# Breakdown of Unitary Octet Symmetry in a Nonlinear Spinor Model of Elementary-Particle Theory*

N. BYRNE, C. IDDINGS, AND E. SHRAUNER
*Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California*
(Received 8 April 1965)

We consider the spontaneous breakdown of unitary octet symmetry in a nonlinear spinor model of elementary-particle theory. Our model is an adaptation to unitary octet symmetry of a model originated by Nambu and Jona-Lasinio and contains $SU_3 \otimes SU_3$ symmetry in the same way that their model contained $SU_2 \otimes SU_2$ symmetry. We derive an exact formula for the physical baryon mass that reduces to the usual superconductor-type formula in the lowest order approximation. Nonperturbative solutions that leave the physical $\Lambda$ and $\Sigma$ masses degenerate are obtained. We confirm the presence in our nonperturbative solutions of massless pseudoscalar and scalar mesons transforming as components of $F$-type octets as predicted by the Goldstone theorem. We also find massless pseudoscalar and scalar mesons that transform as components of unitary spin decimets. We explain that only the $F$ octets are Goldstone mesons associated with the spontaneous breakdown of octet symmetry in our model, whereas the massless decimets are associated with the invariance of a restricted part of our Lagrangian under a larger group and their masslessness is not a consequence of symmetry breakdown in our solutions.

## 1. INTRODUCTION

THE idea that the strong interactions may obey some higher symmetries than those observed to hold exactly in nature has a strong appeal. It is also an attractive idea to incorporate the higher symmetries within the framework of a specific dynamical model of elementary-particle theory in which all the observable states are derivable from a single, or a few fundamental fields.

Nambu and Jona-Lasinio developed such a model based on a nonlinear self-interaction of a fundamental fermion field.[1] They emphasized nonperturbative solusions that involve spontaneous breakdown of the symmetry group of the Lagrangian. We consider a nonlinear spinor model of elementary-particle theory that is an adaptation of their model to unitary octet symmetry. The symmetry group of the Lagrangian in our model includes $SU_3 \otimes SU_3$ in analogy to the way Nambu and Jona-Lasinio's model contained $SU_2 \otimes SU_2$ symmetry, our fundamental fermion field being a unitary spin octet where theirs was an isospin doublet.[2]

The primary Lagrangian of our model is described in Sec. 2. The unitary spin formalism required in it is introduced there, and the restrictions imposed on it by crossing and certain chiral symmetries are discussed. Having established the notation and form of our model, we then derive in Sec. 3 an exact formula for the dynamical mass of the physical baryon solutions in terms of a spectral representation. This exact equation is shown to reduce directly to the usual lowest order formula when simple pole approximations are assumed for the propagators. The relationship of the lowest order approximation to the exact formulation is thus exhibited.

In the lowest order approximation we obtain self-consistent, nonperturbative solutions of the baryon mass formula. Solutions are found which fit the observed baryon masses to within a few percent. It is a feature of our solutions that the physical masses of the $\Lambda$ and $\Sigma$ are left degenerate, and therefore they do not generally satisfy the Gell-Mann–Okubo mass formula.

In Sec. 4 the lowest order solutions are used as a basis for our study of the meson states of our model. By using formulas that we have developed in the accompanying article[3] we confirm the presence of massless pseudoscalar and scalar mesons which transform as components of $F$-type unitary spin octets in accordance with the predictions of the Goldstone theorem. In lowest order approximations we also obtain massless pseudoscalar and scalar mesons transforming as components of unitary spin decimets. These mesons appear because the approximation that we use in calculations for zero-mass mesons involves only a part of our Lagrangian and this part happens to be invariant under a larger group ($R_8 \otimes R_8$) than the symmetry group of our complete Lagrangian ($SU_3 \otimes SU_3$). In addition to explaining these complications of spontaneous symmetry breakdown, in Sec. 4 we also discuss the search for other meson states of small, nonvanishing masses. In Sec. 5 we discuss the effects of introducing a slight intrinsic symmetry breakage into our model with small bare mass terms. We also relate our results to a discussion of spontaneous breakdown of chiral symmetries given by Freund and Nambu.[4]

---

* Work supported by the U. S. Air Force through Air Force Office of Scientific Research Contract AF 49(638)-1389. Parts of this material appear in a doctoral dissertation of N. Byrne, Stanford University, 1965 (unpublished).

[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

[2] M. Baker and S. Glashow have also discussed an extension of the Nambu model to $SU_3$, but they based their model on a fundamental fermion field which is a unitary triplet instead of an octet; Phys. Rev. **128**, 2462 (1962). S. Glashow, *ibid.* **130**, 2132 (1963) discusses certain general properties of an octet model such as ours.

[3] N. Byrne, C. Iddings, and E. Shrauner, following paper, Phys. Rev. **139**, B933 (1965); we shall hereafter refer to it as article II.

[4] P. G. O. Freund and Y. Nambu, Phys. Rev. Letters **13**, 221 (1964).

## 2. THE LAGRANGIAN AND ITS SYMMETRIES

To accommodate the isospin and hypercharge degrees of freedom in our model, we consider a primary fermion field $\psi$ which transforms as a unitary spin octet and interacts with itself in a Lagrangian that is invariant under the group of unitary spin transformations, $SU_3$. The dynamics of our model are to be determined from a quartic Lorentz-invariant self-interaction of this primary field. Further specifications of our model Lagrangian are not dictated by general physical principles. In order to restrict the arbitrariness of the Lagrangian, we will later impose certain chiral symmetries on it, although these are not required by such profound physical principles as Lorentz invariance. The model finally arrived at in this way corresponds to a direct generalization of Nambu and Jona-Lasinio's model from isospin symmetry to unitary spin symmetry.

We first consider the unitary spin couplings required for our four-field interaction. For convenience, we will separate the unitary spin part of the wave function as a direct factor from the part that depends on all other coordinates. Suppressing for now these other coordinates, we write the unitary spin wave function of our primary fermion octet field in the form of a traceless $3\times3$ tensor $\psi_k{}^m$ the lower covariant index labeling the row and the upper contravariant index labeling the column of the elements in a matrix array. An explicit $3\times3$ representation of the octet $\psi$ and the generators of the group of unitary spin transformations, as well as a description of the behavior of our quartic couplings according to the irreducible representations of this group, is given in the Appendix.

For describing the unitary spin couplings in our model it is convenient to consider couplings of four distinguishable unitary octets instead of a single octet quartically self-coupled. The invariant tensor couplings of four octets, $A$, $B$, $C$, $D$, are those which saturate all the tensor indices. We introduce our notation and definitions for these couplings with the following example and Table I:

$$A^\alpha{}_A B^\beta{}_B C^\gamma{}_C D^\delta{}_D (\delta^A{}_\delta \delta^B{}_\gamma \delta^C{}_\beta \delta^D{}_\alpha) = \mathrm{Spur}(AD)\ \mathrm{Spur}(CB) \equiv (AD)(CB) \equiv \mathcal{T}_9. \quad (2.1)$$

For each invariant $\mathcal{T}_l$ we define the corresponding invariant tensor $T(l)$ as follows:

$$\begin{aligned} A^\alpha{}_A B^\beta{}_B C^\gamma{}_C D^\delta{}_D T(9)^{ABCD}{}_{\alpha\beta\gamma\delta} &\equiv \mathcal{T}_9 \\ T(9)^{ABCD}{}_{\alpha\beta\gamma\delta} &\equiv \delta^A{}_\delta \delta^B{}_\gamma \delta^C{}_\beta \delta^D{}_\alpha \end{aligned} \quad (2.2)$$

and similarly, for the other $T(l)$ and $\mathcal{T}_l$. These nine couplings are not all independent. Due to the restrictions of the tensor indices to the values 1, 2, 3, and the tracelessness of the octets, $A^\alpha{}_\alpha=0$, the $\mathcal{T}_l$ are related by the condition

$$\mathcal{T}_1+\mathcal{T}_5+\mathcal{T}_9-\mathcal{T}_2-\mathcal{T}_3-\mathcal{T}_4-\mathcal{T}_6-\mathcal{T}_7-\mathcal{T}_8=0. \quad (2.3)$$

TABLE I. The invariant couplings of four unitary spin octets.

| | | |
|---|---|---|
| $\mathcal{T}_1\equiv(AB)(CD)$ | $\mathcal{T}_4\equiv(ABDC)$ | $\mathcal{T}_7\equiv(ABCD)$ |
| $\mathcal{T}_2\equiv(ADCB)$ | $\mathcal{T}_5\equiv(AC)(BD)$ | $\mathcal{T}_8\equiv(ACBD)$ |
| $\mathcal{T}_3\equiv(ACDB)$ | $\mathcal{T}_6\equiv(ADBC)$ | $\mathcal{T}_9\equiv(AD)(BC)$ |

Our Lagrangian does not couple four distinguishable octet fields. When the expressions above are transcribed into the quartic self-couplings of our model, $A$ and $C$ both are replaced by $\bar\psi$ while $B$ and $D$ are both replaced by $\psi$. The matrix element of an interaction Lagrangian of our type for the scattering process $B+D\to A+C$ is of the form

$$\langle AC|\bar\psi\Gamma\psi\bar\psi\Gamma\psi|BD\rangle = \bar u_A\Gamma u_B\bar u_C\Gamma u_D-\bar u_A\Gamma u_D\bar u_C\Gamma u_B. \quad (2.4)$$

The "exchange" term here takes account of the antisymmetry under exchange of identical Fermi particles. This may also be considered as the consequence of anticommutation relations of quantized fermion field operators. We want to take account explicitly of this (anti) symmetrization in the construction of our Lagrangian in order to take advantage as early as possible of the resulting simplifications. We are therefore led to consider the effects of the exchange of two identical fields in our interaction. That is, we must consider the effects of interchanging $A\leftrightarrow C$ and/or $B\leftrightarrow D$ in the above quartic invariants. It is, of course, not essential that we symmetrize the Lagrangian with respect to interchange of identical fields at this point. The exchange symmetry would be taken into account in the correct calculation of the matrix elements and contributions of the wrong symmetry would drop out of the final results.

The usual "crossing symmetry" corresponds to evenness or oddness of matrix elements under the exchange of the fields $A\leftrightarrow C$, $B\to B$, $D\to D$ in the four-field expressions in Table I. For example, for the coupling $\mathcal{T}_9$ in Eq. (2.1), this exchange gives:

$$\mathcal{T}_9=(AD)(CB)\to(AB)(CD)=\mathcal{T}_1.$$

Proceeding in this way we get

$$\begin{gathered}\mathcal{T}_1\leftrightarrow\mathcal{T}_9,\quad \mathcal{T}_2\leftrightarrow\mathcal{T}_7,\quad \mathcal{T}_3\leftrightarrow\mathcal{T}_8,\\ \mathcal{T}_4\leftrightarrow\mathcal{T}_6,\quad \mathcal{T}_5\to\mathcal{T}_5.\end{gathered} \quad (2.5)$$

We are here considering the fields $A$, $B$, $C$, $D$ as $c$ numbers. An alternative description of this interchange is to leave the field labels unchanged but interchange the corresponding indices on the coupling tensors $T(l)$, which is equivalent to just relabeling $T(l)\to T(l')$ in the same way as $\mathcal{T}_l\to\mathcal{T}_{l'}$ in Eq. (2.5). Obviously the combinations $(\mathcal{T}_1\pm\mathcal{T}_9)$, $(\mathcal{T}_2\pm\mathcal{T}_7)$, $(\mathcal{T}_3\pm\mathcal{T}_8)$, $(\mathcal{T}_4\pm\mathcal{T}_6)$, and $\mathcal{T}_5$ diagonalize the crossing matrix.

The matrix element Eq. (2.4) is also invariant under the exchange of the labels $A\leftrightarrow C$, $B\leftrightarrow D$; so clearly, the corresponding interchange of identical fields in our Lagrangian can have no physical effect. Under this interchange, $\mathcal{T}_3 \leftrightarrow \mathcal{T}_4$ and $\mathcal{T}_6 \leftrightarrow \mathcal{T}_8$, while the other $\mathcal{T}_l$ remain invariant. Therefore only the combinations $(\mathcal{T}_3+\mathcal{T}_4)$ and $(\mathcal{T}_6+\mathcal{T}_8)$, but not $(\mathcal{T}_3-\mathcal{T}_4)$ nor $(\mathcal{T}_6-\mathcal{T}_8)$, give physical contributions from our quartic self-interaction. In terms of the irreducible representations of unitary spin discussed in the Appendix to this section, these restrictions are equivalent to allowing $(DF-FD)$ and $(X\bar{X}+\bar{X}X)$ couplings, but excluding contributions from $(DF+FD)$ and $(X\bar{X}-\bar{X}X)$. Instead of eight unitary spin couplings, our symmetrization of the Lagrangian shows that only six of them can appear in it.

