to the partial waves with $l>1$, thus manifesting themselves only in the angular distributions corresponding to the first excited state.

The form of the experimental angular correlations corresponding to the second excited state is well explained by the statistical model, confirming the conclusions drawn on the basis of the angular distributions about the preponderance of the compound-nucleus mechanism. The angular correlations corresponding to the first Mg^{26} excitation level come nearer to the predictions of the statistical model than did the angular distributions, suggesting that the interference effects are less prominent in the angular correlations. The greater deviations of the angular-correlation curves from the predictions of the statistical model at energies higher than 6 MeV could be caused by some contribution of the direct interaction. The probability for a D.I. contribution is also stronger, owing to the fact that at energies higher than 6 MeV the integrated inelastic cross section is smaller than that at energies lower than 6 MeV, which could correspond to a decrease of the C.N. weight.

The data in this work indicate that an important part in determining the shape of the angular distributions and angular correlations is played by the height of the excitation energy in the compound nucleus and by the value of the excitation energy of the residual nucleus level. The nearness to the predictions of the statistical model, found in many of the measured cases, is due to the excitation of a comparatively large number of levels in the Al^{27} compound nucleus.

On the other hand, the good agreement of the angular distributions and correlations corresponding to the second Mg^{26} excitation level with the predictions of the statistical model is due to the higher value of the excitation energy of the residual nucleus level, which leads to the decrease in importance of the interference terms. This fact was also observed in the angular distributions of the protons scattered on the first two P^{31} excited states measured in this laboratory.[14] While the angular distributions corresponding to the P^{31} first-excited state showed shape fluctuations at some energies, the angular distributions connected with the excitation of the second level kept their shape, in good agreement with the predictions of the statistical model.

In the case of the $Mg^{24}(p,p'\gamma)$ reaction theoretically analyzed by Sheldon,[6] the experimental data did not agree with the statistical theory, thus confirming the importance of the compound-nucleus excitation energy, which is in this case about 6 MeV lower than in the case of the $Mg^{26}(p,p'\gamma)$ reaction.

ACKNOWLEDGMENTS

The authors thank the cyclotron staff and I. Huştea for technical assistance.

[14] H. Hulubei, M. Ivaşcu, A. Berinde, I. Neamu, N. Scîntei, I. Francz, N. Martalogu and E. Marincu, Rev. Roumaine Phys. (to be published).

PHYSICAL REVIEW VOLUME 139, NUMBER 4B 23 AUGUST 1965

Electron Scattering From Nuclear Magnetic Moments

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(Received 14 April 1965)

The elastic scattering of high-energy electrons from the magnetic dipole and magnetic octupole moments of light nuclei is calculated using shell-model wave functions. The results of the calculation are compared with recent experimental results for Be^9 and B^{11} and the possibility of obtaining a value for the magnetic octupole moment from an analysis of these experiments is discussed.

I. INTRODUCTION

IT is well known that elastic scattering of high-energy electrons is quite useful in investigating the charge distribution of nuclei.[1] More recently the experiments of Rand *et al.*[2] have shown that one may obtain considerable information regarding the magnetic structure of nuclei by measuring the cross section for electron scattering at 180°. (One chooses a scattering angle of 180° since, from arguments based on time-reversal invariance and parity conservation, one can show that only odd magnetic-multipole moments contribute to the elastic-scattering cross section.[3]) This technique has been used previously to investigate inelastic scattering from various light nuclei[4] and some investigation

* Supported in part by the U. S. Office of Naval Research through Contract No. Nonr-225(67).

† Supported by the U. S. Air Force through Air Force Office of Scientific Research Contract No. 49(638)-1389.

[1] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

[2] R. E. Rand, R. Frosch, and M. R. Yearian, Phys. Rev. Letters **14**, 234 (1965).

[3] R. H. Pratt, J. D. Walecka and T. A. Griffy, Nucl. Phys. **64**, 677 (1965).

