Nonlocal Properties of Stable Particles*

GORDON N. FLEMING†
University of Washington, Seattle, Washington
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Some physical implications of the position operators studied in an earlier paper are presented. Lower limits on the dimensions, perpendicular to the momentum, of spinning particles are obtained and the partial localization of massless particles by the center-of-mass operator is discussed. The lowest order correction to a local potential, from a potential which is spherically symmetric in terms of the nonlocal center of mass, is shown to be exactly the Thomas spin-orbit coupling interaction.

1. INTRODUCTION

IN a recent paper,[1] I presented a study of three physically interesting position operators that can be defined for arbitrary massive closed relativistic systems. In this paper the position operators are used to obtain quantitative results concerning the model-independent internal structure of spinning particles implied by the interpretation of the operators advanced in A. In the last section, I indulge in some speculations on the use of the nonlocal center-of-mass operator for inducing nonlocal interactions in a *natural* way.

The position operators in question were introduced some time ago by Pryce and Møller,[2] and the uniqueness of the local position operator was proved by Newton and Wigner[3] and, more recently, discussed at length by Wightman.[4]

The principle goal of A was the explicit statement, in a manifestly covariant formalism, of the transformation properties of the operators (they transform like the spatial parts of appropriately defined four vectors). It has been well known since the original work of Pryce and Møller that the entities $(\mathbf{x}(t),ct)$, where $\mathbf{x}(t)$ is a position operator at the time t, do not transform into one another like a four-vector. This is *one* sense in which the position operators *are not* covariant. However, if the position operators are interpreted as locating some dynamical property of an extended system rather than *the* position of a point particle then there is no reason to require such transformation rules, even in classical relativity. Upon introducing the possible dependence of a position measurement on the space-like hyperplane in which the measurement is carried out (thereby generalizing the definition of the position operator), a four-vector transformation rule can be derived from special relativity and is, in fact, satisfied by the generalized position operators studied in A. In *this* sense the position operators *are* covariant and that is the sense intended here.[5]

The secondary goal of A was the presentation of the view that each of the position operators studied does locate a definite and distinct dynamical property of the system and should be regarded, notwithstanding possible nonlocality,[6,7] as legitimate candidates for the title of *position observable*. The two nonlocal operators were referred to in A as the center of mass and the center of inertia and it was suggested that the local Newton-Wigner operator be regarded as a center of spin.

In Sec. 2 of this paper the position operators are defined anew without recourse to the covariant formalism of A in the hope that the physical interpretation may be made even clearer. The algebraic relations between the operators are presented and used in Sec. 3 to obtain definite lower limits on the transverse dimensions of

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† Address after 1 August 1965: Physics Department, The Pennsylvania State University, University Park, Pennsylvania.

[1] G. N. Fleming, Phys. Rev. **137**, B188 (1965); hereafter to be referred to as A.

[2] M. H. L. Pryce, Proc. Roy. Soc. (London) **A195**, 62 (1948); C. Møller, Commun. Dublin Inst. for Adv. Studies A No. 5 (1949).

[3] T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949). The proof of uniqueness applies only to the case of single stable "elementary" particles, i.e., irreducible representations of the Poincaré group.

[4] A. S. Wightman, Rev. Mod. Phys. **34**, 845 (1962).

[5] There is a *second* sense in which the local Newton-Wigner position operator is said to be noncovariant. Let $|\Phi\rangle$ be a normalizable superposition of position eigenstates, $|\mathbf{x},t=0\rangle$, where $\mathbf{x}$ is confined to some bounded region S. Then the probability density, $|\langle\mathbf{x}',t'|\Phi\rangle|^2/\langle\Phi|\Phi\rangle$, for finding the position eigenvalue to be $\mathbf{x}'$ at the time t' is nonvanishing even when $(\mathbf{x}',ct')$ is space like with respect to every point in S at $t=0$. However, if $|\Phi\rangle$ is now allowed to approach a position eigenstate as a limit, then $\langle\mathbf{x}',t'|\Phi\rangle$ becomes singular on the light cone of the point of localization and finite everywhere else. The square of the norm, $\langle\Phi|\Phi\rangle$, at the same time becomes infinite and a close look at the probability density shows that it *vanishes* everywhere except on the light cone where it yields a total probability of unity. This result appears to conflict with a remark on p. 826 of A. S. Wightman and S. Schweber, Phys. Rev. **98**, 812 (1955) concerning the probability of finding a particle at a space-like interval from its original *point* of localization. The present author prefers to call the aforementioned behavior of normalizable states *noncausal* rather than noncovariant, reserving covariance to describe only the transformation properties of operators and state vectors.

