Calculation of Inelastic Alpha-Particle Scattering by Ni^{58}

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A distorted-wave calculation is presented for excitation of the 1.45-MeV 2^+ level of Ni^{58} by inelastic scattering of 43-MeV alpha particles. The collective one-phonon state is taken to be an eigenfunction of the pairing plus $Q\cdot Q$ potential, and the finite-range alpha-nucleon interaction is taken from (p,α) elastic-scattering analysis. It is shown how the collective enhancement and the surface-peaked interaction form factor arise. Qualitative agreement with experimental differential cross sections and with the predictions of the vibrational nuclear model are obtained, but quantitative agreement for the angular distributions is lacking. The cause of the disagreement is discussed. The calculated absolute cross sections are in good agreement with experimental data.

I. INTRODUCTION

BOTH elastic and inelastic alpha-particle scattering from nuclei at energies above the Coulomb barrier are characterized by sharp diffraction patterns. The angular distribution for elastic scattering was first obtained theoretically by Blair[1] using the sharp-cutoff model. Blair has also succeeded in explaining the angular distribution for inelastic alpha scattering,[2] in particular the phase rule, with the use of adiabatic approximation. Recent work of Blair, Sharp, and Wilets[3] using the adiabatic approximation of Bassel, Satchler, Drisko, and Rost[4] and Rost[5] using the distorted-wave Born approximation (DWBA) has successfully explained both angular distribution and magnitudes of cross sections. Buck[6,7] has shown that although the first-order DWBA is not adequate for describing transitions to two-phonon collective vibrational states, it is accurate enough for the one-phonon transitions provided that the distortion parameter β is not too large. Perey and Satchler[8] have shown that the range β for which the DWBA is valid is greater than the work of Buck[7] might lead one to believe.

All of these treatments of inelastic alpha scattering have relied on the use of a collective coordinate, either rotational or vibrational, to describe the initial and final states of the nucleus. It has recently become possible to describe the collective vibrational states as coherent combinations of independent particle motions.[9,10] Kisslinger[11] has formulated the inelastic scattering in terms of these states. He restricted the inelastic-scattering calculation to a plane-wave Born approximation (PWBA) with zero-range forces and a lower cutoff for the radial integrals. The bound-state radial wave functions were taken to be those of a harmonic oscillator. He found for excitation of the 1.45-MeV 2^+ state in Ni^{58} that satisfactory agreement could be obtained with the experimental angular distributions, with an appropriate choice for the cutoff radius. However, as Kisslinger points out, the coupling constant for the alpha particle-nucleon interaction which was required to fit the absolute cross section is unreasonably large; in fact his delta-function force has the same volume integral as a square well of radius 2F and depth 340 MeV. Analysis of the (p,α)[12] scattering data indicates that the alpha-nucleon potential has a depth of about 50 MeV and that the range is less than 2F. The use of a zero-range potential with volume integral equal to that of the realistic interaction potential would therefore lead to an absolute cross section low by about a factor of 50.

It is well known that for direct reactions involving strongly absorbed particles the angular distribution can be fairly well reproduced at angles less than 90° by a plane-wave Born approximation with a cutoff. The characteristic sharp diffraction pattern is reproduced provided that the interaction is confined to a narrow region of the surface. In the PWBA calculation the combination of cutoff integrals and the rapid falloff of the bound-state wave functions gives the required narrow-surface shell. It is therefore not surprising that Kisslinger was able to fit the angular distribution and

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[1] J. S. Blair, Phys. Rev. **95**, 1218 (1954).
[2] J. S. Blair, Phys. Rev. **115**, 928 (1959).
[3] J. S. Blair, D. Sharp, and L. Wilets, Phys. Rev. **125**, 1625 (1962).
[4] R. H. Bassel, G. R. Satchler, R. M. Drisko, and E. Rost, Phys. Rev. **128**, 2693 (1962).
[5] E. Rost, Phys. Rev. **128**, 2708 (1962).
[6] B. Buck, Phys. Rev. **127**, 940 (1962).
[7] B. Buck, Phys. Rev. **130**, 712 (1963).
[8] F. Perey and G. R. Satchler, Phys. Letters **5**, 212 (1963).
[9] M. Baranger, Phys. Rev. **120**, 957 (1960); T. Morumori, Progr. Theoret. Phys. (Kyoto) **24**, 331 (1960); D. J. Thouless, Nucl. Phys. **22**, 78 (1960); R. Arview and M. Veneroni, Compt. Rend. **250**, 992, 2155 (1960).
[10] S. Yoshida, Nucl. Phys. **38**, 380 (1962).
[11] L. S. Kisslinger, Phys. Rev. **129**, 1316 (1963).
[12] J. L. Gammel and R. M. Thaler, Phys. Rev. **109**, 2041 (1958).

that the fit was not sensitive to the nuclear-structure parameters.

