Approximate Solution of Partial-Wave Dispersion Relations

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An exact expression is derived for the solution of a partial-wave dispersion relation. This expression has the property that, when the zero-width-resonance approximation is made in all integrals where its use is valid, the solution of the dispersion relation is given by simple integrals and no further approximation is required. The result is shown to be independent of the subtraction point and symmetric when applied to a many-channel dispersion relation. Although it utilizes the inverse amplitude, the approximation can be used in some cases of zeros of the amplitude. As a sample calculation, the approximation is applied to the simple π-π-ρ bootstrap and a self-consistent solution is obtained.

I. INTRODUCTION

IN the application of the bootstrap philosophy[1] to strong interactions, the general procedure has involved partial-wave dispersion relations using unitarity to determine the imaginary part of the amplitude in the physical region and some form of crossing symmetry to approximate the imaginary part in the unphysical region. The zero-width-resonance approximation has generally been used in the application of crossing to the unphysical region to determine "driving forces" $V(s)$, assuming the existence of resonances with given energies and widths. A self-consistent bootstrap is achieved when the resonance parameters determined using a set of driving forces agree with those parameters used to determine the driving forces.

There are several weak links in the calculational chain connecting input resonance parameters to output parameters. In this paper we will concentrate on just one weak link—the problem of how to determine the output resonance parameters assuming you know the imaginary part of the amplitude along its unphysical cuts from a given use of crossing and the zero-width approximation.

As usually formulated, this problem involves the solution of a nonlinear integral equation for the amplitude. The ND^{-1} method[2–4] can be used to convert this to a linear integral equation for either D[2] or N,[3] but these integral equations are often not convergent and the solution then depends on an arbitrary cutoff parameter. Pole approximations[5] have been used to simplify the N integral equation but these generally involve additional approximations and are still cutoff-dependent.

There have been several attempts to achieve approximate solutions of partial-wave dispersion relations. The most popular, because of its simplicity, has been the determinantal method[6–8] which simply approximates $N(s)$ by the driving force $V(s)$. However, this method has limited validity, the solution depends on an unphysical parameter (the subtraction point), and is not symmetric when applied to a many-channel problem. Recently, Shaw[9] and Hassoun and Kang[10] have derived approximate solutions which are symmetric and independent of a subtraction point. However, these solutions also involve approximations whose validity is not clear. We will discuss these solutions further, in an Appendix, where we use the methods of this paper to extend them.

In the method presented here, we derive an exact expression for the partial-wave amplitude which has the property that, when the zero width resonance approximation (or any other reasonable approximation using crossing symmetry) is made for $V(s)$, the consistent use of this approximation results in the amplitude being given in terms of simple convergent integrals and no further approximations need be made. The method is applicable to single- or many-channel problems where a self-consistent solution is required. The motivation is to achieve the best possible extrapolation into the physical region of an amplitude that is reasonably well known (within the validity of the zero width approximation) below threshold. The approximate solution is designed to make this extrapolation while preserving the correct discontinuities of the amplitude across its physical and unphysical cuts.

In Sec. II we derive an exact expression for the partial-wave amplitude and use the zero width resonance approximation to derive our approximate solution. In Sec. III we prove that the approximate solution is independent of the subtraction point and symmetric. We also discuss the validity of the method in cases where $\det V=0$ at some point in the unphysical region. We apply the procedure to a single-channel bootstrap problem in Sec. IV. In the Appendix, we use our method to extend the approximations of Shaw[9] and Hassoun and Kang,[10] and compare their type of approximate solution to the one derived in Sec. II.

[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[3] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).
[4] J. L. Uretzky, Phys. Rev. **123**, 1459 (1961).
[5] L. A. P. Balázs, Phys. Rev. **134**, B1315 (1964); **137**, 3168 (1965).
[6] M. Baker, Ann. Phys. (N.Y.) **4**, 271 (1958).
[7] F. Zachariasen, Phys. Rev. Letters **7**, 112, 268(E) (1961).
[8] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[9] G. Shaw, Phys. Rev. Letters **12**, 345 (1964).
[10] G. Q. Hassoun and Kyungsik Kang, Phys. Rev. **137**, B955 (1965).