[4] W. C. Barber, Ann. Rev. Nucl. Sci. **12**, 1 (1962).

of the magnetic structure of the deuteron has been performed.[5]

One of the interesting results of the experiments of Rand *et al.* is that, for nuclei with spin $J\geq\frac{3}{2}$, the contribution to the elastic-scattering cross section from the magnetic-octupole moment is very important at large values of the momentum transfer. It appears then that one can obtain some information regarding the size of the magnetic octupole moments of light nuclei, in particular the $p_{3/2}$ nuclei, from the elastic scattering of high-energy electrons. This method complements the more usual method of obtaining nuclear magnetic octupole moments from an investigation of electronic hyperfine structure which is useful only in heavy nuclei.[6]

The purpose of the present paper is to present some simple calculations of the magnetic-moment structure of the $p_{3/2}$ nuclei. In particular we calculate the magnetic octupole moment and the magnetic dipole and octupole form factors for the two nuclei Be^9 and B^{11} which have been studied in detail by Rand *et al.*

II. CROSS SECTION FOR ELASTIC SCATTERING

The general form of the cross section for elastic electron scattering from a nucleus with charge Z and spin J, assuming that only one virtual photon is exchanged (Born approximation), has been given by Pratt *et al.*[3]

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{\rm Mott}\times\Bigg\{Z^2|F_c(q^2)|^2+\frac{q^4}{180}\frac{(J+1)(2J+3)}{J(2J-1)}Q^2|F_Q(q^2)|^2+\frac{q^2}{4M^2}\left(1+2\tan^2\frac{\theta}{2}\right)\left[\frac{1}{3}\left(\frac{J+1}{J}\right)\mu^2|F_{M1}(q^2)|^2+\frac{2q^4}{4725}\frac{(J+1)(J+2)(2J+3)}{J(J-1)(2J-1)}\Omega^2|F_{M3}(q^2)|^2\right]+\text{higher multipoles}\Bigg\}. \quad (1)$$

The effect of nuclear recoil is neglected in writing Eq. (1). It is also assumed that the energy of the electrons is much greater than their rest mass ($E\gg m$).

The notation used in Eq. (1) is as follows: Z is the charge of the nucleus, Q the electric quadrupole moment in units of F^2, μ the magnetic dipole moment in units of nuclear magnetons and Ω the magnetic octupole moment in units of (nuclear magnetons)$\times F^2$. The form factors F_c, F_Q, F_{M1}, and F_{M3} describe the distribution of the charge, electric quadrupole moment, magnetic dipole moment and magnetic octupole moment, respectively. All form factors are normalized to unity at $q^2=0$ where q^2 is the square of the momentum

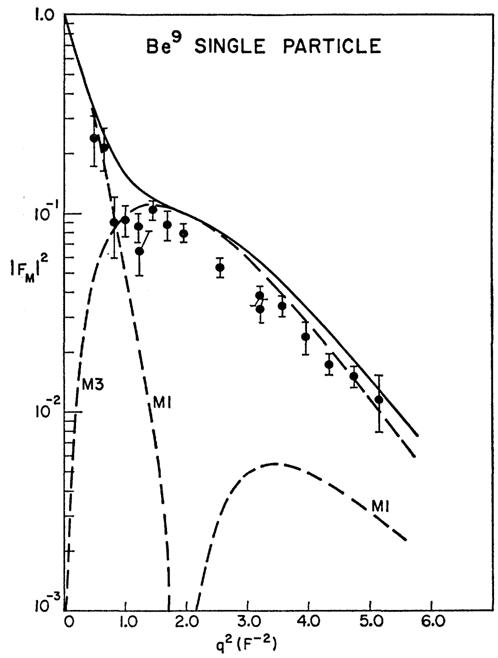

FIG. 1. Magnetic dipole ($M1$) and octupole ($M3$) form factors for Be^9 in single-particle shell model. The radial wave function is that of a harmonic oscillator with $\beta=0.39$ F^{-1}. The solid curve shows the combined form factor, $|F_M|^2$, of Eq. (6).

transfer from the electron to the nucleus; $q^2=4E^2\sin^2\theta/2$ where θ is the electron scattering angle and E the electron energy. The Mott cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Mott}=\frac{\alpha^2\cos^2(\theta/2)}{4E^2\sin^4(\theta/2)}. \quad (2)$$

Our units are such that $\hbar=c=1$ and $\alpha=1/137$.