It is interesting to see what sort of results can be obobtained with less drastic approximations. The purpose of this paper is to present results of a calculation in which distorted waves have been used for the alpha-particle continuum wave functions and the alpha-nucleon interaction has a finite range. The differential cross section is calculated for the excitation of the one-phonon 1.45-MeV state of Ni^{58} using Kisslinger's[11] nuclear-structure parameters.

From a purely theoretical point of view the use of the particle model for the nucleus is interesting in that it shows how in DWBA the enhancement of the cross section for excitation of a collective state arises and how a surface-peaked bound-state form factor occurs even though the projectile is allowed to interact throughout the nucleus.

The formalism developed by Kisslinger for treating one-phonon excitation is reviewed in an appendix. The result and its relation to collective enhancement is examined in Sec. II. In Sec. III, we discuss the question of the interaction between alpha particle and bound nucleons and of the binding parameters for the nucleons. In Sec. IV, we present the calculated collective cross section for excitation of the 1.45 MeV 2^+ state in Ni^{58} from the single-particle amplitudes.

II. THE COLLECTIVE FORM FACTOR

The differential cross section for inelastic scattering in DWBA is

$$d\sigma/d\Omega=(2m/4\pi\hbar^2)^2(k'/k)\,|A\,|^2, \tag{1}$$

where m is the reduced mass, $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final projectile wave vectors, and

$$A=\int d\mathbf{r}_\alpha x_{k'}{}^{(-)*}(\mathbf{r}_\alpha)K(\mathbf{r}_\alpha)x_k{}^{(+)}(\mathbf{r}_\alpha). \tag{2}$$

The form factor $K(\mathbf{r}_\alpha)$ is defined by the expression

$$K(\mathbf{r}_\alpha)=\langle\Phi_f(\xi)\,|\,V(\mathbf{r}_\alpha,\xi)\,|\,\Phi_i(\xi)\rangle, \tag{3}$$

where ξ represents all nuclear coordinates, and Φ_i and Φ_f are the initial and final states of the nucleus. The alpha-particle continuum functions $x_k{}^{(\pm)}$ are taken to be eigenfunctions of complex optical-model Hamiltonians which reproduce the elastic-scattering differential cross section. In the absence of Coulomb distortion they each approach asymptotically a plane wave of unit amplitude plus a spherical scattered wave.

The calculation of the form factor using approximate eigenfunctions of the pairing plus $Q\cdot Q$ or $O\cdot O$ Hamiltonian is included in the Appendix. The result is a linear combination of single-particle transition terms:

$$K(\mathbf{r}_\alpha)=[i^LY_L{}^M(\hat{r}_\alpha)]^*(\tfrac{1}{2}N_L/(2L+1)^{1/2})\times\sum_{jj'}\langle j'\|i^LY_L\|j\rangle^2u_{j'j}{}^2\frac{E_j+E_{j'}}{(E_j+E_{j'})^2-(\hbar\omega)^2}\times\langle j'n'l'\,|\,w_L\,|\,jnl\rangle\langle j'n'l'\,|\,v_L(r_\alpha,r_n)\,|\,jnl\rangle, \tag{4}$$

where N_L is a normalization factor, $u_{j'j}$ is the combination $U_jV_{j'}+V_jU_{j'}$ of occupation parameters, E_j is the j-level quasiparticle energy, the two matrix elements are single-particle radial integrals of the multipole operator $w_L=[(m\omega_0/\hbar)^{1/2}r]^L$ and the radial multipole of the alpha-nucleon interaction $v_L(r_\alpha,r_n)$, and $\hbar\omega$ is the energy of a particular eigenfunction of the pairing plus long-range interaction.

It is interesting to examine in some detail the terms of the form factor. For the one collective state which is depressed below all of the two-quasiparticle states,[10] the relation $\hbar\omega<E_j+E_{j'}$ holds for all jj'. Therefore, since $\langle j'\|i^LY_L\|j\rangle$ and $u_{jj'}$ are real, the sign of the $j'j$ term depends entirely on the signs for the two radial matrix elements of w_L and v_L. Now, since the function r^L weights the region of large radial distance, the sign of the first radial integral $\langle j'n'l'\,|\,w_L\,|\,jnl\rangle$ should be[13] $(-1)^{n+n'}$. The radial integral of the interaction potential,

$$\langle j'n'l'\,|\,v_L\,|\,jnl\rangle=\int dr\,r^2R_{n'j'l'}v_L(r_\alpha,r)R_{njl} \tag{5}$$

does not have a definite sign since it is a function of r_α. Nevertheless, for inelastic alpha scattering the important region in the calculation of the scattering amplitude is that of large r_α. Furthermore, since for a short-range force v_L is peaked at $r=r_\alpha$ [$v_L(r_\alpha,r)=\delta(r_\alpha-r)/r_\alpha r$ if $V(r_\alpha-r)$ is a delta-function interaction], it will be true that for large r_α the sign of Eq. (21) will again be $(-1)^{n+n'}$. This means that every term of the form factor is *positive* in the surface region and, therefore, that all the single particle contributions to the scattering amplitude add constructively.[14] Thus we expect a large cross section. Note that this argument no longer holds for excitation of those states other than the one state whose energy is lowered by the long range interaction, since for all the other states the energy $\hbar\omega$ is greater than the lowest two-quasiparticle energy $(E_j+E_{j'})_{\min}$. This means that for these other states the factor $E_j+E_{j'}/[(E_j+E_{j'})^2-(\hbar\omega)^2]$ will sometimes be positive and sometimes negative, and there will therefore be cancellations among the terms of the form factor. As Brown and Bolsterli[15] have shown for the case of dipole

[13] This phase is due to use of the convention that the bound state radial wave function is positive near the origin.