Next, we consider the Dirac-spin character of the four-field coupling. The Lorentz-invariant couplings of the Dirac spinors are well known. For definiteness, they and their behavior under crossing, which is the familiar Fierz transformation, are summarized in the explicit matrix equation:

$$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{S}^2 \\ \tilde{V}^2 \\ \tilde{T}^2 \\ \tilde{A}^2 \\ \tilde{P}^2 \end{bmatrix} = \begin{bmatrix} S^2 \\ V^2 \\ T^2 \\ A^2 \\ P^2 \end{bmatrix} \equiv \begin{bmatrix} (\bar{\psi}\psi)(\bar{\psi}\psi) \\ (\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma_\mu\psi) \\ (\bar{\psi}\sigma_{\mu\nu}\psi)(\bar{\psi}\sigma_{\mu\nu}\psi)\frac{1}{2} \\ (\bar{\psi}i\gamma_\mu\gamma_5\psi)(\bar{\psi}i\gamma_\mu\gamma_5\psi) \\ (\bar{\psi}\gamma_5\psi)(\bar{\psi}\gamma_5\psi) \end{bmatrix}, \tag{2.6}$$

with $\gamma_\mu=\gamma_\mu{}^\dagger=\gamma_\mu{}^{-1}$ for $\mu=1, 2, 3, 4, 5$.

Considering the combined Dirac-spin and unitary-spin dependence of our Lagrangian, we expect that there are 15 possible couplings of the appropriate symmetry which might occur in our interaction Lagrangian, each with an independent weight.[5] To further reduce the arbitrariness in our model, we impose on it the restrictions of a chiral symmetry.

Our model, as described so far, is invariant under the gauge transformations

$$\text{I}:\ \psi \to e^{i\alpha}\psi,\quad \bar{\psi} \to \bar{\psi}e^{-i\alpha}, \tag{2.7}$$

and

$$\text{II}:\ \psi \to e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\beta}}\psi,\quad \bar{\psi} \to \bar{\psi}e^{-i\boldsymbol{\lambda}\cdot\boldsymbol{\beta}}. \tag{2.8}$$

Invariance under the group of constant phase transformations I means that our model conserves fermion number. The transformations II are the unitary spin rotations, where $\boldsymbol{\lambda}$ represents the eight operators which act on the octet $\psi$ to generate these transformations (see Appendix). $\boldsymbol{\beta}$ is an arbitrary octet which characterizes the particular rotation in unitary spin space. It is a "gauge vector" in unitary spin space; hence, the vector notation $\boldsymbol{\lambda}\cdot\boldsymbol{\beta}$. In addition to invariance under the above gauge transform groups, we now consider symmetry under chiral transformations of the types

$$\text{III}:\ \psi \to e^{i\delta\gamma_5}\psi,\quad \bar{\psi} \to \bar{\psi}e^{i\delta\gamma_5}, \tag{2.9}$$

$$\text{IV}:\ \psi \to e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\theta}\gamma_5}\psi,\quad \bar{\psi} \to \bar{\psi}e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\theta}\gamma_5}. \tag{2.10}$$

$\boldsymbol{\theta}$ is a unitary octet gauge vector, as $\boldsymbol{\beta}$. Simultaneous transformations of the types II and IV are equivalent to independent unitary spin transformations on left- and right-handed components of $\psi$:

$$\begin{aligned} \text{V}:\ &\psi_L \to e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\eta}}\psi_L,\quad \bar{\psi}_R \to \bar{\psi}_R e^{-i\boldsymbol{\lambda}\cdot\boldsymbol{\eta}}, \\ &\psi_R \to e^{i\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}\psi_R,\quad \bar{\psi}_R \to \bar{\psi}_R e^{-i\boldsymbol{\lambda}\cdot\boldsymbol{\xi}}, \end{aligned} \tag{2.11}$$

where $\boldsymbol{\eta}\equiv\boldsymbol{\beta}+\boldsymbol{\theta}$ and $\boldsymbol{\xi}\equiv\boldsymbol{\beta}-\boldsymbol{\theta}$, while $\psi_L=\frac{1}{2}(1+\gamma_5)\psi$ and $\psi_R\equiv\frac{1}{2}(1-\gamma_5)\psi$ are respectively the left-handed (l.h.) and right-handed (r.h.) components of $\psi$. Similarly combinations of the transformations I and III are equivalent to

$$\begin{aligned} \text{VI}:\ &\psi_L \to e^{i\nu}\psi_L,\quad \bar{\psi}_L \to \bar{\psi}_L e^{-i\nu}, \\ &\psi_R \to e^{i\rho}\psi_R,\quad \bar{\psi}_R \to \bar{\psi}_R e^{-i\rho}, \end{aligned} \tag{2.12}$$

where $\nu\equiv\alpha+\delta$ and $\rho\equiv\alpha-\delta$. Invariance under the group of transformations VI is equivalent to the conservation of the numbers of left-hand and right-hand fermions separately. In order to consider the effects of these independent transformations on left-hand and right-hand field components, we analyze certain combinations of the Dirac spinor invariants that were exhibited in Eq. (2.6) into products of left-hand and right-hand fields. This chirality, or handedness decomposition, is given in Table II.

TABLE II. Handedness decomposition of the Dirac invariants.

| Handedness decomposition of Dirac spinor invariants | | Fierz conjugate |
|---|---|---|
| $S^2+P^2=2(\bar{\psi}_R\psi_L)(\bar{\psi}_R\psi_L)$ | $+2(\bar{\psi}_L\psi_R)(\bar{\psi}_L\psi_R)$ | $\to \frac{1}{2}(S^2+P^2+T^2)$ |
| $S^2-P^2=2(\bar{\psi}_R\psi_L)(\bar{\psi}_L\psi_R)$ | $+2(\bar{\psi}_L\psi_R)(\bar{\psi}_R\psi_L)$ | $\to \frac{1}{2}(V^2+A^2)$ |
| $V^2+A^2=2(\bar{\psi}_R\gamma_\mu\psi_R)(\bar{\psi}_L\gamma_\mu\psi_L)$ | $+2(\bar{\psi}_L\gamma_\mu\psi_L)(\bar{\psi}_R\gamma_\mu\psi_R)$ | $\to 2(S^2-P^2)$ |
| $V^2-A^2=2(\bar{\psi}_R\gamma_\mu\psi_R)(\bar{\psi}_R\gamma_\mu\psi_R)$ | $+2(\bar{\psi}_L\gamma_\mu\psi_L)(\bar{\psi}_L\gamma_\mu\psi_L)$ | $\to -(V^2-A^2)$ |
| $T^2=2(\bar{\psi}_R\sigma\psi_L)(\bar{\psi}_R\sigma\psi_L)$ | $+2(\bar{\psi}_L\sigma\psi_R)(\bar{\psi}_L\sigma\psi_R)$ | $\to (3/2)(S^2+P^2)-\frac{1}{2}T^2$ |

Under simultaneous independent transformations I and III, or equivalently the transformations VI, only the coupling combinations $S^2-P^2$, $V^2$, and $A^2$ remain invariant. Invariance of our four-field interaction under the transformations II and IV, or equivalently, the independent unitary spin transformations V of left-hand and right-hand fields separately, requires that all the left-hand octets be coupled into a unitary spin invariant and the right-hand octets be coupled into a separate unitary invariant. In all the Dirac couplings in Table II except $(V^2-A^2)$, two left-hand and two right-hand fields occur. For example, in the Dirac coupling $(S^2+P^2)$ the form is $\bar{\psi}_R\psi_L\bar{\psi}_R\psi_L$. So the unitary spin coupling in this case must be $\text{Spur}(\bar{\psi}_R\bar{\psi}_R)$ $\text{Spur}(\psi_L\psi_L)$, where the

[5] Under a Fierz transformation, three of the unitary couplings are even and three are odd. Of the five Dirac coupling factors, three are odd and two are even.

unitary spin octet parts of the $\psi$'s are separately contracted into right-hand unitary singlet and left-hand unitary singlet. By the same type of argument the Dirac spin coupling $T^2$ is associated with the unitary spin coupling $\mathcal{T}_5$, and $(V^2+A^2)$ with $\mathcal{T}_1$, and $(S^2-P^2)$ with $\mathcal{T}_9$. The Dirac spin coupling $(V^2-A^2)$ is of the homogeneous form $\bar{\psi}_R\psi_R\bar{\psi}_R\psi_R$; so it may be associated with any of the unitary spin couplings that are compatible with the requirements of crossing symmetry.

Having analyzed the unitary spin and Dirac spin aspects separately we now combine these factors and consider the over-all crossing symmetry of our complete four-field interaction. Under crossing the unitary spin coupling $\mathcal{T}_5$ is invariant and $\mathcal{T}_1\leftrightarrow\mathcal{T}_9$. Therefore, $\mathcal{T}_5$ must associate in the Lagrangian with the Fierz-antisymmetric Dirac coupling $(S^2+P^2-T^2)$. Similarly, $\mathcal{T}_1$ must associate with $(V^2+A^2)$, and $\mathcal{T}_9$, with $(S^2-P^2)$, so that they occur together in the combination $[(S^2-P^2)\mathcal{T}_9-\frac{1}{2}(V^2+A^2)\mathcal{T}_1]$. So far we have considered the fields as $c$ numbers. The over-all antisymmetry of these combined couplings will be cancelled by the anticommutation symmetry of the quantized field operators, so that our operator Lagrangian is crossing symmetric.[6] The Dirac coupling $(V^2-A^2)$ is Fierz antisymmetric, and so it may associated with $(\mathcal{T}_1+\mathcal{T}_9)$, $(\mathcal{T}_2+\mathcal{T}_7)$, $(\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_6+\mathcal{T}_8)$, and $\mathcal{T}_5$.

Finally we may write the most general quartic self-coupling of a unitary octet fermion field in an interaction Lagrangian that is invariant under Lorentz transformations and under the gauge transformations I, II, and IV, as

$$\begin{aligned}\mathcal{L}_{\rm int}=&\tfrac{1}{2}g[(S^2-P^2)\mathcal{T}_9-\tfrac{1}{2}(V^2+A^2)\mathcal{T}_1]\\&+\tfrac{1}{2}f(S^2+P^2-T^2)\mathcal{T}_5+\tfrac{1}{2}(V^2-A^2)[h_1(\mathcal{T}_1+\mathcal{T}_9)\\&+h_2(\mathcal{T}_2+\mathcal{T}_7)+h_3(\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_6+\mathcal{T}_8)],\end{aligned}\quad(2.13)$$

where $g$, $f$, $h_1$, $h_2$, $h_3$ are real coupling constants. If we further impose invariance under the simple $\gamma_5$ transformations, III, then the coupling constant $f$ in the interaction (2.13) must vanish. It turns out (in Sec. 3) that $f=0$ is incompatible with self-consistent baryon solutions with nondegenerate masses and therefore we will leave $f$ arbitrary here. As it is, the interaction Lagrangian (2.13) that we have finally arrived at corresponds exactly to a direct generalization of the Nambu–Jona-Lasinio model that we would have obtained by replacing the isospin character of their model with the unitary spin character of ours.

Consider now the case in which we set $h_2=h_3=0$ in the above Lagrangian. Unlike the two-component model of article II, our Lagrangian now admits a larger class of symmetries than I, II, and IV. $\mathcal{L}_{\rm int}$ is now invariant under transformations that correspond to real rotations in a real eight-dimensional space. Since this group $R_8$ has 28 generators and since such rotations may be separately applied to either the right- or left-handed components of $\psi$ there are 56 generators in the symmetry group of the Lagrangian, $(R_8)_L\otimes(R_8)_R$. The $SU_3$ symmetries II and IV are included as subgroups of this larger group.[7] We shall see the implications of this symmetry in Sec. 4.