[6] The requirement of locality plays a key role in the investigations of D. G. Currie, T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1962); T. F. Jordan and N. Mukurda, Phys. Rev. **132**, 1842 (1963); and P. M. Mathews and A. Sankaranarayanan, Progr. Theoret. Phys. (Kyoto) **26**, 499 (1961); **27**, 1063 (1962); **32**, 159 (1964). Nonlocal operators have been discussed by M. A. Melvin, Rev. Mod. Phys. **32**, 477 (1960) (Appendix) and recently, in the search for covariance, a concept of localizable state, for which no corresponding Hermitian operator exists, has been introduced by T. O. Philips, Phys. Rev. **136**, B893 (1964).

[7] A. S. Wightman, Ref. 4. This paper contains a careful discussion of the problem of localization at a point as a limiting case of localization in a finite region. It should be noted, however, that the axioms stated there have no solutions in which $E(S)=0$ for some nonempty region, S, no matter how small S may be. $E(S)=0$ is the natural statement of nonlocalizability within S. In this sense the axioms already presume localizability at a point.

massive stable particles. The calculation is entirely independent of any assumptions concerning the nature or even the existence of interactions that the particle may participate in. It is, in fact, the quantum-mechanical analog of the classical calculation by Møller of the minimum extension of a spinning relativistic system.[8] In the case of the electron the lower limit, 2.75×10^{-11} cm, may seem inconsistent with the frequent remark that experimental tests of quantum electrodynamics so far indicate no structure for the electron down to dimensions of the order of the electron Compton wavelength, 3.8×10^{-11} cm. This statement, however, refers to the description of the spatial properties of the electron in terms of the coordinate parameter occurring in the Dirac wave function or the spinor field operator and this parameter has no direct connection with the performance of position measurements on the electron. It is well known that the Foldy-Wouthuysen[9] transformation which takes the Dirac coordinate parameter into the local Newton-Wigner center of spin also transforms the minimal coupling of the Dirac equation into an infinite series of interaction terms corresponding to the interaction of an extended object.

In Sec. 4 the subject of particles with vanishing rest mass is briefly touched on. The nonexistence of a local position operator for most massless particles is known to be due to the absence of a complete set of spin states[10] and the connection of the two circumstances is clearly displayed by the relation between the center of mass, which exists for massless systems, and the local center of spin. The relation involves the raising and lowering operators for helicity and these can exist only in the presence of a complete set of helicity states. The limit to which a massless particle can be localized as a result of the nonlocality of the center of mass is discussed.

Finally in Sec. 5 a first meager step is taken to utilize the position operators in the description of interactions. The language of potential scattering naturally suggests itself and after obtaining general expressions for the velocity operators of interacting systems a simple example of a nonlocal potential is considered. It is shown that if one assumes a potential which is spherically symmetric in terms of the center of mass of the particle, then, upon expanding about the local center of spin (for purposes of familiar comparison), the first order correction to the local spherically symmetric potential is, in the low-energy limit, just the Thomas[11] spin-orbit coupling interaction,

$$\frac{\mathbf{L}\cdot\mathbf{S}}{2m^2c^2}\frac{1}{r}\frac{d}{dr}V(r).$$

[8] C. Møller, Ref. 2.

[9] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[10] The condition of irreducibility restricts the helicity to $\pm S$, where S is the spin of the particle.

[11] L. H. Thomas, Nature **117**, 514 (1926).

2. THE POSITION OPERATORS AND THE SPIN VECTOR

If $\theta_{\mu\nu}(\mathbf{x},t)$ is the symmetric energy-momentum density tensor operator for an arbitrary closed system,[12] then the *center-of-mass position operator* is given by

$$\int d^3x\,\theta_{00}(\mathbf{x},t):\chi_i^M(t)=\int d^3x\,\theta_{00}(\mathbf{x},t)x_i\,, \qquad (2.1)$$

where the double dots on the left side indicate a symmetrized product. The total four-momentum P_μ and the total angular-momentum tensor $M_{\mu\nu}$ are given by

$$P_\mu=\int d^3x\,\theta_{\mu0}(\mathbf{x},t) \qquad (2.2)$$

and

$$M_{\mu\nu}=\int d^3x[x_\mu\theta_{\nu0}(\mathbf{x},t)-x_\nu\theta_{\mu0}(\mathbf{x},t)]. \qquad (2.3)$$