The form factors are defined in terms of the reduced matrix elements of the electric and magnetic multipole operators by[3]

$$eZF_c(q^2)=(4\pi)^{1/2}\langle J\|T_0^e(q)\|J\rangle, \quad (3a)$$

$$\frac{e}{2M}\mu F_{M1}(q^2)=\left(\frac{4\pi}{3}\right)^{1/2}(JJ10|J1JJ)\langle J\|T_1^M(q)\|J\rangle, \quad (3b)$$

$$eQF_Q(q^2)=2\left(\frac{4\pi}{5}\right)^{1/2}(JJ20|J2JJ)\langle J\|T_2^e(q)\|J\rangle, \quad (3c)$$

$$\frac{e}{2M}\Omega F_{M3}(q^2)=-\left(\frac{4\pi}{7}\right)^{1/2}(JJ30|J3JJ)\langle J\|T_3^M(q)\|J\rangle \quad (3d)$$

where $(j_1m_1j_2m_2|j_1j_2JM)$ is the usual Clebsch-Gordan coefficient. The multipole operators are defined in terms of the charge $\rho(\mathbf{r})$, current $\mathbf{J}(\mathbf{r})$ and magnetization $\mathbf{M}(\mathbf{r})$, operators by

$$T_{\lambda\sigma}^e(q)=\frac{(2\lambda+1)!!}{q^\lambda}\int\rho(\mathbf{r})j_\lambda(qr)Y_\lambda^\sigma(\hat{r})d\mathbf{r}, \quad (4a)$$

$$T_{\lambda\sigma}^M(q)=\frac{(2\lambda+1)!!}{q^\lambda}\int\mathbf{L}[j_\lambda(qr)Y_\lambda^\sigma(\hat{r})]\cdot[\mathbf{J}(\mathbf{r})+\nabla\times\mathbf{M}(\mathbf{r})]d\mathbf{r}, \quad (4b)$$

[5] J. Goldemberg and C. Schaerf, Phys. Rev. Letters **12**, 298 (1964).
[6] C. Schwartz, Phys. Rev. **97**, 380 (1955).

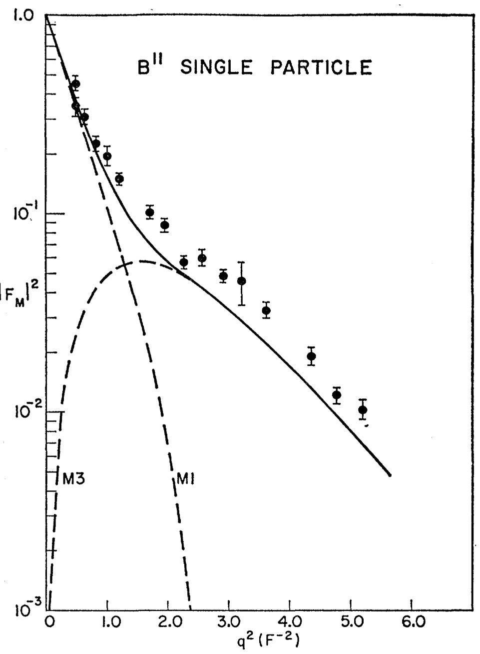

FIG. 2. Same as Fig. 1, except for B^{11} with $\beta=0.42$ F^{-1}.

where $\mathbf{L}=-i\mathbf{r}\times\nabla$, $j_\lambda(qr)$ is the usual spherical Bessel function, and Y_λ^σ are spherical harmonics.

From the Clebsch-Gordan coefficients appearing in Eqs. (3) one sees that to have a multipole contribution of order λ, the inequality $2J\geq\lambda$, must be satisfied, i.e., for a magnetic dipole contribution, $J\geq\frac{1}{2}$, for an electric quadrupole contribution, $J\geq 1$, etc.