It will be a convenience to be able to write our interaction Lagrangian in a more concise and explicit expression than Eq. (2.13). We express the unitary spin invariants $\mathcal{T}_l$ according to their definition in Eq. (2.2), and then rename the combinations of the coupling tensors $T(l)$ that are required in our model as

$$\begin{aligned}&K(1)\equiv T(1)\,,\quad K(3)\equiv T(5)\,,\\&K(2)\equiv T(9)\,,\quad K(4)\equiv T(2)+T(7)\,.\end{aligned}\quad(2.14)$$

The Dirac couplings we likewise label with a running index, $k=1, 2, 3, 4, 5$, corresponding to $S$, $V$, $T$, $A$, $P$, respectively:

$$\begin{aligned}&\Gamma^1=\Gamma^S\equiv 1\,,\quad \Gamma^4=\Gamma^A\equiv i\gamma_\mu\gamma_5\,,\\&\Gamma^2=\Gamma^V\equiv\gamma_\mu\,,\quad \Gamma^5=\Gamma^P\equiv\gamma_5\,.\\&\Gamma^3=\Gamma^T\equiv\sigma_{\mu\nu}\,,\end{aligned}\quad(2.15)$$

The coupling constants, when identified by the running indices of the $K(i)$ and the $\Gamma^k$, we express as $C_{ik}$ in the following matrix form:

$$\|C_{ik}\|=\begin{array}{c}\begin{matrix}S & V & T & A & P\end{matrix}\\ \begin{bmatrix}0 & (-\frac{1}{2}g+h_1+h_3) & 0 & (-\frac{1}{2}g-h_1-h_3) & 0\\ g & (h_1+h_3) & 0 & -(h_1+h_3) & -g\\ f & h_3 & -f & -h_3 & f\\ 0 & (h_2-h_3) & 0 & (h_3-h_2) & 0\end{bmatrix}\begin{matrix}K_1\\K_2\\K_3\\K_4\end{matrix}\end{array}.\quad(2.16)$$

We can then finally write the Lagrangian of our model explicitly as

$$\begin{aligned}&\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\rm int}\,,\\&\mathcal{L}_0=-\frac{1}{4}\left[\bar{\psi}^\alpha{}_\eta,\left(\gamma\frac{\partial\psi^\eta{}_\alpha}{\partial x}+m_0\psi^\eta{}_\alpha\right)\right]-\frac{1}{4}\left[\left(-\frac{\partial\bar{\psi}^\alpha{}_\eta}{\partial x}\gamma+m_0\bar{\psi}^\alpha{}_\eta\right),\psi^\eta{}_\alpha\right],\\&\mathcal{L}_{\rm int}=\tfrac{1}{8}C_{ik}[\bar{\psi}^\alpha{}_A,\Gamma^k\psi^\beta{}_B][\bar{\psi}^\gamma{}_C,\Gamma^k\psi^\delta{}_D]K(i)^{ABCD}{}_{\alpha\beta\gamma\delta}\,.\end{aligned}\quad(2.17)$$

[6] By crossing symmetry of the Lagrangian, we mean that it does not contain any terms whose matrix elements would vanish solely because of the symmetries under exchange of identical fields. See comments associated with Eq. (3.4).

[7] Similar symmetries for nonlinear spinor models have been considered by Marshak and Okubo, Nuovo Cimento **19**, 1226 (1961) and Ne'eman, Phys. Rev. Letters **13**, 769 (1964).

The commutators in our Lagrangian, Eq. (2.17), are just the usual symmetrization on the quantized field operators $\psi$ and $\bar{\psi}$ and are not to be confused with the *crossing* symmetries that we discussed above, which involved only exchanges of $\psi$ with $\psi$ and $\bar{\psi}$ with $\bar{\psi}$. We have included for now a bare mass $m_0$. Of course, unless $m_0=0$ our Lagrangian is not symmetric under the transformations III and IV.

### 3. SELF-CONSISTENT SUPERCONDUCTOR SOLUTIONS FOR THE BARYON MASSES

The dynamical equations of motion for the field operators $\psi(x)$ [and $\bar{\psi}(x)$] are the Euler-Lagrange equations derived from the principle that the action is stationary under variations of $\mathcal{L}$ with respect to $\bar{\psi}(x)$ [and $\psi(x)$, respectively]. [The variations $\delta\psi(x)$ and $\delta\bar{\psi}(x)$ anticommute with all fermion fields.[8]]

$$(\gamma\partial+m_0)\psi^L{}_\lambda=\tfrac{1}{4}C_{ik}\{\Gamma^k\psi^\beta{}_B[\bar{\psi}^\gamma{}_C,\Gamma^k\psi^\delta{}_D]K(i)^{LBCD}{}_{\lambda\beta\gamma\delta}+[\bar{\psi}^\alpha{}_A,\Gamma^k\psi^\beta{}_B]\Gamma^k\psi^\delta{}_D K(i)^{ABLD}{}_{\alpha\beta\lambda\delta}\}\,. \quad (3.1)$$

The problem is to find self-consistent solutions of the dynamical equations of motion for which there exist single-particle states that propagate freely with (physical) mass $m$:

$$(\gamma\partial+m)\langle|\psi(x)|1\rangle=0\,. \quad (3.2)$$

We assume that our procedure provides a self-consistent "physical" vacuum and Hilbert space in which we can take matrix elements. The vacuum arrived at in this program is neither unique nor invariant under the full symmetry group of the Lagrangian. Nambu pointed out that the transformations of this group are not proper operators in a single such Hilbert space, but rather they generate a continuum of such Hilbert spaces and are defined on this extended space. We will not discuss further here these problems of formal field theory, although they will be relevant later to our consideration of mesonic excitations. They are discussed by Nambu and Jona-Lasinio.[1]

To illustrate the method by which we treat the dynamical mass of the physical baryons, it is convenient to employ a simplification of our model that is unencumbered by unitary spin. If these degrees of freedom are completely suppressed for the time being in our model, then the equation of motion (3.1) is correspondingly reduced to a similar nonlinear field equation for a single field:

$$(\gamma\partial+m_0)\psi(x)=\tfrac{1}{4}C_k\{\Gamma^k\psi[\bar{\psi},\Gamma^k\psi]+[\bar{\psi},\Gamma^k\psi]\Gamma^k\psi\}\,. \quad (3.3)$$

Here $k$ still runs over the same $S^2$, $V^2$, $T^2$, $A^2$, $P^2$ Dirac couplings described in Eqs. (2.6) and Table II, with $C_S=(g'+f')$, $C_V=(h'-\frac{1}{2}g')$, $C_T=-f'$, $C_A=-(h'+\frac{1}{2}g')$, and $C_P=(f'-g')$.[9] The reduced interaction is thus still invariant under exchange of identical field operators, as we discussed in Sec. 2. This has the consequence that the two terms on the right-hand side of Eq. (3.3) are equivalent—all of their matrix elements are equal. If $m_0\equiv f'\equiv 0$, then the equation of motion (3.4) is also invariant under the group of chiral transformations

$$\psi\to e^{i\alpha\gamma_5}\psi\,,\quad \bar{\psi}\to\bar{\psi}e^{i\alpha\gamma_5}\,. \quad (3.4)$$

If we also set $h'=0$, then $C_S=-C_P=-\frac{1}{2}C_V=-\frac{1}{2}C_A=g'$ and our model coincides with Nambu and Jona-Lasinio's original model.

In the usual method of calculating the dynamical mass, the equation of motion is linearized by means of a lowest order Tamm-Dancoff, or Hartree-Fock approximation.[10] Applied to the dynamical field equation (3.3) this lowest order approximation gives

$$(\gamma\partial+m_0)\langle|\psi(x)|1\rangle=\tfrac{1}{2}C_k\langle|T\Gamma^k\psi(x)[\bar{\psi}(x),\Gamma^k\psi(x)]|1\rangle\,, \quad (3.5)$$

$$\simeq C_k\{\langle|T(\bar{\psi}\Gamma^k\psi)|\rangle+\Gamma^k\langle|T\psi\bar{\psi}|\rangle\}\Gamma^k\langle|\psi(x)|1\rangle\,. \quad (3.6)$$

Because the couplings have already been restricted in our interaction so that it is crossing symmetric under exchange of pairs of identical fields the two matrix elements on the right-hand side of Eq. (3.5) are equivalent, and so our self-consistency condition is that

$$m_0-m=2C_k\langle|T(\bar{\psi}(x)\Gamma^k\psi(x))|\rangle\Gamma^k=C_S\,\mathrm{Sp}(S'_F(x-x,m)) \quad (3.7)$$

have nontrivial solutions for $m$. The last equality is due to the Lorentz invariance of our theory. When the physical mass $m$ is considered to be generated completely by the dynamical interaction $\mathcal{L}_{\text{int}}$ and $m_0$ is equal to zero in the equation of motion (3.3), then the mass formula (3.7) becomes the one obtained by Nambu and Jona-Lasinio, which they solved by introducing a relativistically invariant cutoff.[11]

Rather than follow the usual derivation, we will formulate a general self-consistent mass equation which reduces directly to the usual mass formula (3.7) in a well-defined dynamical approximation.[12] We assume that the field theory of our model satisfies the usual spectral conditions so that a Lehmann spectral repre-

[8] See G. Källén, *Handbuch der Physik*, edited by S. Flügge (Springer Verlag, Berlin, 1956), Vol. 5; also S. S. Schweber, *An Introduction to Relativistic Field Theory* (Row Peterson and Company, Evanston, Illinois, 1961), Sec. 17a.

[9] The five coupling constants, $C_S$, $C_V$, $C_T$, $C_A$, $C_P$, are combinations of $g'$, $f'$, $h'$, which are the coefficients of the three Fierz-antisymmetric couplings.

[10] In the language of the interaction picture and Feynman diagrams, the approximation (3.6) neglects the connected diagrams in the right-hand side of Eq. (3.5), while in the language of the Heisenberg picture, it consists of ignoring all but the simplest (lightest) intermediate states. The first interpretation describes what we refer to as the Hartree-Fock approximation, and the latter we refer to as lowest order Tamm-Dancoff approximation. Here both of these approximation methods are equivalent in lowest order.

[11] Nambu and Jona-Lasinio (cf. Ref. 1) use the cutoff that we give in Eq. (3.46).

[12] A similar treatment has also been given by G. Pocsik, Nucl. Phys. 49, 286 (1964). We thank Professor Y. Nambu for bringing this work to our attention.

sentation exists for the exact propagators.[13] We explicitly assume local commutation relations.

From the dynamical equation of motion (3.3) we form the anticommutator of both sides with the field operator $\bar{\psi}(y)$:

$$(\gamma\partial+m_0)\{\psi(x),\bar{\psi}(y)\}=\tfrac{1}{4}C_k\{(\Gamma^k\psi(x)[\bar{\psi}(x),\Gamma^k\psi(x)]+[\bar{\psi}(x),\Gamma^k\psi(x)]\Gamma^k\psi(x)),\bar{\psi}(y)\}. \quad (3.8)$$

Taking the vacuum expectation value of the left-hand side of Eq. (3.8), expressing this in the Lehmann spectral representation, and putting $x_0=y_0$, we get

$$\langle|(\gamma\partial+m_0)\{\psi(x),\bar{\psi}(y)\}|\rangle|_{x_0=y_0}$$
$$=-i(\gamma\partial+m_0)S'(x-y,m)|_{x_0=y_0}=-i\int_0^\infty d\mu^2$$
$$\times[(m_0-\mu)\rho_1(\mu^2)+\rho_2(\mu^2)]i\gamma_4\delta^3(\mathbf{x}-\mathbf{y})|_{x_0=y_0}. \quad (3.9)$$

The right-hand side is reduced in this Lorentz frame by the canonical (anti) commutation relations to

$$\tfrac{1}{4}C_k\{(\Gamma^k\psi(x)[\bar{\psi}(x),\Gamma^k\psi(x)]+[\bar{\psi}(x),\Gamma^k\psi(x)]\Gamma^k\psi(x)),\bar{\psi}(y)\}|_{x_0=y_0}$$
$$=\tfrac{1}{2}C_k\gamma_4([\bar{\psi}(x),\Gamma^k\psi(x)]+\Gamma^k[\psi(x),\bar{\psi}(x)])\times\Gamma^k\delta^3(\mathbf{x}-\mathbf{y})|_{x_0=y_0}. \quad (3.10)$$

We take the vacuum expectation value of Eq. (3.10), replacing $\langle|[\bar{\psi},\Gamma\psi]|\rangle$ by $\langle|T[\bar{\psi},\Gamma\psi]|\rangle$, combine with Eq. (3.8), and cancel a factor $\gamma_4\delta^3(\mathbf{x}-\mathbf{y})$ from both sides. We then obtain

$$\int_0^\infty d\mu^2[(m_0-\mu)\rho_1(\mu^2)+\rho_2(\mu^2)]$$
$$=C_k\{\langle|T(\bar{\psi}(x)\Gamma^k\psi(x))|\rangle+\Gamma^k\langle|T\psi(x)\bar{\psi}(x)|\rangle\}\Gamma^k. \quad (3.11)$$

Since our couplings have already been symmetrized under exchange (crossing) of identical field operators, the right-hand side here condenses as in Eq. (3.7) into a single term:

$$2C_k\langle|T(\bar{\psi}(x)\Gamma^k\psi(x))|\rangle\Gamma^k=C_S\,\mathrm{Sp}(S'_F(x-x,m)) \quad (3.12)$$
$$=4C_S\int_0^\infty d\mu^2[-\mu\rho_1(\mu^2)+\rho_2(\mu^2)]\Delta_F(x-x,\mu^2). \quad (3.13)$$

The last two equalities here depend on Lorentz-invariance. Using Eq. (3.13) in the right-hand side of Eq. (3.11) and the normalization condition,[13]

$$1=\int_0^\infty d\mu^2\rho_1(\mu^2), \quad (3.14)$$

Eq. (3.11) can be rewritten as

$$m_0-m=-\int_0^\infty d\mu^2[(m-\mu)\rho_1(\mu^2)+\rho_2(\mu^2)]$$
$$+4C_S\int_0^\infty d\mu^2[-\mu\rho_1(\mu^2)+\rho_2(\mu^2)]\Delta_F(x-x,\mu^2). \quad (3.15)$$

The assumption of the existence of single-particle states which propagate freely with physical mass $m$ is now expressed as the assumption that the spectral weight function $\rho_1(\mu^2)$ has an isolated delta function at $\mu=m$,

$$\rho_1(\mu^2)=Z_2\delta(m-\mu)+\theta(\mu-\mu_0)\sigma_1(\mu^2); \quad (3.16)$$

and that there exist self-consistent solutions of Eq. (3.15) with such a $\rho_1(\mu^2)$. In Eq. (3.16) $\mu_0$ is the threshold of the continuous spectrum, $0<m<\mu_0$.