Consequently, (2.1) can be rewritten as

$$P_0:\chi_i^M(t)=M_{i0}+ctP_i\,. \qquad (2.4)$$

The P_μ and $M_{\mu\nu}$, however, are also the Hermitian generators of the Poincaré group with known commutation relations. These commutation relations allow one to solve (2.4) for $\chi_i^M(t)$ and obtain

$$\chi_i^M(t)=ct(P_i/P_0)+M_{i0}:(1/P_0)\,. \qquad (2.5)$$

This position operator is nonlocal. A straightforward but tedious calculation yields

$$[\chi_i^M(t),\chi_j^M(t)]=-(i\hbar/P_0^2)\epsilon_{ijk}\times(S_k^{\|}+(Mc/P_0)S_k^{\perp})\,, \qquad (2.6)$$

where

$$Mc=|(P_\mu P^\mu)^{1/2}|$$

and $\mathbf{S}=\mathbf{S}^{\|}+\mathbf{S}^{\perp}$ will be called the *spin-vector* operator. The name is suggested by the relations

$$S_{ij}=\epsilon_{ijk}(S_k^{\|}+(P_0/Mc)S_k^{\perp})\,, \qquad (2.7a)$$

$$S_{i0}=-S_{ij}P^j/P_0=(\mathbf{P}\times\mathbf{S})_i/Mc\,, \qquad (2.7b)$$

and

$$S_{\mu\nu}=M_{\mu\nu}-M_{\mu\lambda}:(P^\lambda P_\nu/P^2)+M_{\nu\lambda}:(P^\lambda P_\mu/P^2)\,, \qquad (2.7c)$$

which imply

$$[S_i,S_j]=i\hbar\epsilon_{ijk}S_k\,, \qquad (2.8a)$$

$$[S_i,P_j]=0\,, \qquad (2.8b)$$

$$\mathbf{S}\cdot\mathbf{P}=\mathbf{S}^{\|}\cdot\mathbf{P}\,, \qquad (2.8c)$$

and

$$\mathbf{S}^2=\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}=\hbar^2s(s+1)\,, \qquad (2.8d)$$

$$\mathbf{P}\cdot\mathbf{S}/|\mathbf{P}|=\hbar\lambda\,, \qquad (2.8e)$$

for states of definite spin s and helicity λ.

The *spin tensor* $S_{\mu\nu}$ describes the angular momentum

[12] F. J. Belinfante, Physica **6**, 887 (1939).

of the system relative to the center of mass *in the rest frame*. Thus by direct substitution from (2.5) and (2.7c) one can show that

$$(\chi_\mu{}^M(t):P_\nu-\chi_\nu{}^M(t):P_\mu+S_{\mu\nu})|\mathbf{P}=0;\rangle=M_{\mu\nu}|\mathbf{P}=0;\rangle,$$

where $\chi_0{}^M(t)=ct$. This relation does not hold for states of arbitrary momentum and the reason is that the Lorentz transform of the coordinates of the rest-frame center of mass, at a given time, does not yield the coordinates of the center of mass in the new frame, at the transformed time.[13] Because of this difference it is desirable to have a position operator which does describe the coordinates of the Lorentz transform of the rest-frame center of mass, at a given time. The simplest way to introduce such an operator is to require that

$$M_{\mu\nu}=\chi_\mu{}^I(t):P_\nu-\chi_\nu{}^I(t):P_\mu+S_{\mu\nu} \tag{2.9}$$

hold for the operator in any frame. The superscript I indicates that this position operator is to be called the *center of inertia*. The $(i,0)$ component of (2.9) can be solved for $\chi_i{}^I(t)$ and using (2.4) and (2.7b) yields

$$\boldsymbol{\chi}^I(t)=\boldsymbol{\chi}^M(t)-(\mathbf{P}\times\mathbf{S}/McP_0). \tag{2.10}$$

This position operator is also nonlocal. The commutation relations are

$$[\chi_i{}^I(t),\chi_j{}^I(t)]=(i\hbar/M^2c^2)\times\epsilon_{ijk}(S_k{}^{\parallel}+(Mc/P_0)S_k{}^{\perp}). \tag{2.11}$$