II. APPROXIMATE SOLUTION

The usual matrix dispersion relation for a partial-wave scattering amplitude, $A(s)$, for n coupled channels can be represented as

$$A(s)=V(s)+U(s), \tag{1}$$

where $U(s)$ is the integral over the unitarity cut (C_R),[11]

$$U(s)=\frac{1}{\pi}\int_{C_R}\frac{\mathrm{Im}A(s')}{s'-s}ds', \tag{2}$$

and $V(s)$ is everything else, generally an integral over left-hand cuts (C_L),[12]

$$V(s)=\frac{1}{\pi}\int_{C_L}\frac{\mathrm{Im}A(s')ds'}{s'-s}. \tag{3}$$

In the strip approximation,[13] $V(s)$ would be the "potential." On the right-hand cut, $\mathrm{Im}A(s)$ is given by the unitarity condition,

$$\mathrm{Im}A(s)=A^*(s)\rho(s)\theta(s)A(s), \tag{4}$$

where ρ and θ are diagonal matrices with elements

$$(\rho(s))_{ij}=\rho_i(s)\delta_{ij} \tag{5}$$

and

$$(\theta(s))_{ij}=\theta(s-s_i)\delta_{ij}, \tag{6}$$

where $\rho_i(s)$ is a known kinematical factor and $\theta(s-s_i)$ is the unit step function with s_i being the threshold for channel i. From the unitarity condition, it follows that

$$\mathrm{Im}(A(s)^{-1})=-\rho(s)\theta(s) \quad \text{for } s \text{ on } C_R. \tag{7}$$

To effect a solution of Eq. (1) for $A(s)$, the zero-width-resonance approximation is usually made. This consists of keeping in the integral over C_L only those states coupled in by crossing which are resonant (or bound) and approximating these resonant amplitudes by

$$\mathrm{Im}A(s)=\pi\Gamma\delta(s-m^2), \tag{8}$$

where Γ is a reduced partial-width matrix given by

$$\Gamma_{ij}=(\gamma_i\gamma_j)^{1/2}, \tag{9}$$

with γ_i being the reduced partial width for the resonance to state i. The zero width approximation is a reasonable one when the other factors in the integral over $\mathrm{Im}A(s)$ are slowly varying over the width of the resonance.

Even after making the zero-width approximation a left-hand cut usually remains making the solution of the resulting integral equation for $A(s)$ non trivial. In the method presented here, we derive an exact expression which has the property that, when the zero-width-resonance approximation (or any other reasonable approximation using crossing symmetry) is made for $V(s)$, the same use of this approximation in $U(s)$ below threshold results in the amplitude being given in terms of simple integrals and no further approximations need be made.

We write the partial-wave amplitude as

$$A(s)=V(s)G^{-1}(s), \tag{10}$$

where, for values of s such that $A(s)$ is a nonsingular matrix, $G(s)$ is given by

$$G(s)=A^{-1}(s)V(s). \tag{11}$$

$G(s)$ can then be expressed by [14,15]

$$G(s)=G_0-(s-s_0)\left[\frac{1}{\pi}\int_{C_R}\frac{ds'\rho(s')V(s')}{(s'-s_0)(s'-s)}-\frac{1}{\pi}\int_{C_L}\frac{ds'\,\mathrm{Im}[A^{-1}(s')V(s')]}{(s'-s_0)(s'-s)}\right]. \tag{12}$$

This representation gives $G(s)$ the correct right and left cuts. G_0 is given by

$$G_0=[V(s_0)+U(s_0)]^{-1}V(s_0) \tag{13}$$

and s_0 is an arbitrary subtraction point. We will show later that the amplitude as given by Eq. (10) with $G(s)$ given by Eqs. (12) and (13) does not depend on the subtraction point and is symmetric. This will be so because the amplitude is given the proper discontinuities across both the right- and left-hand cuts. The determinantal method[6] would correspond to setting $G_0=1$ and dropping the second integral in Eq. (12). It does not give the amplitude the proper discontinuity across the left-hand cut and the amplitude then depends on the subtraction point and is not symmetric.