[14] The paper of Kisslinger (Ref. 11) contains an error in sign, which resulted in a reduction in cross section when the $1f_{7/2}$ shell was included.

[15] G. E. Brown and M. Bolsterli, Phys. Rev. Letters **3**, 472 (1959).

radiative transitions in nuclei, this destructive interference can lead to extremely small transition rates. The one state of lowest energy will have a large cross section for inelastic excitation and all the rest will have very small cross sections. The collective enhancement in radiative quadrupole and octupole transitions has been discussed by Yoshida.[10] The collective enhancement of inelastic scattering has been discussed by Yoshida[10] and by Pinkston and Satchler.[16]

It can also be easily seen that Eq. (4) leads to a surface-peaked form factor. We have shown that in the region of the surface every term jj' of the sum of Eq. (4) has a positive sign, so they all add constructively. Since the radial wave function for quantum numbers n and n' will in general differ in their oscillatory character at values of r_α in the interior, some terms will be positive and some negative, resulting in a destructive interference. The form factor $K(\mathbf{r}_\alpha)$ will therefore tend to be large at the surface and small in the interior.

III. THE ALPHA-NUCLEON INTERACTION AND BOUND NUCLEON PARAMETERS

No one form for the alpha-nucleon interaction is really applicable. Because of the momentum distribution of the target nucleons, a continuous distribution of relative kinetic energies of the alpha particle and struck nucleon must be considered. If the interaction were a purely static one this would present no difficulties for the DWBA calculation. That this is not the case is seen in Table I, which shows the interaction potential

TABLE I. Gammel-Thaler proton-alpha potential depths[a] versus energy.

E_{lab} (MeV)	V_+ (MeV)	V_- (MeV)
≤ 9.5	60.6	54.6
17	56.7	52.0
40	45.6	40.0

[a] V_+ and V_-, are, respectively, the strengths for even and odd orbital angular momentum.

depths obtained by Gammel and Thaler[12] by analysis of proton-alpha elastic scattering as a function of laboratory energy. The ratio of the potential depths for even and odd l is about constant, and the shapes of the even and odd terms in the potential are kept fixed. The effect on the interaction of increasing the energy is to lower its strength. The even and odd shape parameters used by Gammel and Thaler are actually very nearly the same. We therefore expect that a simple Wigner potential is adequate for our purposes. Figure 1 shows such a potential used by Gammel and Thaler to fit the scattering data for $E \leq 9.5$ MeV along with an approximation to it consisting of two Yukawa terms:

$$V(r) = (871/r)(e^{-1.09r} - e^{-1.40r}) \text{ MeV}. \quad (6)$$

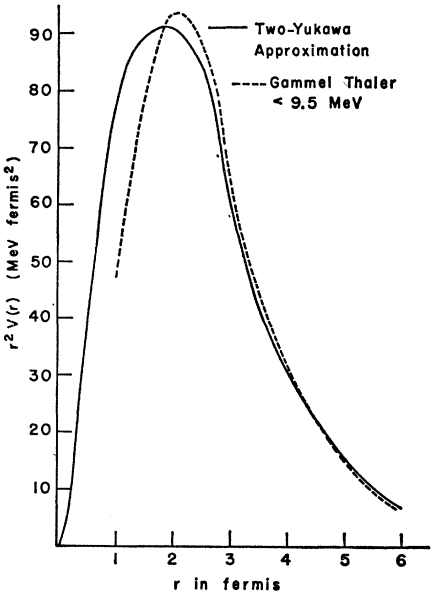

FIG. 1. A comparison of optical-model potentials used to fit proton-alpha elastic-scattering data with a potential made up of the sum of two Yukawa terms. The latter was used for the alpha-nucleon interaction in the distorted-wave Born approximation calculation described in the text.

The agreement between the two curves is quite good in the important region $r > 2$ F.

We expect that aside from the question of magnitude the use of Eq. (6) is fairly realistic. We are also interested in absolute magnitude, but can give only a rough value because of the uncertainty in the strength of the interaction. We expect that Eq. (6) is an overestimate because of the reduction in interaction strength with energy from its value at $E \leq 9.5$ MeV.