We now specialize to the case of scattering at 180°. In this case only the magnetic terms contribute since $(d\sigma/d\Omega)_{\text{Mott}}\to 0$ while $[\tan^2(\theta/2)](d\sigma/d\Omega)_{\text{Mott}}$ is finite. From Eq. (1) and the expression for q^2 we obtain, for $J=\frac{3}{2}$ which is the case of interest here,

$$\frac{d\sigma}{d\Omega}(\theta=180°)=\frac{10}{9}\left(\frac{e}{2M}\right)^2\mu^2\left\{|F_{M1}(q^2)|^2+\frac{2q^4}{75}\left(\frac{\Omega}{\mu}\right)^2|F_{M3}(q^2)|^2\right\}. \quad (5)$$

It is convenient to define a magnetic form factor from Eq. (5) as

$$|F_M(q^2)|^2=|F_{M1}(q^2)|^2+\frac{2q^4}{75}\left(\frac{\Omega}{\mu}\right)^2|F_{M3}(q^2)|^2 \quad (6)$$

and to compare the results of the calculation with the experimental results for this form factor (which is normalized to unity at $q^2=0$) rather than with the measured cross section.

III. CALCULATIONS OF THE MAGNETIC FORM FACTORS

In this section we use some simple models for the two nuclei Be^9 and B^{11} to calculate the magnetic moments μ and Ω and the corresponding form factors $F_{M1}(q^2)$ and $F_{M3}(q^2)$.

TABLE I. Results of the calculation for the magnetic dipole and magnetic octupole moments for Be^9 and B^{11}. The experimental numbers given for the magnetic octupole moments were obtained by fitting the magnetic form factors as described in the text.

Model	μ (nm) Be^9	μ (nm) B^{11}	Ω (nm F^2) Be^9	Ω (nm F^2) B^{11}	(Ω/μ) (F^2) Be^9	(Ω/μ) (F^2) B^{11}
Single particle	−1.91	3.79	−7.30	10.1	3.82	2.67
L-S coupling	−1.5	3.4	−5.6	7.9	3.71	2.31
j-j coupling	−1.2	···	−5.1	···	4.28	···
Experiment	−1.18	2.69	−3.5	8.1	3.0	3.0

The charge, current, and magnetization operators which appear in Eqs. (4) we write as[7]

$$\rho(\mathbf{r})=|e|\sum_{j=1}^{A}\epsilon_j\delta(\mathbf{r}-\mathbf{r}_j),$$

$$\mathbf{J}(\mathbf{r})=\frac{|e|}{2M}\sum_{j=1}^{A}\{\epsilon_j\delta(\mathbf{r}-\mathbf{r}_j)\mathbf{P}_j\}_{\text{symm}}, \quad (7b)$$

$$\mathbf{M}(\mathbf{r})=\frac{e}{2M}\sum_{j=1}^{A}\gamma_j(\mathbf{r}-\mathbf{r}_j)\boldsymbol{\sigma}_j, \quad (7c)$$

where $\epsilon_j=1$ for a proton and 0 for a neutron, and $\gamma_j=2.79$ for a proton and -1.91 for a neutron. For the jth nucleon, $\mathbf{P}_j$ is the momentum operator and $\boldsymbol{\sigma}_j$ the spin operator.

The first (and simplest) model we consider is the strict single-particle model. We assume that all of the magnetic properties of the two nuclei are due to a single unpaired nucleon in the $p_{3/2}$ shell (a single proton for B^{11} and a single neutron for Be^9). The predicted

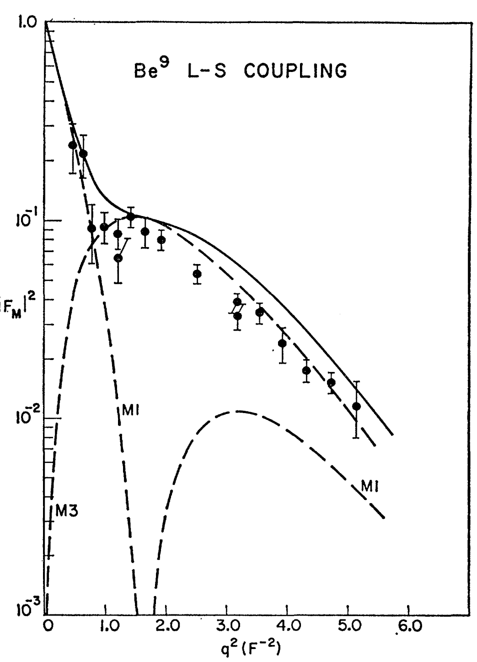

FIG. 3. Same as Fig. 1, except for *L-S* coupling of the p-shell nucleons.