By making use of the identity

$$\mathrm{Re}A^{-1}\,\mathrm{Im}A+\mathrm{Im}A^{-1}\,\mathrm{Re}A=0 \tag{14}$$

which follows immediately from $A^{-1}A=I$ (I is the unit matrix), and of the fact that

$$\mathrm{Im}V(s')=\mathrm{Im}A(s') \quad \text{for } s' \text{ on } C_L \tag{15}$$

we can derive the identity

$$\mathrm{Im}[A^{-1}(s')V(s')]=\mathrm{Im}A^{-1}(s')\times[\mathrm{Re}V(s')-\mathrm{Re}A(s')] \quad \text{for } s' \text{ on } C_L. \tag{16}$$

Then, using the equations

$$\mathrm{Re}V(s')-\mathrm{Re}A(s')=-U(s') \quad \text{for } s' \text{ on } C_L \tag{17}$$

and

$$\mathrm{Im}A^{-1}(s')=-[V^*(s')+U(s')]^{-1}\,\mathrm{Im}V(s')\times[V(s')+U(s')]^{-1} \quad \text{for } s' \text{ on } C_L, \tag{18}$$

[11] $U(s)$ would also contain any bound state poles and $V(s)$ any crossed channel poles.

[12] The term "left-hand cuts" means all cuts but the unitarity cut even if they happen to be on the right.

[13] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

[14] In general, Eq. (12) could also include CDD (Ref. 15) poles corresponding to zeros of det A. We discuss later the effects of certain types of such zeros. As it stands, Eq. (12) is similar in appearance to the one channel form of G. Feldman, P. T. Mathews, and A. Salam, Nuovo Cimento **16**, 549 (1960), except that their form is subtracted at a pole position of $B(s)$.

[15] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

we can rewrite Eq. (12) as

$$G(s)=G_0-(s-s_0)\left[\frac{1}{\pi}\int_{C_R}\frac{ds'\rho(s')V(s')}{(s'-s_0)(s'-s)}-\frac{1}{\pi}\int_{C_L}\frac{ds'[V^*(s')+U(s')]^{-1}\,\mathrm{Im}V(s')[V(s')+U(s')]^{-1}U(s')}{(s'-s_0)(s'-s)}\right], \quad (19)$$

which is our final result for $G(s)$.

At this stage, no approximations have been made and the amplitude will be given exactly by Eq. (10) with $G(s)$ defined by Eqs. (13) and (19). If we now make the zero-width approximation to determine $V(s')$ and, as well, make the same approximation for $U(s')$, the amplitude is given by simple integrals. It is clear that application of the zero-width approximation to $U(s')$ will be at least as good as its application to $V(s')$ since $U(s')$ is only needed under an integral over the left hand cut in Eq. (19) where the validity condition for the zero-width approximation is well satisfied. If any other approximation is made to $V(s')$ (for example, using a Breit-Wigner form for the resonance or even doing the integral over the actual form of the amplitude), then that approximation, too, will be at least as good when applied to $U(s')$. The reason we say "at least as good" is that $V(s')$ generally involves approximations with respect to the use of crossing that are not involved in the determination of $U(s')$.

One point that must be discussed further is the use of $V(s')$ under the second integral in Eq. (19) which is to be taken over the left-hand cut where the application of the zero-width approximation to $V(s')$ is not necessarily valid. The reason why it still is permissable to use the approximate form of $V(s')$ here is that $G(s)$ is only required to be known in regions to the right of the left hand cut so that the denominator $(s'-s)$ in this integral will be slowly varying. Use of the zero-width approximate form for $V(s')$ in this integral will therefore have the same kind of validity as that approximation has in determining $V(s)$ on the right.

Although we will show that our result does not depend on the subtraction point, it should, in a practical calculation, be chosen somewhere between the right- and left-hand cuts. The reason for this is that we want the denominator $(s'-s_0)$ to be slowly varying in integrals over the left- and right-hand cuts if the zero-width approximation is to be valid. Also we have to use the zero width approximate forms for $U(s_0)$ and $V(s_0)$ in Eq. (13) to define G_0 and these will be simultaneously valid only between the cuts. Application could still be made to a problem where the left- and right-hand cuts overlapped, provided there was still a region where the zero-width approximation was simultaneously valid for $U(s)$ and $V(s)$.