In the above discussion we have ignored the fact that the phenomenological proton-alpha potentials are designed to represent on-the-energy-shell two body collision matrix elements. It is clear that off-the-energy-shell matrix elements are important[17] in the inelastic alpha scattering since, even in the absence of distortion, considerable scattering takes place at angles greater than the kinematically allowed 14.5° for free alpha-nucleon scattering. The error made in using the potentials is difficult to estimate. A further difficulty is that the potentials give a good representation of elastic scattering when treated exactly whereas in the DWBA they are used only in first order. Fortunately the Born matrix element of Eq. (6) is not a bad approximation to the elastic scattering.

The target nucleons are taken to be bound in a Saxon well of radius parameter $r_0 = 1.25$ F and diffuseness parameters $a = 0.65$ F. The single-particle levels included in the collective wave functions[11] are the $2p_{1/2}$, $1f_{5/2}$, $2p_{3/2}$, and $1f_{7/2}$ neutrons. All these levels are taken to be bound by the separation energy in Ni^{57}, 10.5 MeV. It

[16] W. T. Pinkston and G. R. Satchler, Nucl. Phys. **27**, 270 (1961).

[17] We are grateful to Dr. J. S. Blair for pointing this out.

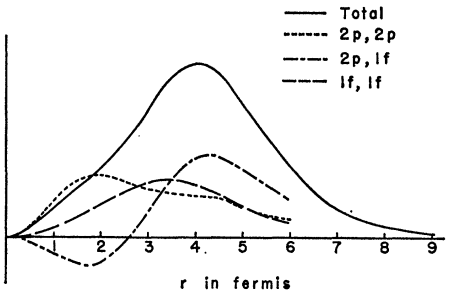

FIG. 2. The particle-model form factor shown with the contributions from the various bound orbital-angular-momentum transitions. The alpha-nucleon interaction used is the two-Yukawa approximation shown in Fig. 1.

would be preferable to have included the splittings of the various single particle levels, but we have found that the results are not very sensitive to small changes in the binding energy, so the error introduced will not be large. Assuming that the $1f_{7/2}$ level is 4 MeV lower than the $2p_{3/2}$, the largest error introduced in any of the single-particle amplitudes, that in the $1f_{7/2}$-$1f_{7/2}$ term, is only about 12%.

IV. RESULTS AND DISCUSSION

The form factor given by the radial part of Eq. (4) is shown in Fig. 2. Also shown are the contributions from the various terms which make up the total (different jj' terms with the same value of ll' are very similar and are lumped together). The destructive interference in the region $r \leq 2.5$ F and the constructive interference in the region $r \geq 2.5$ F is evident from the figure. It should be noted, however, that the alpha-particle optical radius, where the collective vibrational model form factor peaks, is at about 6 F, well beyond the peak of the collective particle form factor shown above.

Figure 3 shows the differential cross section for excitation of the 1.45 MeV one phonon state in Ni^{58} calculated in the DWBA using the D.R.C. code[18] with the form factor shown in Fig. 2 along with experimental points taken from the work of Broek, Braid, Yntema, and Zeidman.[19] The optical parameters are those given by Bassel *et al.*[4] As in the calculations[4] using the derivative Saxon form factor given by the collective vibrational model, a sharp diffraction pattern results from the use of the collective particle form factor Eq. (4). The agreement in magnitude of the calculated curve with the data is very good. Figure 4 shows the angular distribution calculated using the same particle form factor and a different set of optical parameters which give a fit to the elastic scattering data. The magnitude of the cross section is changed slightly and the angular distribution does not fall off so rapidly as in Fig. 3. Evidently the cross section is sensitive to

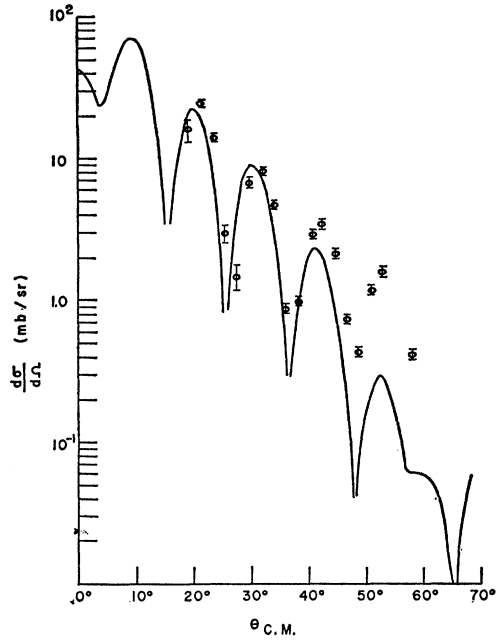

FIG. 3. Comparison with experiment of the DWBA differential cross section for excitation of the 1.45-MeV state of Ni^{58} using the particle-model form factor shown in Fig. 3. Volume absorption optical parameters are used with $V_0=47.6$ MeV, $W_0=13.8$ MeV, $R_0=6.1$ F, $a=0.549$ F.