[7] R. S. Willey, Nucl. Phys. 40, 529 (1963).

magnetic dipole moment is then simply the Schmidt value, 3.79 nm for B^{11} and -1.91 nm for Be^9. To calculate the form factors we use an expression given by Willey[7] for the reduced matrix elements of the magnetic multipole operators for a single nucleon in the state specified by j and l. For a $p_{3/2}$ nucleon the results are

$$\langle l=1,\ j=\tfrac{3}{2}\|T_1{}^M(q)\|l=1\ j=\tfrac{3}{2}\rangle,$$
$$=\frac{e}{2M}\left(\frac{5}{11}\right)^{1/2}\{(\langle j_0\rangle+\langle j_2\rangle)\epsilon_i+(\langle j_0\rangle-\tfrac{1}{5}\langle j_2\rangle)\gamma_i\},\quad (8a)$$

and

$$\langle l=1,\ j=\tfrac{3}{2}\|T_3{}^M(q)\|l=1,\ j=\tfrac{3}{2}\rangle=\mu_i\frac{7!!}{q^2}\frac{3}{\sqrt{\pi}}\langle j_2\rangle.\quad (8b)$$

The quantities $\langle j_\lambda\rangle$ which appear in Eq. (8) are radial integrals defined by

$$\langle j_\lambda\rangle=\int_0^\infty|R_l(r)|^2 j_\lambda(qr)r^2dr,\quad (9)$$

where $R_l(r)$ is the radial part of the single-particle wave function. The q dependence of the magnetic scattering is contained in the radial integrals, which are easily evaluated if one uses harmonic-oscillator wave functions to describe the single unpaired nucleon, e.g.,

$$\langle j_0\rangle=(1-\tfrac{2}{3}x)e^{-x}\quad (10a)$$

and

$$\langle j_2\rangle=\tfrac{2}{3}xe^{-x},\quad (10b)$$

where $x=q^2/4\beta$ and β is the strength parameter in the harmonic-oscillator wave function.

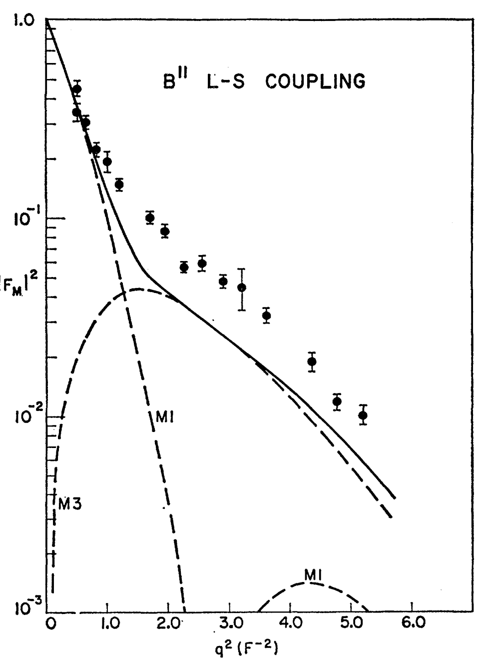

FIG. 4. Same as Fig. 2, except for L-S coupling.