III. GENERAL PROPERTIES OF THE SOLUTION

We now show that $G(s)$ as given by Eq. (19) is independent of the subtraction point provided only that $G(s)$ is well given by

$$G(s)=[V(s)+U(s)]^{-1}V(s) \quad (20)$$

in the region of the subtraction point. Equation (20) is simply the condition that we have achieved a good approximate solution. Let $G_a(s)$ and $G_b(s)$ be two G functions given by Eq. (19) with subtraction points s_a and s_b, respectively. Then we can write [after some algebraic manipulation of Eq. (19)]

$$G_a(s)-G_b(s)=G_{a_0}-G_{b_0}-\frac{s_0-s_a}{\pi}\int\frac{ds'H(s')}{(s'-s_a)(s'-s_b)}, \quad (21)$$

where the integral is considered over both cuts and the explicit form of $H(s')$ would follow from Eq. (19). The only characteristic of $H(s')$ that we need consider here is that it is the same for $G_a(s)$ and $G_b(s)$. Now, from Eq. (19), we see that

$$G_{a_0}-\frac{(s_b-s_a)}{\pi}\int\frac{ds'H(s')}{(s'-s_a)(s'-s_b)}=G_a(s_b) \quad (22)$$

so that Eq. (21) becomes

$$G_a(s)-G_b(s)=G_a(s_b)-G_{b_0}. \quad (23)$$

But $G_a(s_b)$, to the extent it is well given by Eq. (20), is just equal to G_{b_0} as defined by Eq. (13). Thus, Eq. (23) becomes

$$G_a(s)=G_b(s). \quad (24)$$

To show that our result for $A(s)$ is symmetric if $V(s)$ is symmetric, we follow the method used by Bjorken and Nauenberg[16] to show that the ND^{-1} solution to Eq. (1) is symmetric. In each case, the proof depends on the calculated amplitude having the proper discontinuity along both the left- and right-hand cuts. In our case we must also require that the form we use for $U(s')$ under the second integral of Eq. (19) be symmetric but this will be so if the same approximation is used in determining $U(s')$ as was used to determine a symmetric $V(s)$.

If we consider the matrix function of s

$$\tilde{G}(A-\tilde{A})G=\tilde{G}V-\tilde{V}G, \quad (25)$$

then

$$\mathrm{Im}[\tilde{G}(A-\tilde{A})G]=-\tilde{V}\rho V+\tilde{V}\rho V=0 \quad \text{for } s \text{ on } C_R \quad (26)$$

since ρ is a symmetric matrix; and

$$\mathrm{Im}[\tilde{G}(A-\tilde{A})G]=\mathrm{Im}(\tilde{G}V)-\mathrm{Im}(\tilde{V}G) \text{ for } s \text{ on } C_L. \quad (27)$$

[16] J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

By using Eqs. (14–17) and the further identity

$$\mathrm{Re}A^{-1}\,\mathrm{Re}A-\mathrm{Im}A^{-1}\,\mathrm{Im}A=I \tag{28}$$

which also follows from $A^{-1}A=I$, we can reduce Eq. (27) to

$$\mathrm{Im}[\tilde{G}(A-\tilde{A})G]=\tilde{U}(\mathrm{Im}\tilde{A}^{-1})\tilde{U}-U(\mathrm{Im}A^{-1})U=0 \quad \text{for } s \text{ on } C_L \tag{29}$$

since $U(s)$ is symmetric, as is $A^{-1}(s)$ as given by Eq. (18). Thus, the function $\tilde{G}(A-\tilde{A})G$ is analytic everywhere in the s plane and vanishes at ∞. It is therefore identically zero and we obtain for A, as given by our Eqs. (10) and (19),

$$A=\tilde{A}. \tag{30}$$

As with the ND^{-1} case,[15] we would get the same conclusion, if G is a singular matrix, by considering the matrix (det G)$\boldsymbol{A}$.

Ordinarily, an inverse amplitude method would be expected to break down if either of the matrices $A(s)$ or $V(s)$ were singular [(det A)$=0$ or (det V)$=0$]. The method described here, however, could still give reliable results if det V had a zero not too far from the zero in det A and these zeros were not near the physical region.[17] The subtraction point would also have to be picked so as not to be too close to the zeros. In this case the approximation would still be reasonable in the physical region.

The method has several checks on its validity. Eq. (20) should be approximately satisfied between the cuts, the extent to which it is satisfied being a measure of the approximation. In particular, if a zero of det A or det V were causing trouble this would show up in Eq. (20). Further checks are provided by how well the conditions of symmetry and independence of the subtraction point are satisfied.