If $S'_F(x-x,m)$ is approximated by a single pole by setting $\sigma_1(\mu^2)=0$ and $Z_2=1$ in Eq. (3.16), then since $\rho_1(\mu^2)\geq\rho_2(\mu^2)\geq0$ for any $\mu$ as a consequence of the positive definiteness condition in our Hilbert space,[14] Eq. (3.15) reduces directly to the self-consistent mass formula (3.7) obtained by the usual approximation.

When we consider the model with unitary spin the mass operator becomes a matrix in unitary spin space, as we seek solutions for which the physical baryon masses are not all degenerate. The physical mass operator is diagonal between eigenstates of hypercharge $Y$, total isospin $T$, and its third component $T_Z$. We label the eight combinations $(T,T_Z,Y)$ that correspond to the physical baryons by the index $l$ which takes values $p, n, \Xi^-, \Xi^0, \Sigma^+, \Sigma^0, \Sigma^-, \Lambda$, and define projection operators $\mathcal{P}(l)$ for the corresponding states so that

$$(\bar{\psi}\mathcal{P}(n)\psi)=\bar{\psi}^A{}_\alpha\mathcal{P}^{\alpha\beta}{}_{AB}(n)\psi^B{}_\beta=\bar{n}n,\text{ etc.}, \quad (3.17)$$

where $n$ and $\bar{n}$ are the respective neutron components of the octet field operators $\psi$ and $\bar{\psi}$. These projection operators are normalized so as to preserve the normalization of the states:

$$\mathcal{P}^{\alpha\beta}{}_{\gamma\delta}(l)\mathcal{P}^{\delta\sigma}{}_{\beta\mu}(l')=\delta_{ll'}\mathcal{P}^{\alpha\sigma}{}_{\gamma\mu}(l), \quad (3.18)$$

also,

$$\mathcal{P}^{\alpha\beta}{}_{\gamma\delta}(l)\mathcal{P}^{\delta\gamma}{}_{\beta\alpha}(l')=\delta_{ll'}\quad\text{and}\quad 1=\mathcal{P}^{\alpha\beta}{}_{\beta\alpha}(l). \quad (3.19)$$

They also satisfy

$$\mathcal{P}^{\alpha\beta}{}_{\alpha\gamma}(l)=\mathcal{P}^{\alpha\gamma}{}_{\beta\gamma}(l)=0. \quad (3.20)$$

In terms of these projection operators the mass operator in our theory can be written

$$m^{\alpha\beta}{}_{\gamma\delta}=\mathcal{P}^{\alpha\beta}{}_{\gamma\delta}(l)m_l, \quad (3.21)$$

where $m_l$ is the numerical value of the physical mass.

For the solutions that we seek, in which $m$ is degener-

[13] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[14] For calculating numerical results, we will later introduce a cutoff procedure. Such a procedure is, of course, not consistent with the assumption of local anticommutation relations.

ate in the eigenvalue $T_Z$, it is convenient to work with combinations that correspond to these isomultiplets, which we label by $L$ for the four combinations $(I,Y)$: $L=N$, $\Xi$, $\Sigma$, $\Lambda$. The mass operator is then

$$m^{\alpha\beta}{}_{\gamma\delta}=P^{\alpha\beta}{}_{\gamma\delta}(L)m_L\,, \tag{3.22}$$

where $m_N=m_n=m_p$, $m_\Sigma=m_{\Sigma^+}=m_{\Sigma^-}=m_{\Sigma^0}$, etc., and

$$\begin{aligned} P(N)&=\mathcal{P}(n)+\mathcal{P}(p)\\ P(\Sigma)&=\mathcal{P}(\Sigma^+)+\mathcal{P}(\Sigma^-)+\mathcal{P}(\Sigma^0)\,,\text{ etc.}\end{aligned} \tag{3.23}$$

Because of Eqs. (3.18), (3.19), and (3.20) we now have

$$(\bar{\psi}P(N)\psi)=\bar{p}p+\bar{n}n\,,\text{ etc.,} \tag{3.24}$$

$$P^{\alpha\beta}{}_{\gamma\delta}(L)P^{\delta\sigma}{}_{\beta\mu}(L')=\delta_{LL'}P^{\alpha\sigma}{}_{\gamma\mu}(L)\,, \tag{3.25}$$

$$P^{\alpha\beta}{}_{\beta\alpha}(L)=W_L\,, \tag{3.26}$$

where $W_L$ is the multiplicity over which $m_L$ is degenerate, $W_N=W_\Xi=2$, $W_\Sigma=3$, $W_\Lambda=1$.

The baryon propagator in our theory is likewise a matrix operator in unitary spin space. It shares the same degeneracies and diagonalization in this space as the mass operator. They are related through

$$\begin{aligned}\langle|T\psi^\alpha{}_\gamma(x)\bar{\psi}^\beta{}_\delta(y)|\rangle&\equiv-\tfrac{1}{2}S'_F(x-y)^{\alpha\beta}{}_{\gamma\delta}\\ &=-\tfrac{1}{2}P^{\alpha\beta}{}_{\gamma\delta}(L)S'_F(x-y,\,L)\,,\end{aligned} \tag{3.27}$$

where $S'_F(x-y,\,L)$ is the usual propagator function for a single fermion with the Lehmann spectral weights ${}^L\rho_1$ and ${}^L\rho_2$.

It will also be convenient to analyze the mass operator in terms of definite irreducible representations of the unitary octet symmetry group, as was done for the couplings in Sec. 2. The states we seek are not eigenstates of the full unitary spin symmetry group and do not transform as basis elements of an irreducible representation. But they do constitute definite eigenstates of total isospin and hypercharge, both of which are still separately conserved. The mass, and likewise the propagator, has definite unitary spin transformation properties, and although it does not transform as an irreducible representation, it can be reduced to definite combinations of irreducible components:

$$(m-m_0)^{\alpha\beta}{}_{\gamma\delta}=\delta m_LP^{\alpha\beta}{}_{\gamma\delta}(L)=\delta M_JR^{\alpha\beta}{}_{\gamma\delta}(J)\,, \tag{3.28}$$

where $J$ labels the irreducible representation, $J=E$, $D$, $F$, $t$.[15] The operators $R(J)$ are related to the $P(L)$ by the Clebsch-Gordan coefficients $G_{JL}$:

$$\begin{aligned}R(J)&=G_{JL}P(L)\,,\\ P(L)&=G^{-1}{}_{LJ}R(J)\,.\end{aligned} \tag{3.29}$$

Due to the normalization for the $P(L)$, given by Eqs. (3.25) and (3.26), $G$ is not an orthogonal matrix in the usual manner, but rather

$$G^{-1}{}_{LJ}=G_{JL}W_L(\text{no sum on }L)\,. \tag{3.30}$$

$$\|G_{JL}\|=\begin{array}{c} \begin{array}{cccc}(N)&(\Xi)&(\Sigma)&(\Lambda)\end{array}\\ \begin{bmatrix}1/\sqrt{8}&1/\sqrt{8}&1/\sqrt{8}&1/\sqrt{8}\\ -1/2\sqrt{5}&-1/2\sqrt{5}&1/\sqrt{5}&-1/\sqrt{5}\\ -1/2&1/2&0&0\\ -(3/40)^{1/2}&-(3/40)^{1/2}&1/2(30)^{1/2}&3(3/40)^{1/2}\end{bmatrix}\end{array}\begin{array}{c}L/J\\(E)\\(D)\\(F)\\(t)\end{array} \tag{3.31}$$

$$\|G^{-1}{}_{LJ}\|=\begin{bmatrix}1/\sqrt{2}&-1/\sqrt{5}&-1&-(3/10)^{1/2}\\ 1/\sqrt{2}&-1/\sqrt{5}&1&-(3/10)^{1/2}\\ 3/2\sqrt{2}&3/\sqrt{5}&0&(3/40)^{1/2}\\ 1/2\sqrt{2}&-1/\sqrt{5}&0&3(3/40)^{1/2}\end{bmatrix}\,. \tag{3.32}$$

The $R(J)$ satisfy conditions similar to those of $P(L)$:

$$R^{\alpha\beta}{}_{\beta\alpha}(J)=1\,, \tag{3.33}$$

$$R^{\alpha\beta}{}_{\alpha\gamma}(J)=R^{\alpha\beta}{}_{\gamma\beta}(J)=0\,. \tag{3.34}$$

The masses are related by

$$m_L-m_0=G_{JL}\delta M_J\,. \tag{3.35}$$

We now have all the machinery we need to describe the breakdown of unitary spin symmetry in our model. We can proceed with the formulation of the dynamical mass equation in exact correspondence with the development in the simplified model. Starting with the full dynamical equation of motion (3.1), we arrive at the expression corresponding to Eq. (3.11) above

$$\begin{aligned}P^{HN}{}_{\eta\nu}(L)\int_0^\infty d\mu^2[(m_0-\mu)^L\rho_1(\mu^2)+{}^L\rho_2(\mu^2)]\\ =C_{ik}\{\langle|T(\bar{\psi}^\gamma{}_C(x)\Gamma^k\psi^\delta{}_D(x))|\rangle K(i)^{HNCD}{}_{\eta\nu\gamma\delta}\\ +\Gamma^k\langle|T\psi^\beta{}_B(x)\bar{\psi}^\gamma{}_C(x)|\rangle K(i)^{HBCN}{}_{\eta\beta\gamma\nu}\}\,.\end{aligned} \tag{3.36}$$

Generalizing Eqs. (3.12) and (3.13) in accordance with Eq. (3.27) for the propagator, we bring Eq. (3.36) into

[15] In each of these irreducible components $m$ and $S'_F$ transform only as the (unique) $T=Y=0$ elements, i.e., they are diagonal between eigenstates of total isospin and hypercharge. $T=Y=0$ elements occur in $E$, $D$, $F$, and $t$ representations, but not in $X$ nor $\bar{X}$; thus $J$ in Eq. (3.27) needs only to run over four values.

a form corresponding to Eq. (3.15) above for our self-consistent mass formula[16]:

$$(m_0-m_L)P^{HN}{}_{\eta\nu}(L)$$
$$=-P^{HN}{}_{\eta\nu}(L)\int_0^\infty d\mu^2[(m_L-\mu)^L\rho_1(\mu^2)+{}^L\rho_2(\mu^2)]$$
$$+4P^{\delta\gamma}{}_{DC}(L')C_{is}\int_0^\infty d\mu^2[-\mu^{L'}\rho_1(\mu^2)+{}^{L'}\rho_2(\mu^2)]$$
$$\times\Delta_F(x-x,\mu^2)K(i)^{HNCD}{}_{\eta\nu\gamma\delta}. \quad (3.37)$$

On the right-hand side of the mass formula (3.37) we have, due to Eq. (3.29):

$$P^{\delta\gamma}{}_{DC}(L')C_{is}K(i)^{HNCD}{}_{\eta\nu\gamma\delta}$$
$$=G^{-1}{}_{L'J}R^{\delta\gamma}{}_{DC}(J)C_{is}K(i)^{HNCD}{}_{\eta\nu\gamma\delta}; \quad (3.38)$$

and by Eqs. (2.14) and (2.16) this becomes

$$G^{-1}{}_{L'J}[(g+f)R^{HN}{}_{\eta\nu}(J)-2\delta_{FJ}fR^{HN}{}_{\eta\nu}(F)]. \quad (3.39)$$

If we now use the orthogonality conditions (3.25) and the Clebsch-Gordan transformation (3.29) and multiply both sides of Eq. (3.37) by $P^{\nu\eta}{}_{NH}(L)$, we get

$$m_0-m_L=-\int_0^\infty d\mu^2[(m_L-\mu)^L\rho_1(\mu^2)+{}^L\rho_2(\mu^2)]$$
$$+4[(g+f)\delta_{L'L}-2fG^{-1}{}_{L'F}G_{FL}]\int_0^\infty d\mu^2$$
$$\times[-\mu^{L'}\rho_1(\mu^2)+{}^{L'}\rho_2(\mu^2)]\Delta_F(x-x,\mu^2). \quad (3.40)$$

The index $F$ on $G$ refers here to the single representation, and is not summed over. The exact mass formula (3.38) cannot be solved further without assuming further information about the mass spectra, i.e., the ${}^L\rho_i(\mu^2)$'s. The lowest order dynamical approximation,

$${}^L\rho_1(\mu^2)=Z^L{}_2\delta(m_L-\mu)+{}^L\sigma_1(\mu^2)\to\delta(m_L-\mu), \quad (3.41)$$

which we discussed above gives

$$m_L-m_0=4[(g+f)\delta_{LL'}-2fG^{-1}{}_{L'F}G_{FL}]$$
$$\times m_{L'}\Delta_F(x-x,m_{L'}), \quad (\text{no sum on } F) \quad (3.42)$$

in which $\Delta_F(0,m_{L'})$ is assumed to be appropriately tempered to give finite results.[17] The difference in the form of the general mass Eq. (3.40) and the lowest order dynamical approximation Eq. (3.42) is only in the generality of the spectral weight functions, $\rho_1$ and $\rho_2$, and not in the unitary spin structure of the equations.