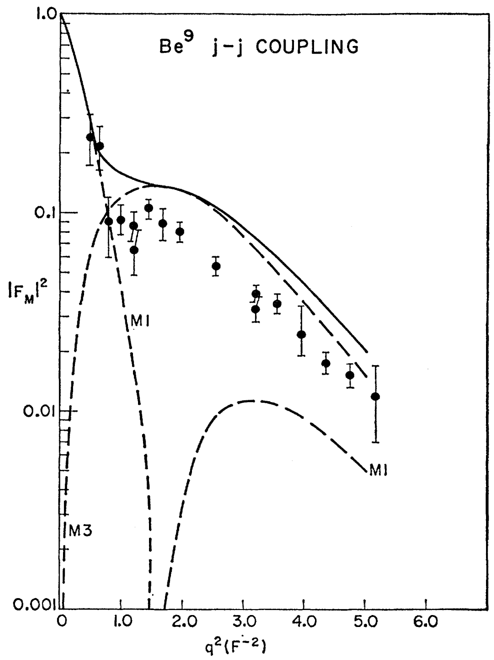

FIG. 5. Same as Fig. 1, except for j-j coupling.

In addition to the strict single-particle model we have evaluated the magnetic moments and form factors using L-S and j-j coupled shell-model wave functions. In both of these cases the reduced-matrix elements can be expressed in terms of the single-particle reduced-matrix elements by expanding the wave function using fractional parentage coefficients.[8] (For B^{11}, since there is only one parent, the results of the calculation for j-j coupling and the single particle model are the same.)

The results of the calculation are given in Table I, which gives the predicted values for the magnetic dipole and octupole moments, and Figs. 1 through 5 which show the calculated magnetic form factors compared with the experimental results of Rand *et al.*[9] The values of the harmonic-oscillator parameter β were taken from an analysis of the charge form factors and the values used were $\beta=0.39$ F^{-1} for Be^9 and $\beta=0.42$ F^{-1} for B^{11}.[10]

We have included in the form factors a correction for the finite size of the nucleons by multiplying the calculated reduced matrix elements by the nucleon form factor, which we took to be[11]

$$G(q^2)=\frac{1}{1+q^2/7.5}.\quad (11)$$

[8] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963).

[9] The data of Rand *et al.* are to be considered preliminary in that the final results may differ from these data by small amounts ($\lesssim 5\%$).

[10] U. Meyer-Berkhout, K. W. Ford, and A. E. S. Green, Ann. Phys. (N.Y.) **8**, 119 (1959).

[11] E. B. Hughes, T. A. Griffy, M. R. Yearian, and R. Hofstadter, Phys. Rev. **139**, B458 (1965).

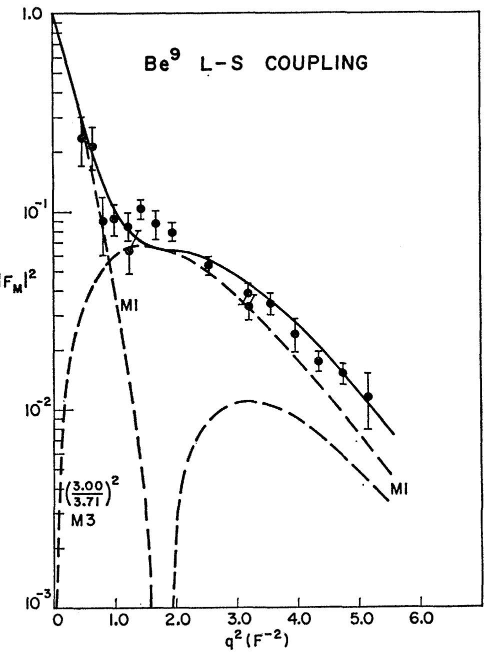

FIG. 6. Same as Fig. 1, except the ratio of the magnetic octupole moment to the magnetic dipole moment, (Ω/μ), of Be^9 is here adjusted to fit the experimental data.

In addition there is a correction due to the motion of the center of mass of the nucleus which has been discussed by Barker and Tassie.[12] This correction is included by multiplying the reduced matrix elements by a factor

$$F(q^2)=\exp[q^2/4\beta A]. \tag{12}$$

Both of these corrections are included in the results shown in Figs. 1 through 5.

It should be emphasized that there are no free parameters in the calculations shown in Figs. 1 through 5. The results of the calculation have however been normalized to 1 at $q^2=0$ which amounts to requiring that the calculations give the correct magnetic dipole moment.