IV. APPLICATION TO π-π BOOTSTRAP

We now consider the application of the VG^{-1} method to the simple one channel example of a self-consistent ρ-meson bootstrap in the p-wave π-π system. There are good reasons to expect other than simple ρ exchange to be important and the usual representation of the left-hand cut by simply crossing a ρ-meson pole is open to question, so we do not take the model too seriously but merely use it as an example of the method.

With no further apologies, we take the following forms[8,10] for $V(s)$, $U(s)$, and $\rho(s)$:

$$\mathrm{Re}V(s)=12\Gamma\frac{(m^2-4+2s)}{(s-4)^2}\left[\left(1+\frac{2m^2}{s-4}\right)\times\ln\left(\left|1+\frac{(s-4)}{m^2}\right|\right)-2\right]=\Gamma v(s), \tag{31}$$

$$\mathrm{Im}V(s)=12\pi\Gamma(m^2-4+2s)(2m^2-4+s)/(s-4)^3=\Gamma w(s) \quad \text{for} \quad s<4-m^2, \tag{32}$$

$$U(s)=4\Gamma/(m^2-s)=\Gamma u(s), \tag{33}$$

and

$$\rho(s)=\tfrac{1}{4}(s-4)^{3/2}/s^{1/2}. \tag{34}$$

s is the square of the total center-of-mass energy and all energies are in units of the pi-meson mass. We are using the form of $A(s)$ that guarantees the correct threshold behavior. Specifically, we have taken

$$A(s)=e^{i\delta(s)}\sin\delta(s)/\rho(s), \tag{35}$$

where $\delta(s)$ is the usual phase shift. Γ is the reduced ρ width and is related to the full width Γ_E of the ρ meson by

$$\Gamma_E=\Gamma(m^2-4)^{3/2}/m^2 \tag{36}$$

and to the π-π-ρ coupling constant γ_ρ[18] by

$$\Gamma=\tfrac{1}{3}(\gamma_\rho^2/4\pi). \tag{37}$$

We follow the self-consistent method of Zachariasen and Zemach[7] and make two determinations of the width at a range of mass values for the ρ. The self-consistent mass is then that value for which the two widths agree. The first width determination is given by requiring Re$G(s)$ to be zero at the ρ mass so that

$$\mathrm{Re}G(s)=G_0-(s-s_0)[\Gamma F(s)+E(s)]=0 \text{ for } s=m^2, \tag{38}$$

where

$$F(s)=\frac{P}{\pi}\int_4^\infty\frac{ds'\rho(s')v(s')}{(s'-s_0)(s'-s)} \tag{39}$$

and

$$E(s)=-\frac{1}{\pi}\times\int_{-\infty}^{4-m^2}\frac{ds'w(s')u(s')}{(s'-s_0)(s'-s)[[v(s')+u(s')]^2+w(s')^2]}. \tag{40}$$

This gives, as the "input" width required to put the resonance at a particular mass position,

$$\Gamma_m=[G_0/(s-s_0)-E(m^2)]/F(m^2). \tag{41}$$

The "output" width is then determined as the width that arises from the application of the zero-width approximation to the resonance:

$$\Gamma_w=-\tfrac{1}{4}V(m^2)/[(\partial/\partial s)\,\mathrm{Re}G(s)]_{s=m^2}, \tag{42}$$

and that value of m^2 for which

$$\Gamma_w=\Gamma_m \tag{43}$$

defines the self-consistent mass of the ρ meson.

It should be pointed out that most of the equations written down are only correct when Eq. (43) is satis-

[17] This does turn out to be the case in the π–π example considered in this paper.

[18] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

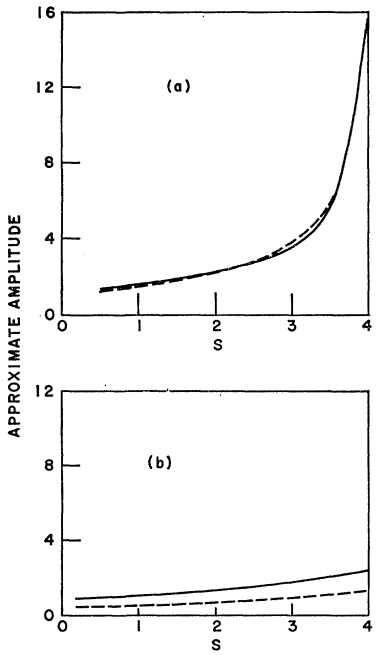

FIG. 1. Comparison of "input" ($A=V+U$, solid curve) and "output" ($A=VG^{-1}$, dashed curve) self-consistent solutions for (a) the VG^{-1} method with $s_0=2$, (b) the determinantal approximation with $s_0=4-m^2$.

fied, but, since this is the only case of interest, the procedure does make sense. Because it is possible to factor out the Γ dependence and solve directly for Γ_m in terms of the integrals $E(m^2)$ and $F(m^2)$ [Eq. (41)] the procedure is quite simple. For each value of m^2, only four integrals (E and F at $s=m^2$ and at an adjacent point to determine the derivative) have to be performed.