It is immediately obvious from our mass formulas (3.40) or (3.42) that solutions of the type we seek are incompatible with the simple $\gamma_5$ symmetry that would obtain by setting $m_0=f\equiv 0$. This would result in four uncoupled mass equations which are all identical and could have only degenerate mass solutions as long as only a single cutoff were used throughout.[18]

Writing the matrix equation (3.42) in detail as four coupled equations with the help of Eq. (3.31), we have

$$\begin{aligned}\tfrac{1}{4}(m_N-m_0)&=gm_N\Delta_F(0,m_N)+fm_\Xi\Delta_F(0,m_\Xi),\\ \tfrac{1}{4}(m_\Xi-m_0)&=gm_\Xi\Delta_F(0,m_\Xi)+fm_N\Delta_F(0,m_N),\\ \tfrac{1}{4}(m_\Sigma-m_0)&=(g+f)m_\Sigma\Delta_F(0,m_\Sigma),\\ \tfrac{1}{4}(m_\Lambda-m_0)&=(g+f)m_\Lambda\Delta_F(0,m_\Lambda).\end{aligned} \quad (3.43)$$

The last two of these equations have nontrivial solutions for which

$$m_\Sigma=m_\Lambda, \quad (3.44)$$

which in terms of Eq. (3.35) is equivalent to

$$\delta M_D=(\sqrt{2}/\sqrt{3})\delta M_{27}. \quad (3.45)$$

The degeneracy of the $\Sigma$ and $\Lambda$ mass equations (3.43) leaves us with only three equations instead of four. If we now solve these equations by explicitly introducing a relativistically invariant cutoff then, for the case with $m_0\equiv 0$, there are three parameters in our mass formulas that are at our disposal: the coupling parameters, $g$ and $f$, and the cutoff, $\kappa$. Using Nambu and Jona-Lasinio's cutoff procedure, we replace $m_L\Delta_F(0,m_L)$ in Eqs. (3.42) by

$$m\Delta_F(0,m)\to\frac{m\kappa^2}{8\pi^2}\left[1-\frac{m^2}{\kappa^2}\ln\left(1+\frac{\kappa^2}{m^2}\right)\right]$$
$$\equiv\tfrac{1}{4}\kappa^3F\left(\frac{m}{\kappa}\right). \quad (3.46)$$

All our equations (3.42) can then be written in terms of the dimensionless mass parameters $z_L\equiv m_L/\kappa$ and $F(z_L)$. Recombining them somewhat, we then have

$$z_\Sigma/F(z_\Sigma)=(z_N+z_\Xi)[F(z_N)+F(z_\Xi)]^{-1}, \quad (3.47)$$
$$=(g+f)\kappa^2, \quad (3.48)$$
$$(z_N-z_\Xi)[F(z_N)-F(z_\Xi)]^{-1}=(g-f)\kappa^2. \quad (3.49)$$

Equation (3.47) is an implicit formula for the cutoff parameter $\kappa$ in terms of the self-consistent mass solutions $m_L$. Having solved (3.47) for $\kappa$, Eqs. (3.48) and (3.49) are then explicit formulas for the coupling parameters.

Although the condition $m_\Sigma=m_\Lambda$ imposed by our model does not conform exactly with nature, such a baryon spectrum is still not entirely unreasonable, considering the simplicity of our model. Subject to this condition,

[16] Again, as in the simplified model, we have made use of the crossing symmetry of the interaction Lagrangian. See remarks after Eq. (3.11). Also as in the case of the simplified model, only the scalar Dirac coupling contributes because of the Lorentz invariance of $S'_F$ and $m$.

[17] See Ref. 11 above, and Eq. (3.46) below.

[18] Of course, if one were to introduce a separate cutoff parameter for each mass parameter determined, then nondegenerate mass solutions could still be obtained.

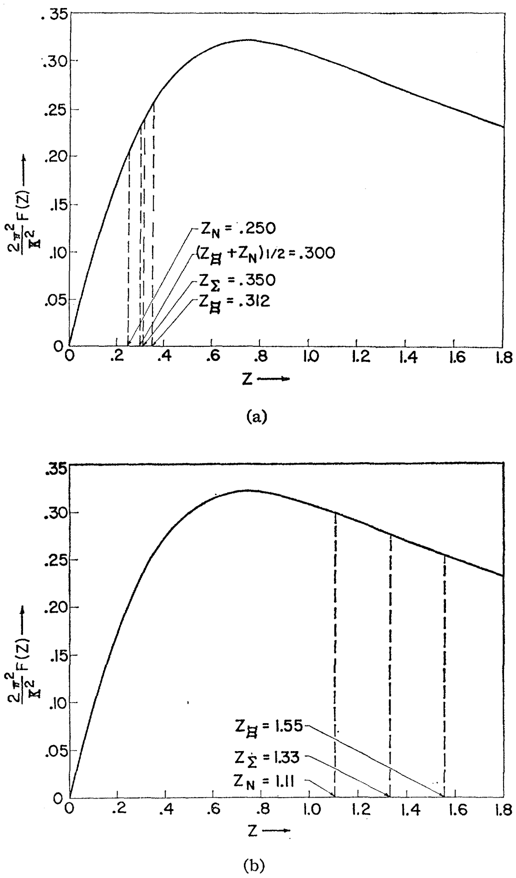


FIG. 1. Two specific graphical solutions of our self-consistent mass equations [(3.47)–(3.49)]: (a) displays the specific solution which is the best fit to the real baryon mass spectrum. This solution is described in Eqs. (3.50) to (3.52). (b) displays the specific solution which is the best fit to the real baryon mass spectrum, subject to the Gell-Mann-Okubo relation, as described in Eqs. (3.53) to (3.55). A graphical solution method is as follows: First, we plot the dimensionless function $2\pi^2/\kappa^2\, F(z)$, described in Eq. (3.46) against its dimensionless argument $z$. Second, we determine the cutoff $\kappa$ by finding the scale this curve must have in order that the three points, $z_N$, $z_\Xi$, $z_\Sigma$, lie on it. The points are related by the self-consistency equations (3.50) [for the solution shown in (a)] and (3.54) [for the solution shown in (b)]. Equation (3.47) requires that the line from the origin 0 to the point $\{z_\Sigma, F(z_\Sigma)2\pi^2/\kappa^2\}$ must pass through the point $\{\frac{1}{2}(z_\Xi+z_N), \frac{1}{2}[F(z_\Xi)+F(z_N)]2\pi^2/\kappa^2\}$, which is the median between points $\{z_\Xi, F(z_\Xi)2\pi^2/\kappa^2\}$ and $\{z_N, F(z_N)2\pi^2/\kappa^2\}$. The form of the graph of $F(z)2\pi^2/\kappa^2$ versus $z$ allows these conditions to be met uniquely as shown in (a), for the conditions (3.50), and in (b) for the conditions (3.54). In the case shown in (a) we find $\frac{1}{2}(z_N+z_\Xi)=0.33$; whence $\kappa=\frac{1}{2}(M_N+M_\Xi)/0.33=2.94M_{\text{av}}$. In the case shown in (b) we find $\frac{1}{2}(z_N+z_\Xi)=z_\Sigma=1.33$, whence $\kappa=1.33M_{\text{av}}$. In each case, putting these $\kappa$'s into Eqs. (3.48) and (3.49) gives Eqs. (3.51) and (3.52) and, respectively, (3.55). Note that the solution shown in (a) is in a steep section of the curve, so that it is not sensitive to the cutoff becoming large, while the solution shown in (b) does not have this property.

the most meaningful physical quantities to which we can fit our mass solutions are: the average of the eight real baryon masses, $M_{\text{av}}$; the splitting between the average $\Xi$ and $N$ masses, $M_\Xi - M_N$, and the average mass of the quartet of baryons of zero hyperchange, $\frac{1}{4}(3M_\Sigma+M_\Lambda)$. Our self-consistency condition is then that Eqs. (3.47)–(3.49) have solutions with

$$\begin{aligned} z_\Sigma\kappa &\equiv m_\Sigma = \tfrac{1}{4}(3M_\Sigma+M_\Lambda) = 1173\text{ MeV} \\ (z_\Xi - z_N)\kappa &\equiv m_\Xi - m_N = M_\Xi - M_N = 376\text{ MeV} \\ \tfrac{1}{2}(z_\Xi+z_N)\kappa &\equiv \tfrac{1}{2}(m_\Xi+m_N) = 2M_{\text{av}} \\ &\quad - \tfrac{1}{4}(3M_\Sigma+M_\Lambda) = 1127\text{ MeV}, \end{aligned} \tag{3.50}$$

where $m_\Sigma$, $m_\Xi$, $m_N$ are the parameters of our model and $M_\Sigma, M_\Xi, M_N$ are the numerical values of the corresponding real physical masses to which the $m_l$ are now equated. These solutions are illustrated in Fig. 1(a), where the abscissas, $z_\Sigma$, $z_\Xi$, $z_N$, related by Eqs. (3.50) and the ordinates $F(z_\Sigma)$, $F(z_\Xi)$, $F(z_N)$, related by Eq. (3.47), (3.49), are fitted with a curve of $F(z)$ versus $z$. The values of the coupling parameters determined by Eqs. (3.48) and (3.49) for this solution are

$$\begin{aligned} (\kappa^2/2\pi^2)(g+f) &\simeq 1.38, \\ (\kappa^2/2\pi^2)(g-f) &\simeq 2.29. \end{aligned} \tag{3.51}$$

The cutoff in this case is

$$\kappa = 2.94M_{\text{av}}. \tag{3.52}$$

It is also of interest to see what fit our model can give to the Gell-Mann–Okubo mass formula. For our purposes the Gell-Mann–Okubo mass formula is expressible as $\delta M_{27}=0$. This condition, plus the condition $m_\Sigma = m_\Lambda$, restricts the baryon mass spectrum to the form

$$m_\Sigma = m_\Lambda = \tfrac{1}{2}(m_N + m_\Xi). \tag{3.53}$$

If for our self-consistency condition we fit the parameters of our model to the real masses with the condition (3.53) so that

$$\begin{aligned} m_\Sigma = m_\Lambda = \tfrac{1}{2}(m_N+m_\Xi) &= M_{\text{av}} \\ m_\Xi - m_N &= M_\Xi - M_N, \end{aligned} \tag{3.54}$$

then the solutions (illustrated in Fig. 1b) are

$$\begin{aligned} (\kappa^2/2\pi^2)(g+f) &= 4.8, \\ (\kappa^2/2\pi^2)(g-f) &= -10, \\ \kappa &= 0.75M_{\text{av}}. \end{aligned} \tag{3.55}$$

So our model does have solutions which satisfy the Gell-Mann–Okubo mass formula, of which this is the most nearly realistic. However, we note that it would have been futile to impose this condition by setting equal to zero the strength parameter of the 27-type coupling in our interaction Lagrangian, $(g+f)=0$. Since the singlet and $D$-type octet coupling coefficient is required by the symmetry to be the same as that for the 27-type coupling, setting $(g+f)=0$ would allow solutions whose masses transform only as pure $F$-type unitary spin octet. Such a spectrum has negative masses: either $m_N$ or $m_\Xi$ is negative, and $m_\Sigma = m_\Lambda = 0$.

The general solutions of Eqs. (3.47)–(3.49) for various mass splittings and coupling constants external conditions are represented in Fig. 2.