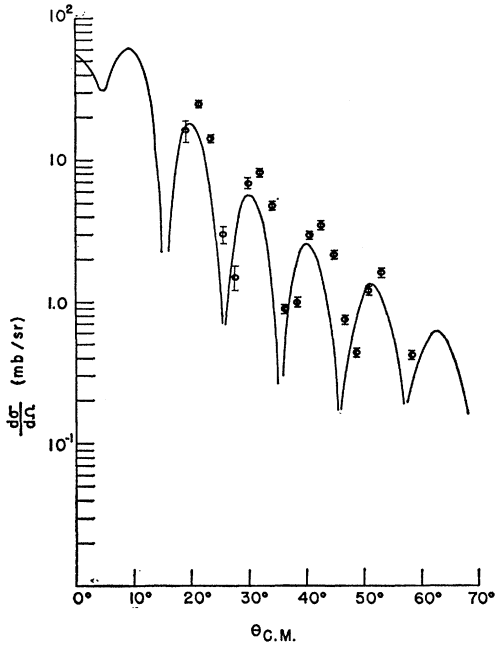

FIG. 4. Same comparison as in Fig. 3 using a different set of optical parameters $V_0=59.3$ MeV, $W_0=16.2$ MeV, $R_0=6.1$ F, $a=0.5$ F.

[18] W. R. Gibbs, V. A. Madsen, J. A. Miller, W. Tobocman, E. C. Cox, and L. Mowry, NASA Technical Note TN D-2170 (unpublished).

[19] H. W. Broek, T. H. Braid, J. L. Yntema, and B. Zeidman, Phys. Rev. **126**, 1514 (1962).

optical parameters. There is a persistent tendency of the calculated diffraction pattern to peak earlier than the data. In contrast to this result, angular distributions using the collective vibrational model but with identical optical parameters is in virtually perfect agreement with experiment for the first several peaks. Our angular distribution is characteristic of a larger object, which seems peculiar since the particle form factor peaks well inside the Saxon derivative form factor.

In order to understand the reason for the frequency discrepancy, we have studied the angular distribution as a function of nucleon binding parameters and found it not to be very sensitive for reasonable ranges of parameters.

We have also studied the diffraction pattern as a function of alpha-nucleon range using a Yukawa potential $V=V_0\exp(-\alpha r_{\alpha n})/(\alpha r_{\alpha n})$. Results are shown in Fig. 5 for the positions of the first two peaks as a function of the inverse range parameter α. It is clear that for a small enough range agreement can be obtained. One qualification must be made, however. The calculations using $\alpha>1$ are all made with a lower radial cutoff of 6.45 F. Figure 6 shows the cutoff and the noncutoff angular distributions for $\alpha=1.6$ along with the results from a calculation using the vibrational model surface interaction form factor (4). We see that when no cutoff is used the diffraction pattern is washed out and bears little resemblance to the surface interaction results. The reason is clear. The short-range potential enhances the contribution from inside the optical radius to such

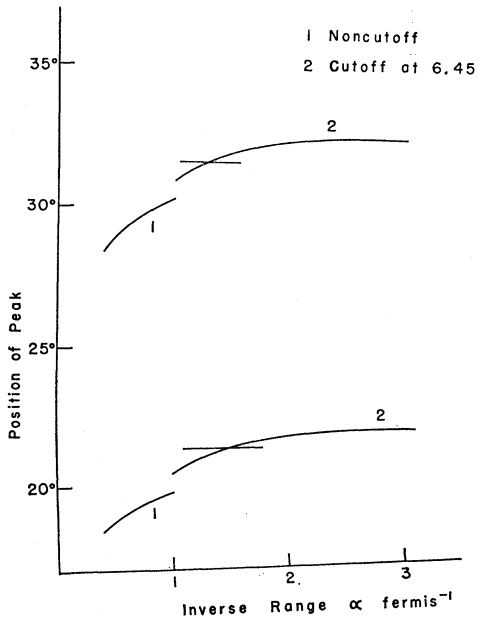

FIG. 5. Position of the 20° and 30° peaks in the diffraction pattern as a function of inverse range parameter α in the alpha-nucleon interaction. The straight lines show the experimental curve peak positions. The first part of each curve is computed using the full radial integrals and the second part with a lower cutoff at 6.45 F.

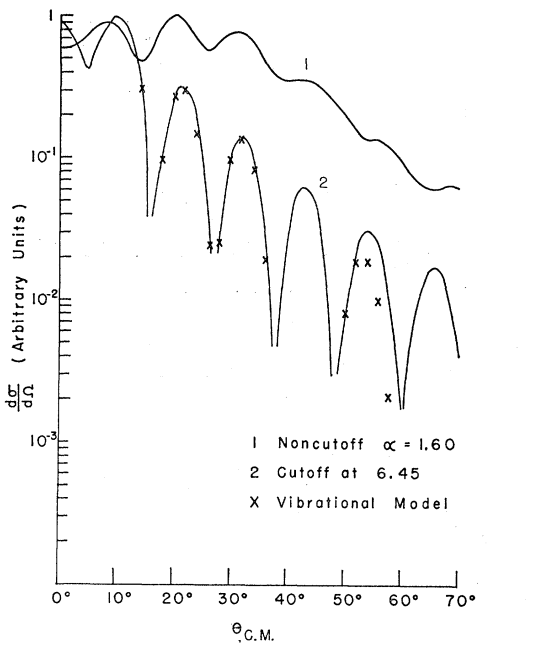

FIG. 6. A comparison of cutoff, noncutoff, and surface-interaction angular distributions for a short-range Yukawa alpha-nucleon potential, $\alpha=1.6$ F^{-1}.

an extent that a substantial fraction of the interaction takes place in the inner region in spite of strong absorption and phase cancellation in the interior. The interaction is no longer localized in the region just outside the optical radius but also occurs in the interior, and the sharp diffraction pattern characteristic of localization of the interaction is no longer present.