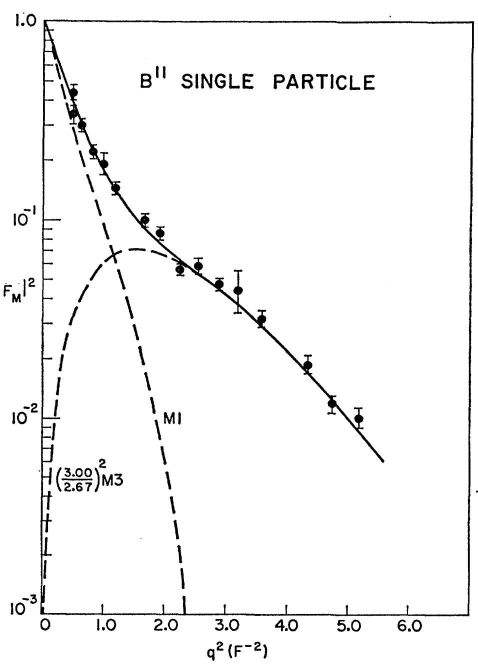

FIG. 7. Same as Fig. 6, except for B^{11}.

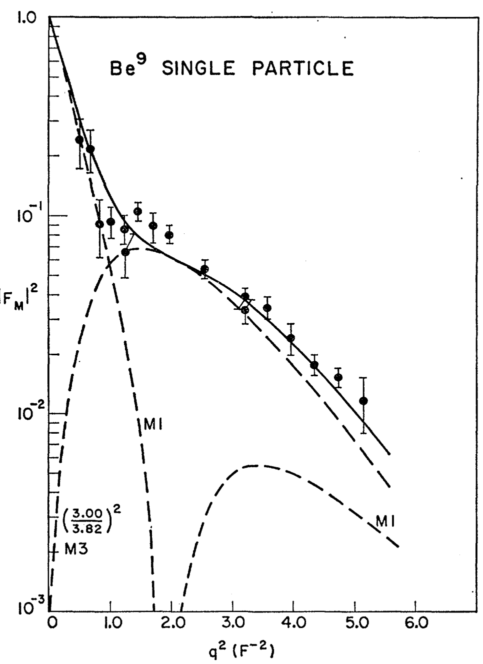

FIG. 8. Same as Fig. 6, except for *L-S* coupling.

IV. DISCUSSION

It is apparent from a comparison of the experimental results with the calculations (Figs. 1 through 5) that the size of the magnetic octupole contribution is not given correctly. We have attempted to obtain a better fit to the experimental results by adjusting the ratio (Ω/μ) which multiplies the magnetic octupole contribution. We assume then that the *shape* of the magnetic octupole contribution is correct and adjust the *size* of the magnetic octupole moment to obtain the best agreement with the experimental data. There is some justification for this procedure since one knows that the size of the magnetic dipole moment is not given correctly by any of the shell-model calculations and hence there is no reason to expect that the magnetic octupole moment would be correct.

The results of shifting the magnetic octupole contribution to obtain the best agreement with the experimental points are shown in Figs. 6 through 10. In each case it is possible to obtain rather good agreement and in each case the ratio $(\Omega/\mu)=3.0$ F^2 seems to give

[12] L. J. Tassie and F. C. Barker, Phys. Rev. **111**, 940 (1958).

a good fit. If one assumes that the *shapes* of the magnetic dipole and magnetic octupole form factors are correct, then this fitting procedure allows us to determine the ratio (Ω/μ) from the experimental magnetic form factors. Since the magnetic dipole moment is well known, one can then obtain an experimental value for the magnetic octupole moment. Using $(\Omega/\mu)=3.0$ F^2 one obtains the magnetic octupole moments given in Table I.

A word of caution is necessary regarding the above procedure for determining the magnetic octupole moment. Although the fact that one obtains the same ratio of (Ω/μ) independent of the coupling scheme (single-particle, *L-S*, or *j-j* coupling), lends some support to the assertion that the magnetic octupole moment can be extracted in a model-independent way, all of these calculations have in common the fact that harmonic-oscillator wave functions are used to describe the independent particle motion. To investigate how sensitive the conclusions are to the assumption of harmonic-oscillator wave functions, we have also calculated the magnetic form factors for Be^9 and B^{11} assuming that the magnetic properties of these nuclei are due to a single particle bound in a square well (a single proton for B^{11} and a single neutron for Be^9). The depth and radius of the square well were adjusted to give the best fit to the magnetic form factors as well as the correct binding energy of the single particle.