The calculations were performed on one of the IBM 7094 computers at the Lawrence Radiation Laboratory at Livermore and the results are summarized in Table I along with the corresponding results of the determinantal method. It is seen that the VG^{-1} method gives no appreciable variation of the self consistent mass and coupling constant as the subtraction point, s_0, is varied over the range of validity, while the determinantal results do show a dependence on the subtraction point. As a further check, we have plotted

TABLE I. Self-consistent ρ parameters for the VG^{-1} and determinantal methods for several subtraction points s_0.

	VG^{-1}		Determinantal	
s_0	m^2 [a]	Γ	m^2 [a]	Γ
1	4.02	0.72	13.0	1.07
2	4.10	0.74	18.9	1.14
3	4.16	0.76	25.5	1.19
4	4.18	0.76	38.5	1.22
$4-m^2$	—[b]	—[b]	5.84	0.87

[a] The ρ mass varies from 281 to 286 MeV for the VG^{-1} results and from 500 to 870 MeV for the determinantal results as s_0 varies from 1 to 4.
[b] The VG^{-1} method is not valid for this subtraction point.

in figure (1a) a comparison of the two forms, $A=V+U$ and $A=VG^{-1}$, of the amplitude which should agree in the region between the cuts. This is equivalent to Eq. (20). In the particular case plotted ($s_0=2$), the two curves are, of course, normalized to cross at $s=2$. The extent to which they agree in the interval plotted indicates that we have approximated the zero-width approximate amplitude by a function (VG^{-1}) which gives a good fit in the region between the cuts and has the correct imaginary parts along both cuts. In this example, $B(s)$ has a zero at $s=2-m^2/2=-0.05$ and A has a zero to the left of this. The close agreement of the two curves of Fig. 1(a) thus indicates that the existence of these zeros does not invalidate the method. For purposes of comparison, we have also plotted the equivalent curves for the determinantal method in Fig. (1b). In this case we have taken $s_0=4-m^2$ (the start of the left-hand cut) since arguments can be made[8] for this to be a "preferred" subtraction point for the determinental method.

V. DISCUSSION

Although the result of the sample calculation is in disagreement with experiment this should not be taken as a judgment of the method. As was pointed out, the crossing approximations used to determine $V(s)$ are open to serious question. It is also likely that this simple π-π bootstrap model is not a good model of the ρ meson. The importance of other channels has been emphasized by several authors[5,8,19,20] as has the exchange of particles other than the ρ meson.[5,20]

Actually, because the calculated self-consistent mass of the ρ does come out so close to threshold, we do happen to be in the region where the arguments for the original strip approximation[13] are valid. This means that we might have a relatively valid solution of the simple π-π bootstrap and the severe disagreement with experiment probably does indicate a weakness of the physical model and not of the calculational scheme. From this point of view, it is encouraging that the present result is in rough agreement with the result of a calculation[21] using the same physical model but proceeding from fixed momentum transfer dispersion relations to generate different crossing approximations.

Although the example considered was a single-channel problem the method of this paper is immediately applicable to the solution of many-channel dispersion relations. The method is only slightly more complicated than the determinental approximation but is symmetric and independent of subtraction parameters. It is considerably simpler and is more convergent than the ND^{-1} method while making no additional approximations other than those which are

[19] R. Blankenbecler, Phys. Rev. **125**, 755 (1962). R. E. Kreps, L. F. Cook, J. J. Brehm and R. Blankenbecler, *ibid.* **133**, B1526 (1964).
[20] J. Franklin, Phys. Rev. **137**, B944 (1965).
[21] J. Franklin, D. Land, R. Piñon, Phys. Rev. **137**, B172 (1965).

usually included in using ND^{-1}. An added advantage of the increased convergence of the method is that the result is less dependent on $V(s)$ for the upper ranges of the right-hand integration where it becomes increasingly unreliable. A possible drawback of the method, the vanishing of det V or det A near the physical region, can be checked for in any particular calculation by testing Eq. (20).