The baryon mass solutions that we have considered

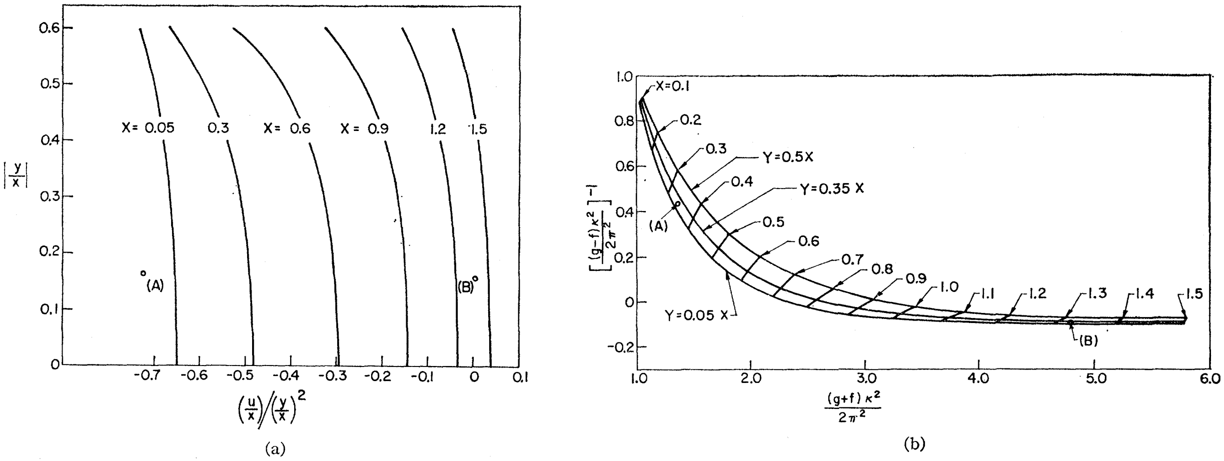


FIG. 2. Mapping of the general solutions of the mass Eqs. (3.43) using the cutoff procedure represented in Eq. (3.46). The mass variables in (3.48) have been parameterized here according to $z_N = x+y+u$, $z_\Xi = x-y+u$, $z_\Sigma = x-u$. In (a) are shown curves of $|y/x|$ versus $(y/x)/(u/x)^2$ for various fixed values of the parameter $x$. The cutoff parameter $\kappa$ and the coupling parameters $g$ and $f$ are implicit variables in the display in (a). In (b) the dependence of the coupling and cutoff parameters $g$, $f$, and $\kappa$ is shown for various values of the mass parameters $x$ and $y$. The parameter $u$ is an implicit variable in (b). The main point here is to exhibit the unique determination of the parameters $g$, $f$, $\kappa$ of our model in terms of the numerical values of the mass parameters $x$, $y$, $u$ (or equivalently, $z_N$, $z_\Xi$, $z_\Sigma$). The solution described in Eqs. (3.50) to (3.52), which represents the best fit to the real baryon mass spectrum, is designated on these curves by the points A. The solution described in Eqs. (3.53) to (3.55), which represents the best fit to the real baryon mass spectrum, subject to the Gell-Mann–Okubo relations, is designated on these curves by the points B.

determined only the coupling parameters $g$ and $f$, and the cutoff parameters $\kappa$. The symmetries that we have imposed on our model in Sec. 2 still allow three other arbitrary parameters in our model: the coupling parameters, $h_1$, $h_2$, and $h_3$. To say anything about these parameters we must consider vector or axial-vector states.[19] We include a brief treatment of these states in the following section, so that all the parameters of our model are ultimately fixed.

## 4. THE MESON STATES

In the accompanying article[3] we have exhibited the techniques by which models of this type may be treated self-consistently in lowest order dynamical approximation for the case of mass-splitting spontaneous symmetry breakdown. The model considered in the other article is a simpler analog of the one we consider here, the basic fermion field there being an isotopic doublet instead of a unitary octet. However, all the techniques of our dynamical approximation, and even the specific equations, may be taken over with only trivial substitutions from the isospin case to the unitary spin case.

In article II we have shown that the description of the meson states as simple chains of baryon-antibaryon *bubbles* is compatible with the lowest order dynamical approximation for the fermion masses. The compatibility of this treatment of the meson states with respect to the symmetries of our model has been shown there by confirming with explicit calculations the occurrence of the zero-mass Goldstone mesons[20] and the Goldberger-Treiman relations[21] that are predicted on more general grounds. Also, in article II we have shown that it is possible to vary continuously away from the solution of incomplete symmetry breakdown where the chiral symmetry, IV, but not the unitary spin symmetry, II, is spontaneously broken, to solutions of more complete spontaneous breakdown of both the chiral and the unitary symmetries. This allows us to solve for the meson states of our model with a particular choice of parameters which gives unitary spin degeneracy and then we can go to solutions having part or all the unitary spin degeneracy removed by continuation in the parameter which characterizes the unitary symmetry breakdown, i.e., the baryon mass splitting. That is, we can then treat the mass splittings of the mesons by perturbation theory and assume that they differ only nonqualitatively from the case of unitary spin degeneracy.

What meson states do we expect in our model? First, according to a theorem of Goldstone,[20] there will be spinless, massless mesons. The essential point of this theorem is that in order to conserve the current that is the generator of a certain group of gauge transformations under which the Lagrangian is symmetric, a specific set of zero-mass bosons must accompany the occurrence of symmetry-breaking, nonperturbative solutions. Each Goldstone meson will transform with the unitary spin quantum numbers of one of the generators of the spontaneously broken symmetry group.

[19] These parameters would also be determined by self-consistency conditions on the baryon mass solutions only if there were mass contributions which transformed as Lorentz vectors. This would represent a spontaneous breakdown of the Lorentz invariance of our model, and is not of interest to us at present.

[20] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962); S. Bludman and A. Klein, *ibid.* **131**, 236 (1963).

[21] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1478 (1958); Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).

According to the Goldstone theorem, then, there will occur in our model massless pseudoscalar mesons that are associated with the spontaneous breakdown of the chiral symmetry, IV, and the nonzero baryon masses. This octet is analogous to the isospin triplet of massless pions in the simple isospin model considered in article II.

There should also occur massless scalar mesons that transform as components of a unitary spin $F$-type octet. The scalar octet of Goldstone mesons are those associated with the spontaneous breakdown of the unitary spin symmetry, II. However, we are presently interested in solutions for which the unitary spin symmetry is broken only with respect to strangeness changing transformations, but is not broken with respect to the transformations of the isospin subgroup. For these solutions [cf. Eqs. (3.50)] only the four components of the scalar meson octet which have nonzero strangeness, the $K$- and $\bar{K}$-like, $Y=\pm 1$ isodoublets, are required by the Goldstone theorem to be present with zero mass. For the baryon mass solutions discussed in Sec. 2, which were split according to hypercharge, but not within isomultiplets, the Goldstone theorem says nothing about the four zero-strangeness scalar mesons, as to whether they exist or whether they are massless. However, we have shown in article II that as we pass among solutions, continuously reducing the baryon mass splittings to zero, all the components of the scalar meson octet will be present with zero mass in the limiting solution where each unitary multiplet, baryonic and mesonic, has been reduced to degeneracy.

As mentioned in Sec. 2, if $h_2=h_3=0$ our Lagrangian is invariant under $R_8$, a larger group of transformations than $SU_3$. This group has 28 generators. Ne'eman[22] has also considered a model involving the extension of the $SU_3$ octet symmetry into $R_8$ symmetry, and has pointed out that the $SU_3$ content of the generators of the $R_8$ algebra is

$$28=8\oplus 10\oplus \overline{10}. \tag{4.1}$$

We can now understand why it is that when $h_2=h_3=0$ the remainder of $\mathfrak{L}_{\text{int}}$ is invariant under $R_8$. Clearly, under equivalence transformations by members of $R_8$ symmetric eight-by-eight matrices go into symmetric matrices and antisymmetric matrices go into antisymmetric matrices. If the couplings $E^2$, $D^2$, $t^2$ are written in terms of eight-by-eight matrices we see that they must all be symmetric and the couplings $F^2$ and $(X\bar{X}+\bar{X}X)$ must be antisymmetric. The two sets of couplings $\{F^2,X\bar{X},X\bar{X}\}$ and $\{E^2,D^2,t^2\}$ transform into themselves under $R_8$. When $h_2=h_3=0$ we see by Eqs. (A.5) of the Appendix that the Lagrangian then has just the unitary spin composition

$$A(F^2+X\bar{X}+\bar{X}X)+B(E^2+D^2+t^2)$$

TABLE III. Couplings of external fields of definite unitary spin character to the baryons of our model. The third column lists the subsidiary conditions satisfied by the external fields. All indices shown in the table run over the values 1, 2, 3. $\epsilon_{\mu\sigma\rho}$ is the antisymmetric three-index tensor. For each unitary spin channel the Dirac-matrix part of the coupling, $\Gamma$, may be $S$, $P$, $V$, $T$, or $A$ [cf. Eq. (2.6)].

| Unitary spin channel | Coupling to external field | Subsidiary condition on external field |
|---|---|---|
| Singlet | $\delta\mathfrak{L}_E=\eta(\bar{\psi}^\sigma{}_\rho\Gamma\psi^\rho{}_\sigma)$ | |
| $F$-octet | $\delta\mathfrak{L}_F=\eta^\alpha{}_\beta(\bar{\psi}^\sigma{}_\alpha\Gamma\psi^\beta{}_\sigma-\bar{\psi}^\beta{}_\sigma\Gamma\psi^\sigma{}_\alpha)$ | |
| $D$-octet | $\delta\mathfrak{L}_D=\eta^\alpha{}_\beta(\bar{\psi}^\sigma{}_\alpha\Gamma\psi^\beta{}_\sigma+\bar{\psi}^\beta{}_\sigma\Gamma\psi^\sigma{}_\alpha)$ | $\eta^\sigma{}_\sigma=0$ |
| Decimet | $\delta\mathfrak{L}_X=\eta^{\alpha\beta\gamma}\epsilon_{\gamma\sigma\rho}(\bar{\psi}^\sigma{}_\alpha\Gamma\psi^\rho{}_\beta)$ | $\eta^{\alpha\beta\gamma}$ symmetric in $\alpha$, $\beta$, $\gamma$ |
| Tensor | $\delta\mathfrak{L}_t=\eta^{\alpha\beta}{}_{\gamma\delta}(\bar{\psi}^\gamma{}_\alpha\Gamma\psi^\delta{}_\beta)$ | $\eta^{\alpha\beta}{}_{\alpha\delta}=0$; $\eta^{\alpha\beta}{}_{\gamma\delta}$ symmetric in $\alpha$, $\beta$ and symmetric in $\gamma$, $\delta$ |

and is invariant under $(R_8)_L\otimes(R_8)_R$. It is also clear from this table that when either $h_2\neq 0$ or $h_3\neq 0$ there is no $R_8$ invariance. Only if $h_2=h_3=0$ does the Goldstone theorem predict 28 massless pseudoscalar mesons, an $F$-octet, a 10, and a $\overline{10}$. In this case the theorem also predicts massless scalar mesons which are the components of nonzero strangeness of an $F$-octet, a 10, and a $\overline{10}$. In the limiting solution with degenerate baryon masses all 28 scalar mesons would be present with zero mass.

To find a meson of a specified unitary spin character, we consider the response of our system to an external probe field having the appropriate unitary spin transformation properties, and look for freely propagating collective excitations. This is the method used in article II for finding mesons of given isotopic spin ($T=0$, or 1). The interactions between these external fields and the $\psi$ field are shown in Table III. The baryon propagator used for computing the bubble matrix (cf. II) simplifies considerably when the baryon masses are all degenerate (and equal to $m$). In this case the general form, Eq. (3.27), reduces to

$$\langle|T\psi^\alpha{}_\gamma(x)\bar{\psi}^\beta{}_\delta(y)|\rangle$$
$$=(\delta^\alpha{}_\delta\delta^\beta{}_\gamma-\tfrac{1}{3}\delta^\alpha{}_\gamma\delta^\beta{}_\delta)[-\tfrac{1}{2}S'_F(x-y,m)]. \tag{4.2}$$

Using the interaction Lagrangian (2.13) with the $\mathcal{T}_i$ written in terms of four Kronecker $\delta$'s [see Eq. (2.2)], the propagator 4.2, and the couplings of Table III, it is straightforward to compute the bubble matrix elements for each type of meson. The analogous calculation has been thoroughly explained in article II. Here the kinematics and Dirac algebra are the same as in the simplified model of article II; the only difference comes from the unitary spin couplings $\mathcal{T}_i$ in the interaction Lagrangian that replace the corresponding isospin factors in the simplified model. Because the calculations for the bubble approximation are essentially the same in the $SU_2$ and the $SU_3$ models, we shall avoid tedious repetition and simply summarize our results for the meson states in the $SU_3$ model.

[22] Y. Ne'eman, Phys. Rev. Letters 13, 769 (1964). We observe with Ne'eman that the 28 generators represented in Eq. (4.1) correspond to the antisymmetric (generator-like) part of the reduction of $8\otimes 8$ in $SU_3$.