Using these square-well wave functions one can indeed obtain an adequate fit to the experimental magnetic form factors. The ratio (Ω/μ) which gives the best fit is again 3.0 F^2 for B^{11}; however, the ratio turns out to be approximately 5.5 F^2 for Be^9. (The large ratio for Be^9 is due to the fact that the binding energy of the last neutron in Be^9 is quite small, about 1.7 MeV.) This last result casts some doubt on the value of the magnetic octupole moment obtained for Be^9.

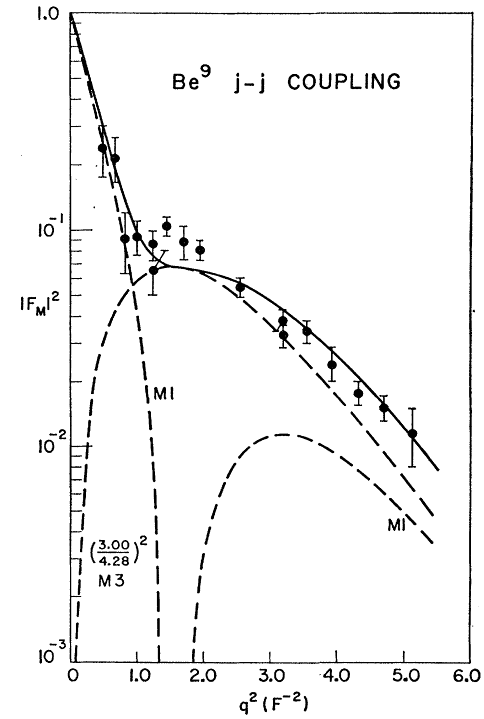

FIG. 10. Same as Fig. 6, except for *j-j* coupling.

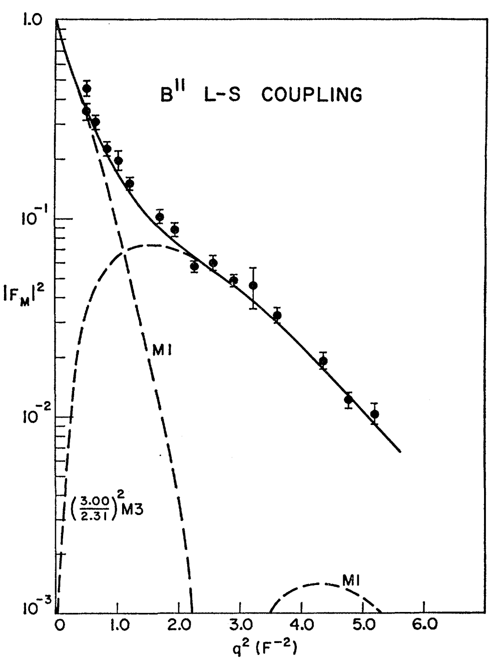

FIG. 9. Same as Fig. 7, except for *L-S* coupling.

V. CONCLUSIONS

The above analysis indicates that the measurement of the elastic-electron-scattering cross section at 180° is a powerful tool for investigating the magnetic properties of the light nuclei. The experimental results of Rand *et al.* can be adequately explained in terms of shell-model wave functions for Be^9 and B^{11}. By adjusting the ratio of the magnetic octupole and magnetic dipole moments (Ω/μ) to obtain the best fit, one can obtain the magnetic octupole moment. The magnitude obtained for the magnetic octupole moment depends to some extent on the wave functions used in the analysis.

ACKNOWLEDGMENTS

We wish to thank Dr. R. Rand, Dr. R. Frosch, and Professor M. Yearian for allowing us to use their data prior to publication and for many useful discussions regarding their experiment. We also wish to acknowledge several helpful discussions with Professor J. D. Walecka.