In view of this result it seems worthwhile to comment on the calculations of Bassel *et al.*,[4] in which the alpha inelastic angular distribution was calculated as a function of the form-factor position, all other parameters being left the same. They found for $Ni^{58}(\alpha,\alpha')$ very little difference in the diffraction pattern with the form factor peak radius at 7.14, 6.14, 5.14 F. However, if the form factor is moved in to 4.14 F, the diffraction pattern becomes washed out.

The characteristic of the particle model responsible for the frequency discrepancy when a reasonable alpha-nucleon range is used can now be understood. Figure 7 shows a comparison of the collective particle and collective vibrational model form factors. As noted before, the former peaks well inside the optical radius, but it also falls off much more slowly than the derivative Saxon form factor. Also shown is the large r part of the particle form factor using the Yukawa alpha-nucleon interaction with range parameter $\alpha=1.6$. This value of α gives about the right angular-distribution peak positions as can be seen in Fig. 6. It falls almost on top of the derivative Saxon form factor. It is clear from these results that aside from optical-model parameters the frequency of the diffraction pattern depends only on the

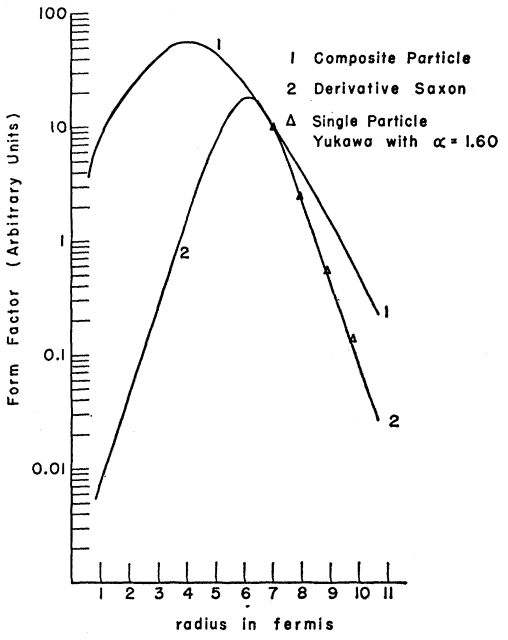

FIG. 7. A comparison of the collective-particle form factor calculated using the two-Yukawa alpha-nucleon interaction, Eq. (6), the derivative Saxon form factor and a particle form factor calculated using a Yukawa alpha-nucleon interaction of short range, $\alpha=1.6$. The surface form factor is normalized to give experimental absolute cross sections. The particle form-factor normalization results directly from the calculation described in the text. The short-range potential form factor is normalized to the surface result.

large r behavior of the form factor. The particle form factor is in disagreement with the derivative Saxon form factor at large r, and the latter gives angular distributions in better agreement with experiment.

A frequency discrepancy is also seen in the comparison of the calculations of Wall[20] on inelastic alpha excitation of the first 3^- state in Ca^{40} with the experimental results of Bauer *et al.*[21] A similar effect was seen in the analysis of 18 MeV inelastic proton scattering[22] using a pure Wigner potential of a Yukawa form which fits the triplet scattering length and effective range.

The extremely close agreement between the calculated absolute differential cross sections must be regarded as fortuitous because of the uncertainty in the alpha-nucleon interaction strength and because of the neglect of the $1f_{7/2}$ protons in the calculation of the wave function for the collective state.[11] The two effects tend to compensate. The inclusion of the protons would increase the number of constructively interfering terms in Eq. (4), and the alpha-nucleon interaction strength should probably be weaker than that given by the 10-MeV proton-alpha elastic-scattering data.

We emphasize that the calculations which we have presented include a considerable contribution from the closed $1f_{7/2}$ shell. If only the two valence nucleons are used in the vibrational wave function,[11] so the filled $1f_{7/2}$ shell is left intact, the cross sections are reduced compared to those shown in Figs. 3 and 4 by a factor of approximately 2.5. Contributions from the $1f_{7/2}$ part of the nuclear core are therefore essential to the description of the one-phonon wave function.

[20] N. S. Wall, ANL-6848, 1964, p. 108 (unpublished).
[21] R. W. Bauer, A. M. Bernstein, G. Heymann, E. P. Lippencott, and N. S. Wall, Phys. Letters 14, 129 (1965).
[22] S. F. Eccles, H. F. Lutz, V. A. Madsen (to be published).