TABLE IV. Coupling parameters to be substituted into the Eqs. (4.3) to (4.10) for determining the possible presence of meson states in the various channels. The columns labeled $T=0$ and $T=1$ refer to the $SU_2$ model discussed in article II and have been included here to facilitate comparison with the explicit treatment of meson states given there.

| Spin and parity \ Unitary spin | | $F$ | $E$ | $D$ | $X$ | $t$ | $T=0$ | $T=1$ |
|---|---|---|---|---|---|---|---|---|
| $0^-$ | $R=$ | $-\frac{1}{4}(g+f)$ | $-\frac{1}{4}(g-f)$ | $-\frac{1}{4}(g-f)$ | $-\frac{1}{4}(g+f)$ | $-\frac{1}{4}(g-f)$ | $-\frac{1}{2}(\alpha-\beta)$ | $-\frac{1}{2}(\alpha+\beta)$ |
| | $Q=$ | $\frac{1}{4}(-h_1-3h_2+3h_3)$ | $-g-(9/4)h_1-\frac{4}{3}h_2-(7/6)h_3$ | $-\frac{1}{4}[h_1-(5/3)h_2-\frac{1}{3}h_3]$ | $-\frac{1}{4}h_1$ | $-\frac{1}{4}h_1-\frac{1}{2}h_3$ | $\frac{1}{2}(3h-\alpha)$ | $\frac{1}{2}h$ |
| $0^+$ | $R=$ | $\frac{1}{4}(g-f)$ | $\frac{1}{4}(g+f)$ | $\frac{1}{4}(g+f)$ | $\frac{1}{4}(g-f)$ | $\frac{1}{4}(g+f)$ | $\frac{1}{2}(\alpha+\beta)$ | $\frac{1}{2}(\alpha-\beta)$ |
| | $Q=$ | $\frac{1}{4}(h_1+3h_2-3h_3)$ | $-g+(9/4)h_1+\frac{4}{3}h_2+(7/6)h_3$ | $\frac{1}{4}[h_1+(5/3)h_2+\frac{1}{3}h_3]$ | $\frac{1}{4}h_1$ | $\frac{1}{4}h_1+\frac{1}{2}h_3$ | $-\frac{1}{2}(3h+\alpha)$ | $-\frac{1}{2}h$ |
| $1^-$ | $R=$ | $\frac{1}{4}(h_1+3h_2-3h_3)$ | $-g+(9/4)h_1+\frac{4}{3}h_2+(7/6)h_3$ | $\frac{1}{4}[h_1+(5/3)h_2+\frac{1}{3}h_3]$ | $\frac{1}{4}h_1$ | $\frac{1}{4}h_1+\frac{1}{2}h_3$ | $-\frac{1}{2}(3h+\alpha)$ | $-\frac{1}{2}h$ |
| | $Q=$ | $\frac{1}{4}f$ | $-\frac{1}{4}f$ | $-\frac{1}{4}f$ | $\frac{1}{4}f$ | $-\frac{1}{4}f$ | $\frac{1}{6}\beta$ | $-\frac{1}{6}\beta$ |
| $1^+$ | $R=$ | $-\frac{1}{4}(h_1+3h_2-3h_3)$ | $-g-(9/4)h_1-\frac{4}{3}h_2-(7/6)h_3$ | $-\frac{1}{4}[h_1+(5/3)h_2+\frac{1}{3}h_3]$ | $-\frac{1}{4}h_1$ | $-\frac{1}{4}h_1-\frac{1}{2}h_3$ | $\frac{1}{2}(3h-\alpha)$ | $\frac{1}{2}h$ |
| | $Q=$ | $\frac{1}{4}f$ | $-\frac{1}{4}f$ | $-\frac{1}{4}f$ | $\frac{1}{4}f$ | $-\frac{1}{4}f$ | $\frac{1}{6}\beta$ | $-\frac{1}{6}\beta$ |

For the $J^P=0^-$ meson channels, the relevant part of the Lagrangian can be written as

$$\mathcal{L}(0^-)=R(\bar{\psi}\gamma_5\psi)^2+Q(\bar{\psi}i\gamma_5\gamma_\mu\psi)^2, \tag{4.3}$$

where $R$ and $Q$ are coupling parameters given in Table IV for each unitary spin channel. In the bubble approximation the pseudoscalar meson masses are determined as solutions $q^2=-\mu^2$ of the equation

$$[1-2RI(q^2)][1+(8m^2Q/q^2)(I(q^2)-I(0))] +(16RQm^2/q^2)[I(q^2)-I(0)]^2=0. \tag{4.4}$$

$I(q^2)$ is a function of $q^2$ which depends only on the dynamics of the bubble approximation, the degenerate baryon mass $m$ and not on unitary spin factors. This function also appears in the same way in the calculation of the mesons in the $SU_2$ model of article II and is described there in Eq. (18). Since an equation of the same form as Eq. (4.4) determines the masses of the $J^P=0^-$ mesons in the $SU_2$ model, we have included for comparison $R$ and $Q$ for the $SU_2$ model in Table IV.

For the $J^P=0^+$ states the relevant part of the interaction Lagrangian is

$$\mathcal{L}(0^+)=R(\bar{\psi}\psi)^2+Q(\bar{\psi}\gamma_\mu\psi)^2. \tag{4.5}$$

The scalar meson masses are determined as $q^2=-\mu^2$ solutions of

$$1-2RJ(q^2)=0. \tag{4.6}$$

$J(q^2)$ is a function of $q^2$ depending only on the dynamics of the bubble approximation, and is defined in Eq. (47) of II. The vector and axial-vector states follow analogously:

$$\mathcal{L}(1^-)=R(\bar{\psi}\gamma_\mu\psi)^2+Q\tfrac{1}{2}(\bar{\psi}\sigma_{\mu\nu}\psi)^2, \tag{4.7}$$

$$\left\{1-\tfrac{2}{3}R\left[2I(q^2)-\frac{4m^2}{q^2}(I(q^2)-I(0))\right]\right\} \times\left\{1-\tfrac{2}{3}Q\left[I(q^2)-\frac{8m^2}{q^2}(I(q^2)-I(0))\right]\right\} +16RQ\frac{m^2}{q^2}[I(q^2)-I(0)]^2=0 \tag{4.8}$$

and

$$\mathcal{L}(1^+)=R(\bar{\psi}i\gamma_\mu\gamma_5\psi)^2+Q\tfrac{1}{2}(\bar{\psi}\sigma_{\mu\nu}\psi)^2, \tag{4.9}$$

$$[1-\tfrac{4}{3}RJ(q^2)][1-\tfrac{2}{3}QJ(q^2)]=0. \tag{4.10}$$

First we consider the massless mesons. Using the baryon mass equation (3.43), we find

$$I(q^2=0)=-8\Delta_F(0,m)=-2/(g+f) \tag{4.11}$$

$$J(q^2=0)=8(\partial/\partial m)\Delta_F(0,m)=2/(g-f), \tag{4.12}$$

which are analogous to Eqs. (21) and (49) of II and are derived in exactly the same manner. From Table IV and Eqs. (4.4) and (4.6) we see that there are massless $F$-type octets of $J^P=0^-$ and $J^P=0^+$ mesons. We also see from Table IV that there will be massless deciments of $J^P=0^+$ and $J^P=0^-$ mesons. Since the meson states are self-conjugate both the 10 and $\overline{10}$ channels have zero-mass $J^P=0^+$ and similarly $J^P=0^-$ meson states. Why do these mesons occur? They are Goldstone bosons only if $h_2=h_3=0$, as discussed above. They occur here because the $h$ terms don't contribute to the approximate boson mass equations when $q^2=0$ [cf. Eqs. (4.4), (4.6), and Table IV]. But since the 10 and $\overline{10}$ are Goldstone bosons for certain choices of $h$, they must appear as solutions of these equations for all $h$. They are therefore a result of our lowest order dynamical approximation rather than any exact symmetry. Presumably, in a better approximation, they would not have zero mass if either $h_2$ or $h_3$ were not zero.

Next, we shall look for solutions of the Eqs. (4.3)–(4.9) with small mesonic masses, $\mu^2 \ll m^2$, and ignore other (possible) collective states. In this approximation we shall take $I(q^2)\simeq I(0)+q^2I'(0)$ and $J(q^2)\simeq J(0)+q^2J'(0)$, where with the use of Eq. (73) of article II

$$-J'(0)=I'(0)=\frac{\partial I(q^2)}{\partial q^2}\bigg|_{q^2=0}=-\frac{I(0)+J(0)}{4m^2}. \quad (4.13)$$

This is the analog of Eq. (63) of II. It shows that in this approximation the mesons are determined by the coupling parameters $f$, $g$, $h_1$, $h_2$, $h_3$.[23] In order to consider mesons of higher mass we would have to know more than just the values and slopes of the functions $I$ and $J$ at $q^2=0$. These details depend on the dynamics and the cutoff procedure and would lead us beyond results that are primarily due to the symmetry structure of the model. Once we specify $h_1$, $h_2$, and $h_3$ we can use Table IV and Eqs. (4.4)–(4.10) to find if any low-mass state is present in any given channel.

For example, we consider the case for which $h_1=h_3=0$, and $h_2=-0.1m^{-2}$ is determined so as to give a $J^P=1^-$ $F$-type octet of mass $\mu=\frac{1}{2}m$. It is interesting to note in this case that except for the massless mesons discussed above, none of the remaining channels shows a meson of low mass. We have summarized the results for this case in Table V. As in the $SU_2$ model, the use of local four-fermion couplings in the bubble approximation restricts us to mesons of spin 0 and 1. The results in Table V are therefore complete and there are no other meson states to consider in this approximation. There are, of course, many other ways to complete the model in which the $h_1$, $h_2$, $h_3$ are chosen by other criteria.

TABLE V. Results of a search for the existence of meson states of small mass for the simplified case in which we have arbitrarily set the parameters of our Lagrangian $h_1$ and $h_3$ equal to zero. $Z$ denotes a meson of zero mass; $U$ indicates that the presence of a small-mass meson is not found with the power-series expansion technique; $\frac{1}{2}$ means that the mass of vector meson octet has been set equal to one-half the (degenerate) baryon mass, just as the baryon masses also were given their value(s) as external conditions.

| Spin and parity \ Unitary spin | $F$ | $E$ | $D$ | $X$ | $t$ | $T=0$ | $T=1$ |
|---|---|---|---|---|---|---|---|
| $0^-$ | $Z$ | $U$ | $U$ | $Z$ | $U$ | $U$ | $Z$ |
| $0^+$ | $Z$ | $U$ | $U$ | $Z$ | $U$ | $U$ | $Z$ |
| $1^-$ | $\frac{1}{2}$ | $U$ | $U$ | $U$ | $U$ | $U$ | $\frac{1}{2}$ |
| $1^+$ | $U$ | $U$ | $U$ | $U$ | $U$ | $U$ | $U$ |

[23] Alternatively, we can take the cutoff $\kappa$, baryon mass $m$, and $h_1$, $h_2$, $h_3$ as the independent parameters of Eqs. (4.10), (4.11), and (3.46).

## 5. SUMMARY AND CONCLUSIONS

In a specific dynamical model we have investigated several of the recent proposals concerning higher symmetries. We have assumed the most general four-fermion interaction Lagrangian that is invariant under crossing and chiral unitary transformations on a fundamental octet field $\psi$. We have obtained lowest order solutions which fit the observed physical baryon masses to within $\sim 6\%$, although they do not satisfy the Gell-Mann–Okubo mass formula.

Freund and Nambu[4] have indicated that spontaneous breakdown of the chiral octet symmetry which is present in our model must lead to the Gell-Mann–Okubo mass relations. Our solutions involve mixings of the octet currents with 10, $\overline{10}$, and 27 type currents which are not small compared to the baryon mass differences, as was assumed for the solutions considered by Freund and Nambu. Specificially, the analogous situation for the $SU_2$ model of article II is the mixing of the $T=0$ and $T=1$ channels, and Eq. (46) of II shows that this mixing is proportional to the mass splittings. When the mixing of the octets with 10's, $\overline{10}$'s, and 27's in the meson channels is as important as in our solutions then there are more form factors that must be considered in the generalized Goldberger-Treiman relations than were used by Freund and Nambu. In our case then there are too many free parameters to obtain mass and coupling constant relations by their method.

All the baryon and meson states which can occur in our model are determined in terms of a few parameters $(g,f,h_1,h_2,h_3)$. We have shown that solutions in the lowest order dynamical approximation are completely described with two functions, $I(q^2)$ and $J(q^2)$, in addition to these parameters. If we confine ourselves to low-mass meson states, we need know only the slope and value at $q^2=0$ of these two functions. The parameters of our model can be chosen so as to give the vector meson octet with the mean mass which is observed.

The occurrence of the zero-mass mesons is a consequence of the spontaneous breakdown of *exact* symmetries. If these symmetries were not exact in the original Lagrangian, but were slightly broken by a small asymmetric bare mass of the form of Eq. (3.22), then these meson states might no longer occur in our model with zero physical masses. Using techniques similar to those of Nambu and Jona-Lasinio we can show that a unitary singlet component of bare fermion mass amounting to only a few ($\sim$2–4)MeV results in a mean mass for the pseudoscalar meson octet that is about that observed in nature. We have remarked above that the mean mass of the pseudoscalar octet is associated with the breakdown of the chiral symmetry, IV, and the mass of the scalar octet is associated with the breakdown of the unitary spin symmetry, II. By analogy with calculations for the simplified model of article II, we estimate that the introduction of splittings in the bare fermion mass results in nonzero masses for the scalar mesons $\mu_S^2$ that are related to the pseudoscalar meson masses $\mu_P^2$ by

$$\frac{\mu_S^2}{\mu_P^2} \simeq \frac{m}{\delta m}\left(\frac{\delta m_0}{m_0}\right)\left(\frac{g+f}{g-f}\right), \tag{5.1}$$

where $m$ and $m_0$ are the mean physical and bare masses, respectively, of the baryons and $\delta m$ and $\delta m_0$ are representative of the corresponding mass-splittings. If the chiral symmetry breakdown is mainly spontaneous, $m/m_0 \gg 1$, but the unitary spin breakdown is mainly intrinsic, so that $m/m_0 \gg \delta m/\delta m_0$, then the scalar meson masses could turn out to be much larger than those of the pseudoscalar mesons.

