\gamma^2+26\sqrt{2}\alpha\beta}{105}\right\} & \frac{13}{105G}\{\beta\gamma+2\sqrt{2}\alpha\gamma\} \\ -\frac{13}{105G}\{\alpha\gamma-2\sqrt{2}\beta\gamma\} & \frac{13}{105G}\{\beta\gamma+2\sqrt{2}\alpha\gamma\} & -\frac{1}{G}\left\{\frac{32\alpha^2+19\beta^2+6\gamma^2-26\sqrt{2}\alpha\beta}{105}\right\}, \end{matrix} \tag{A2.9}$$

where α, β, and γ are defined in Eq. (A2.8). In the case of four equally spaced single-particle levels, this matrix can be split further and leads to the simple eigenvalues: $-8/7$, $-8/7$, $-60/7$, all in units of ϵ^2/G. Similar 2, 3, and 6-fold matrices for the remaining states can be split into smaller matrices in the case of four equally spaced single-particle levels. The results of the calculations are shown in Figs. 6, 7, and 8.