V. SUMMARY

We have shown that a particle model of inelastic-alpha scattering with excitation of the one-phonon vibrational state using eigenfunctions of the pairing plus $Q\cdot Q$ potential and DWBA has many of the features of the phenomenological surface vibration model of the scattering. Constructive interference of the contributions from various single-particle transition amplitudes near the nuclear surface and destructive interference in the interior lead to a surface peaked form factor and to collective enhancement of the excitation cross section. Provided that a finite-range interaction with realistic parameters is used for the alpha nucleon interaction potential, the differential cross section shows the sharp diffraction pattern characteristic of both the collective vibrational model and experiment.

Yet the similarities are mostly qualitative. For a reasonable alpha-nucleon range the particle model form factor extends both farther into the nuclear interior and farther beyond the optical radius than the derivative Saxon form factor given by the collective vibrational model. The longer tail results in an angular distribution with too high a frequency.

In spite of the differences the particle-model calculation is able to reproduce the main features of the experimental differential cross section. The absolute cross section given by the model is in satisfactory agreement with the experiment.

Note added in proof. Since this manuscript was completed two more papers dealing with calculation of inelastic-scattering form factors on the basis of a nuclear particle model have appeared [Daphne Jackson, Phys. Letters **14**, 118 (1965) and N. K. Glendenning and M. Veneroni, Phys. Letters **14**, 228 (1965)].

ACKNOWLEDGMENTS

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APPENDIX

The form factor which we wish to evaluate is given in Eq. (3). In the following discussion of the nuclear states, the notation and several of the results of Yoshida will be used. No attempt is made to discuss the approximations used to obtain the nuclear wave function. The zero-phonon ground state and the one-phonon excited states are approximate eigenfunctions of the pairing

potential plus the long-range $Q\cdot Q$ and $O\cdot O$ potential. The one-phonon excited states are given by

$$\Psi_{\lambda\mu a}=Q_{\lambda\mu a}{}^{\dagger}\Psi_0, \tag{A1}$$

where

$$Q_{\lambda\mu a}{}^{\dagger}=\tfrac{1}{2}\sum_{j_1j_2}[\psi_{j_1j_2a}{}^{\lambda}A^{\dagger}(j_1j_2\lambda\mu) -(-1)^{\lambda-\mu}\phi_{j_1j_2a}{}^{\lambda}A(j_1j_2\lambda-\mu)], \tag{A2}$$

with

$$A^{\dagger}(j_1j_2\lambda\mu)=\sum_{m_1m_2}C(j_1j_2\lambda;m_1m_2\mu)\alpha_{j_1m_1}{}^{\dagger}\alpha_{j_2m_2}, \tag{A3}$$

and

$$\psi_{j_1j_2a}{}^{\lambda}=\tfrac{1}{2}N_{\lambda}\frac{\langle j'\|i^{\lambda}Y_{\lambda}w_{\lambda}\|j\rangle u_{j'j}}{(E_{j'}+E_j)-\hbar\omega_a}, \tag{A4}$$

$$\phi_{j_1j_2a}{}^{\lambda}=\tfrac{1}{2}N_{\lambda}\frac{\langle j'\|i^{\lambda}Y_{\lambda}w_{\lambda}\|j\rangle u_{j'j}}{(E_{j'}+E_j)+\hbar\omega_a}, \tag{A5}$$

$$N_{\lambda}=\left[\sum_{j'j}\frac{2\hbar\omega_a(E_j+E_{j'})\langle j'\|i^{\lambda}Y_{\lambda}w_{\lambda}\|j\rangle^2u_{j'j}{}^2}{(E_j+E_{j'})^2-(\hbar\omega_a)^2}\right]^{-1/2}, \tag{A6}$$

$$w_{\lambda}=[(m\omega_0/\hbar)^{1/2}r]^{\lambda}, \tag{A7}$$

$$u_{j'j}=U_jV_{j'}+V_jU_{j'}. \tag{A8}$$

Here E_j and $E_{j'}$ are quasiparticle energies measured from the quasiparticle vacuum and $\hbar\omega_a$ is the energy measured from the ground state of the excited collective state, a, obtained by diagonalizing the long-range potential. The quantities U_j and V_j and the operators α_{jm} are defined by the Bogoliubov transformation:

$$a_{jm}=U_j\alpha_{jm}+(-1)^{j-m}V_j\alpha_{j-m}{}^{\dagger}. \tag{A9}$$

The operator a_{jm} is the destruction operator for independent particle state j, m, and α_{jm} that of the corresponding quasiparticle operator.