The method is most useful for making a self-consistent calculation where all "input forces" are made consistent with "output" resonances and bound states. It could also be used, however, with a specified set of input forces, in which case self consistency would be required for the output resonances and their zero-width contribution to $U(s)$ on the left. In the case in which there are no output resonances, the method would reduce to the determinantal approximation.

APPENDIX

In this Appendix we use the application of the zero-width resonance approximation to the integral over the unitarity cut to extend the approximate methods suggested by Shaw[9] and by Hassoun and Kang.[10] We first give a brief derivation of an exact equation which can be related to their results.

If we write the partial-wave amplitude as

$$A(s)=V(s)C^{-1}(s)V(s), \tag{A1}$$

then the dispersion relation

$$C(s)=-\frac{1}{\pi}\int_{C_R}\frac{ds'V(s')\rho(s')V(s')}{s'-s}+\frac{1}{\pi}\int_{C_L}\frac{ds'\,\mathrm{Im}[V(s')A^{-1}(s')V(s')]}{s'-s} \tag{A2}$$

can be written for $C(s)$. Now repeated use of the identities of Eqs. (14)–(17) and (28) leads to

$$\mathrm{Im}[V(s')A^{-1}(s')V(s')]=\mathrm{Im}A(s')+U(s')\,\mathrm{Im}A^{-1}(s')U(s'). \tag{A3}$$

Then, using Eq. (18) and the definition of $V(s)$ [Eq. (3)] we can write

$$C(s)=V(s)-\frac{1}{\pi}\int_{C_R}\frac{ds'V(s')\rho(s')V(s')}{s'-s}-\frac{1}{\pi}\int_{C_L}\frac{ds'U(s')[V^*(s')+U(s')]^{-1}\,\mathrm{Im}V(s')[V(s')+U(s')]^{-1}U(s')}{s'-s} \tag{A4}$$

as our final result for $C(s)$.

Equation (A4) is still an exact equation for $C(s)$. It is an extension of Eqs. (17) and (18) of Ref. 10 which are derived by a different method. Kang and Hassoun then use an expansion in powers of $V(s')$ to approximate what amounts to the second integral in Eq. (A4). This approximation is questionable where $V(s')$ is not small and, further, leads to a double integral which is computationally cumbersome. Shaw's approximation[9] would correspond to dropping the second integral of Eq. (34). This would be valid when there are no resonances or when the unitarity integral $U(s)$ is negligible on the left.

If, as before, we now make the zero-width-resonance approximation to $V(s')$ and to $U(s')$ in the exact Eq. (A4), we have $C(s)$ given by simple integrals with no further approximations being necessary. The form $A=VC^{-1}V$ is obviously symmetric and independent of any subtraction point, but this is no real advantage over the form $A=VG^{-1}$ because we have proven the latter form also to be symmetric and independent of the subtraction point to the extent that the approximations are valid.

Another suggested advantage[10] of the form $A=VC^{-1}V$ is that this form of the amplitude will equal V at all infinite singularities of V. However, this, too, is not really an advantage because the infinities in V are generally due to a breakdown of the zero-width approximation at that point and would not be there if a less singular approximation (or any approximation which had $\mathrm{Im}V(s)\to 0$ at the end of the left-hand cut as it should) were used. In fact, the exact amplitude, if it could be determined without the use of the zero-width approximation, would generally not show any enhancement near the start of the cut. As an example, if we consider the case of π-π scattering with ρ meson exchange, $V(s)$ has a logarithmic singularity at the start of the left-hand cut. However, if we average $V(s)$ over a Breit-Wigner shape of reasonable width it no longer shows any particular enhancement at that point.

The form $VC^{-1}V$ also has the serious disadvantage that a simple zero in det V would lead to a double zero in $\det(VC^{-1}V)$ and, even if this occurs in an unphysical region, the trend of the amplitude would be so badly represented that a reasonable extrapolation to the physical region would be difficult. The simpler form $A=VG^{-1}$, therefore seems to be generally preferable to the $VC^{-1}V$ form and is the form we have chosen to use in the example of Sec. IV.