When the pseudoscalar and scalar meson masses are nonzero then the bubble approximation by which the mass is determined will involve the coupling coefficients $h$ in the Lagrangian Eq. (2.13). As noted above, this part of the Lagrangian does not have the $R_8$ symmetry which the $f,g$ terms do. Therefore when the bare mass terms are added to the Lagrangian the relative shifts of the octets and 10's will depend on the choice of parameters and details of the dynamics. It is quite possible that in this case the decimet mass becomes very much larger than the octet mass. The bare masses might represent the approximate effects of some weak couplings of the fundamental field to other fields which do not have the symmetries of our Lagrangian.[24]

*Note added in proof.* We are grateful to Ján Pišút for calling our attention to errors in Eqs. (A4) and (A5) as they appeared in the preprint form of this article. These changes do not affect the conclusions of our work.

## APPENDIX

### A 3×3 Working Representation of Unitary Spin

In practical calculations it is convenient to have in mind a definite representation of the unitary spin functions that occur in our model. An explicit example of a 3×3 working representation of the unitary spin octets, $\psi$ and $\bar{\psi}$, is:

$$\psi=\begin{bmatrix} \left(\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right) & \Sigma^+ & p \\ \Sigma^- & \left(-\frac{\Sigma^0}{\sqrt{2}}+\frac{\Lambda}{\sqrt{6}}\right) & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{bmatrix}, \quad \bar{\psi}=\begin{bmatrix} \left(\frac{\bar{\Sigma}^0}{\sqrt{2}}+\frac{\bar{\Lambda}}{\sqrt{6}}\right) & \bar{\Sigma}^+ & \bar{\Xi}^+ \\ \bar{\Sigma}^- & \left(-\frac{\bar{\Sigma}^0}{\sqrt{2}}+\frac{\bar{\Lambda}}{\sqrt{6}}\right) & \bar{\Xi}^0 \\ \bar{p} & \bar{n} & -\frac{2\bar{\Lambda}}{\sqrt{6}} \end{bmatrix}. \tag{A1}$$

Similar matrix representations of the pseudoscalar and vector meson octets can be obtained by substituting $(\pi,K,\bar{K},\eta)$ or $(\rho,K^*,\bar{K}^*,\omega)$, respectively, for $(\Sigma,N,\Xi,\Lambda)$ in the $\psi$ matrix. A corresponding 3×3 matrix representation of the eight operators $\boldsymbol{\lambda}$ which generate the unitary spin transformations of $SU_3$ on these $\psi$'s, is

$$\lambda_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\quad \lambda_2=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix},\quad \lambda_3=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix},\quad \lambda_4=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},$$
$$\lambda_5=\begin{bmatrix}0&0&-i\\0&0&0\\i&0&0\end{bmatrix},\quad \lambda_6=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad \lambda_7=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\quad \lambda_8=\frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}. \tag{A2}$$

The two diagonal matrices $\lambda_3$ and $\lambda_8$ are, respectively, proportional to the third component of isospin $I_3$ and hypercharge $Y$.

### The Irreducible Tensorial Representation of Unitary Spin

When the transformations of $SU_3$ are represented by 3×3 matrices (unitary and unimodular), as in (A2), the tensorial representatives of unitary octet symmetry are the direct tensor products of the octet representatives $A^m{}_k(k, m=1, 2, 3)$. The irreducible tensorial representatives of unitary octet symmetry are the direct tensor products which have definite symmetries under permutations among the covariant or contravariant indices, and are traceless under contraction of any covariant

[24] This idea was suggested by Nambu and Jona-Lasinio. Cf. Ref. 1.

index with a contravariant index, $A^{\alpha}{}_{k}\cdots A^{m}{}_{\alpha}\cdots=0$.[25] Symmetrizing and contracting in all possible ways the direct tensor product of the pair of octets $A^{\alpha}{}_{\gamma}B^{\beta}{}_{\delta}$, this prescription reduces the product into the irreducible tensorial representatives of 1, 8, 8′, 10, $\overline{10}$, 27 dimensions, respectively:

$$
\begin{aligned}
E&=(1/8^{1/2})(AB)\equiv(1/8^{1/2})(A^{\sigma}{}_{\mu}B^{\mu}{}_{\sigma})\,,\\
D^{\alpha}{}_{\beta}&=(3/10)^{1/2}(A^{\sigma}{}_{\beta}B^{\alpha}{}_{\sigma}+B^{\sigma}{}_{\beta}A^{\alpha}{}_{\sigma})-(2/15)^{1/2}(AB)\delta^{\alpha}{}_{\beta}\,,\\
F^{\alpha}{}_{\beta}&=(1/6^{1/2})(A^{\sigma}{}_{\beta}B^{\alpha}{}_{\sigma}-B^{\sigma}{}_{\beta}A^{\alpha}{}_{\sigma})\,,\\
\left.\begin{array}{l}X^{\alpha\beta}{}_{\gamma\delta}\\ \\ \bar{X}^{\alpha\beta}{}_{\gamma\delta}\end{array}\right\}&=\left\{\begin{array}{l}-(1/4)(A^{\alpha}{}_{\gamma}B^{\beta}{}_{\delta}\pm A^{\alpha}{}_{\delta}B^{\beta}{}_{\gamma}\mp A^{\beta}{}_{\gamma}B^{\alpha}{}_{\delta}-A^{\beta}{}_{\delta}B^{\alpha}{}_{\gamma})\\
-(1/12)[(A^{\sigma}{}_{\delta}B^{\beta}{}_{\sigma}-A^{\beta}{}_{\sigma}B^{\sigma}{}_{\delta})\delta^{\alpha}{}_{\gamma}\mp(A^{\sigma}{}_{\delta}B^{\alpha}{}_{\sigma}-A^{\alpha}{}_{\sigma}B^{\sigma}{}_{\delta})\delta^{\beta}{}_{\gamma}\\
\qquad-(A^{\sigma}{}_{\gamma}B^{\alpha}{}_{\sigma}-A^{\alpha}{}_{\sigma}B^{\sigma}{}_{\gamma})\delta^{\beta}{}_{\delta}\pm(A^{\sigma}{}_{\gamma}B^{\beta}{}_{\sigma}-A^{\beta}{}_{\sigma}B^{\sigma}{}_{\gamma})\delta^{\alpha}{}_{\delta}]\,,\end{array}\right.\\
t^{\alpha\beta}{}_{\gamma\delta}&=(1/4)(A^{\alpha}{}_{\gamma}B^{\beta}{}_{\delta}+A^{\alpha}{}_{\delta}B^{\beta}{}_{\gamma}+A^{\beta}{}_{\gamma}B^{\alpha}{}_{\delta}+A^{\beta}{}_{\delta}B^{\alpha}{}_{\gamma})+(1/40)(AB)(\delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta}-\delta^{\alpha}{}_{\delta}\delta^{\beta}{}_{\gamma})\\
&\quad-(1/20)[(A^{\sigma}{}_{\delta}B^{\beta}{}_{\sigma}+A^{\beta}{}_{\sigma}B^{\sigma}{}_{\delta})\delta^{\alpha}{}_{\gamma}+(A^{\sigma}{}_{\delta}B^{\alpha}{}_{\sigma}+A^{\alpha}{}_{\sigma}B^{\sigma}{}_{\delta})\delta^{\beta}{}_{\gamma}\\
&\quad+(A^{\sigma}{}_{\gamma}B^{\alpha}{}_{\sigma}+A^{\alpha}{}_{\sigma}B^{\sigma}{}_{\gamma})\delta^{\beta}{}_{\delta}+(A^{\sigma}{}_{\gamma}B^{\beta}{}_{\sigma}+A^{\beta}{}_{\sigma}B^{\sigma}{}_{\gamma})\delta^{\alpha}{}_{\delta}]\,.
\end{aligned}
\tag{A3}
$$

The over-all normalization convention here is the same as that used by Neville (cf. Ref. 25).

The invariant couplings of four unitary spin octets can be formed by fully contracting pairs of the irreducible tensors given in Eq. (A3). For example, to contract two $t$-type tensors, $(t^{\alpha\beta}{}_{\gamma\delta})(t^{\alpha\beta}{}_{\gamma\delta})^{\dagger}=t^{\alpha\beta}{}_{\gamma\delta}t^{\delta\gamma}{}_{\beta\alpha}\equiv t^2$. When the octets combined in the first quadratic tensor are $AB$, and those in the other factor are $(CD)^{\dagger}$, then the quartic invariants are of the form $(AB)(CD)$ and $(ABCD)$. In terms of the couplings $\mathcal{T}_i$, that we have already discussed in Sec. 2, the invariants that couple irreducible quadratic tensor representations of unitary octet symmetry, Eq. (A3), are

$$
\begin{aligned}
E^2&=(1/8)\mathcal{T}_{1\gamma}\,,\\
D^2&=(3/10)(\mathcal{T}_2+\mathcal{T}_7+\mathcal{T}_3+\mathcal{T}_4)-(2/5)\mathcal{T}_1\,,\\
F^2&=(1/6)(-\mathcal{T}_2-\mathcal{T}_7+\mathcal{T}_3+\mathcal{T}_4)\,,\\
DF&=(1/2\sqrt{5})(\mathcal{T}_2-\mathcal{T}_7-\mathcal{T}_3+\mathcal{T}_4)\,,\\
FD&=(1/2\sqrt{5})(-\mathcal{T}_2+\mathcal{T}_7-\mathcal{T}_3+\mathcal{T}_4)\,,\\
\bar{X}X&=\tfrac{1}{4}(-\mathcal{T}_1-2\mathcal{T}_9+2\mathcal{T}_6)\\
&\qquad+\tfrac{1}{6}(\mathcal{T}_3+\mathcal{T}_4)+\tfrac{1}{3}(\mathcal{T}_2+\mathcal{T}_7)\,,\\
X\bar{X}&=\tfrac{1}{4}(-\mathcal{T}_1-2\mathcal{T}_9+2\mathcal{T}_8)\\
&\qquad+\tfrac{1}{6}(\mathcal{T}_3+\mathcal{T}_4)+\tfrac{1}{3}(\mathcal{T}_2+\mathcal{T}_7)\,,\\
t^2&=(-9/40)\mathcal{T}_1+(1/2)(\mathcal{T}_6+\mathcal{T}_8)\\
&\qquad+(1/5)(\mathcal{T}_2+\mathcal{T}_7+\mathcal{T}_3+\mathcal{T}_4)\,.
\end{aligned}
\tag{A4}
$$

[25] See M. Hamermesh, *Group Theory and Its Applications to Physical Problems* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), Chap. 10; see also D. E. Neville, Phys. Rev. **132**, 844 (1963), where the reduction of the tensorial couplings according to irreducible representations of $SU_3$ are thoroughly treated in terms of the same prescription that we have summarily described above.

The notation $DF$ refers to the coupling of the first two octets, $A$ and $B$, into a $D$-type octet representation and the last two octets, $C$ and $D$, into an $F$-type octet representation. In addition to the pure couplings, $D^2$ and $F^2$, the mixed couplings $DF$ and $FD$ also occur because the symmetric and antisymmetric octets $D$ and $F$ transform similarly under $SU_3$. The inverses of these equations are:

$$
\begin{aligned}
\mathcal{T}_1&=8E^2\,,\\
\mathcal{T}_9&=E^2+D^2-F^2+t^2-(X\bar{X}+\bar{X}X)\,,\\
\mathcal{T}_5&=E^2+D^2+F^2+t^2+(X\bar{X}+\bar{X}X)\,,\\
\mathcal{T}_3+\mathcal{T}_4&=\frac{16}{3}E^2+\frac{5}{3}D^2+3F^2\,,\\
\mathcal{T}_2+\mathcal{T}_7&=\frac{16}{3}E^2+\frac{5}{3}D^2-3F^2\,,\\
\mathcal{T}_6+\mathcal{T}_8&=-\frac{2}{3}E^2-\frac{4}{3}D^2+2t^2\,,\\
\mathcal{T}_3-\mathcal{T}_4&=-(\sqrt{5})(DF+FD)\,,\\
\mathcal{T}_2-\mathcal{T}_7&=(\sqrt{5})(DF-FD)\,,\\
\mathcal{T}_6-\mathcal{T}_8&=-2(X\bar{X}-\bar{X}X)\,.
\end{aligned}
\tag{A5}
$$