In second quantized notation, we may write the interaction as

$$V(\mathbf{r}_{\alpha},\xi)=\sum_{\lambda\mu}\sum_{n}v_{\lambda}(r_{\alpha},r_n)Y_{\lambda}{}^{\mu*}(\hat{r}_{\alpha})Y_{\lambda}{}^{\mu}(\hat{r}_n) =\sum_{\lambda\mu}Y_{\lambda}{}^{\mu*}(\hat{r}_{\alpha})\sum_{jj'mm'}\langle j'm'|v_{\lambda}Y_{\lambda}{}^{\mu}|jm\rangle \times a_{j'm'}{}^{\dagger}a_{jm}. \tag{A10}$$

Applying the Boguliubov transformation, Eq. (11), results in the form

$$V=V_{11}+\sum_{\lambda\mu}Y_{\lambda}{}^{\mu*}(\hat{r}_{\alpha})\sum_{jj'mm'}\langle j'm'|v_{\lambda}Y_{\lambda}{}^{\mu}|jm\rangle \times(U_{j'}V_j(-1)^{j-m}\alpha_{j'm'}{}^{\dagger}\alpha_{j-m}{}^{\dagger} +V_{j'}U_j(-1)^{j'-m'}\alpha_{j'-m'}\alpha_{jm}), \tag{A11}$$

where V_{11} contains terms with one creation operator and one destruction operator, which will have zero matrix elements between the ground state and the quadrupole states, Eq. (A1). Using the definition Eq. (A3) we rewrite Eq. (A11) in terms of A and $A^{\dagger}$; summing over m, m' results in the expression

$$V-V_{11}=\sum_{\lambda\mu}Y_{\lambda}{}^{\mu*}(\hat{r}_{\alpha})\sum_{jj'}\langle j'\|v_{\lambda}Y_{\lambda}\|j\rangle(2\lambda+1)^{-1/2} \times[U_{j'}V_jA^{\dagger}(j'j\lambda\mu)+V_{j'}U_j(-1)^{j'-j-\mu} \times A(jj'\lambda-\mu)], \tag{A12}$$

or in terms of phonon operators, Eq. (A2)

$$V-V_{11}=\sum_{\lambda\mu a}Y_{\lambda}{}^{\mu*}(\hat{r}_{\alpha})\langle j'\|v_{\lambda}Y_{\lambda}\|j\rangle(2\lambda+1)^{-1/2} \times\sum_{jj'}(\psi_{j'ja}{}^{\lambda}U_{j'}V_j+\phi_{j'ja}{}^{\lambda}V_{j'}U_j)Q_{\lambda\mu a}{}^{\dagger} \tag{A13}$$

+(phonon destruction terms).

The operators $Q_{\lambda\mu a}$ satisfy approximately boson commutation relations (8).

$$[Q_{\lambda\mu a},Q_{\lambda'\mu'a'}{}^{\dagger}]=\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{aa'}. \tag{A14}$$

From the form Eq. (A13) of the interaction using Eq. (A14) one can carry out the evaluation of the form factor Eq. (3) for a final state LM very simply:

$$K(\mathbf{r}_{\alpha})=\langle Q_{LM}{}^{\dagger}\Psi_0|\sum_{\lambda\mu a}[i^{\lambda}Y_{\lambda}{}^{\mu}(\hat{r}_{\alpha})]^*\sum_{jj'}\langle j'\|i^{\lambda}v_{\lambda}Y_{\lambda}\|j\rangle \times(2\lambda+1)^{-\frac{1}{2}}\{\psi_{jj'a}{}^{\lambda}U_{j'}V_j+V_jU_{j'}(-1)^{\lambda+j'-j}\phi_{jj'a}\} \times Q_{\lambda\mu a}{}^{\dagger}|\Psi_0\rangle \tag{A15}$$

$$=\sum_{jj'}D_{j'j}[i^LY_L{}^M(\hat{r}_{\alpha})]^*\langle j'l'n'|v_L|jnl\rangle,$$

where $\langle j'l'|v_L|jl\rangle$ is the radial integral and

$$D_{j'j}=\frac{\langle j'\|i^LY_L\|j\rangle}{(2L+1)^{1/2}}[\psi_{j'j}{}^{L}+\phi_{j'j}{}^{L}]\frac{u_{j'j}}{2}. \tag{A16}$$

Aside from the coefficient $D_{j'j}$ each of the terms in the jj' sum is the form factor of a single particle transition, and the form factor $K(\mathbf{r}_{\alpha})$ is just a linear combination of them.

Inserting the amplitudes $\psi_{j'ja}{}^{\lambda}$ and $\phi_{j'ja}{}^{\lambda}$ from Eqs. (A4) and (A5) into Eq. (A15) gives finally the result:

$$K(\mathbf{r}_{\alpha})=[i^LY_L{}^M(\hat{r}_{\alpha})]^*\frac{\frac{1}{2}N_L}{(2L+1)^{1/2}}\sum_{jj'}\langle j'\|i^LY_L\|j\rangle^2u_{j'j}{}^2 \times\frac{E_j+E_{j'}}{(E_j+E_{j'})^2-(\hbar\omega_a)^2}\langle j'n'l'|w_L|jnl\rangle \times\langle j'n'l'|v_L(r_{\alpha},r_n)|jnl\rangle, \tag{4}$$

where the matrix elements are just radial integrals.