Upon introducing the familiar notation,

$$J_i=\tfrac{1}{2}\epsilon_{ijk}M_{jk}, \tag{2.12}$$

and using (2.7a) and (2.9) the relation,

$$\mathbf{J}=\boldsymbol{\chi}^I(t)\times\mathbf{P}+\mathbf{S}^{\parallel}+(P_0/Mc)\mathbf{S}^{\perp}, \tag{2.13a}$$

for the vector angular momentum of the system follows. The point to notice is that the angular momentum relative to the center of inertia is *greater* than the spin $\mathbf{S}$ of the system. If (2.10) is substituted into (2.13a) there results,

$$\mathbf{J}=\boldsymbol{\chi}^M(t)\times\mathbf{P}+\mathbf{S}^{\parallel}+(Mc/P_0)\mathbf{S}^{\perp}, \tag{2.13b}$$

so that the angular momentum relative to the center of mass is *less* than the spin of the system. Consequently, somewhere on the *line* joining the center of inertia to to the center of mass one may expect a *point* such that the angular momentum relative to that point is just $\mathbf{S}$. Such a point may be called the center of spin $\boldsymbol{\chi}^S(t)$, and the corresponding operator would satisfy

$$\mathbf{J}=\boldsymbol{\chi}^S(t)\times\mathbf{P}+\mathbf{S}. \tag{2.13c}$$

In fact, such an operator does exist. It is

$$\boldsymbol{\chi}^S(t)=\boldsymbol{\chi}^I(t)+(\mathbf{P}\times\mathbf{S}/Mc(P_0+Mc)) \tag{2.14}$$

and, remarkably enough, it is local;

$$[\chi_i{}^S(t),\chi_j{}^S(t)]=0. \tag{2.15}$$

The center of spin is just the local position operator discussed by Newton and Wigner and proved unique (as regards its locality) by them. The locality of this position operator has made it the unquestioned favorite of workers desiring to discuss *the position* of relativistic particles. The importance, however, of keeping in mind the distinction between the local center of spin and the center of mass is illustrated clearly upon considering an *incoming* direct product state,

$$|\Phi_1,\Phi_2\,\text{in}\rangle\equiv|\Phi_1\rangle\otimes^{\text{in}}|\Phi_2\rangle,$$

for a system which, in the distant past, is composed of two (widely separated) systems described separately by $|\Phi_1\rangle$ and $|\Phi_2\rangle$ and having no interaction between them. For such a state the generators of the Poincaré group satisfy

$$P_\mu|\Phi_1,\Phi_2\,\text{in}\rangle=(P_\mu|\Phi_1\rangle)\otimes^{\text{in}}|\Phi_2\rangle+|\Phi_1\rangle\otimes^{\text{in}}(P_\mu|\Phi_2\rangle)$$

and

$$M_{\mu\nu}|\Phi_1,\Phi_2\,\text{in}\rangle=(M_{\mu\nu}|\Phi_1\rangle)\otimes^{\text{in}}|\Phi_2\rangle+|\Phi_1\rangle\otimes^{\text{in}}(M_{\mu\nu}|\Phi_2\rangle).$$

Consequently, from (2.4)

$$P_0:\boldsymbol{\chi}^M(t)|\Phi_1,\Phi_2\,\text{in}\rangle=(P_0:\boldsymbol{\chi}^M(t)|\Phi_1\rangle)\otimes^{\text{in}}|\Phi_2\rangle+|\Phi_1\rangle\otimes^{\text{in}}(P_0:\boldsymbol{\chi}^M(t)|\Phi_2\rangle); \tag{2.16}$$

a result which must be required of any sensible notion of the center of mass. The operators $\boldsymbol{\chi}^S(t)$ and $\boldsymbol{\chi}^I(t)$ have no such property.

3. MASSIVE SPINNING PARTICLES

Let $|\Phi_\lambda\rangle$ be a state vector for a single stable particle of mass m, spin s, and helicity λ. Since $\mathbf{P}\cdot\mathbf{S}/|\mathbf{P}|$ is the helicity operator, it follows that, as regards the spin, $\mathbf{P}\times\mathbf{S}$ is a linear combination of the raising and lowering operators for helicity states. Consequently, from (2.10) and (2.14), the expectation values of the three position operators must be equal for helicity eigenstates,

$$\langle\Phi_\lambda|\boldsymbol{\chi}^I(t)|\Phi_\lambda\rangle=\langle\Phi_\lambda|\boldsymbol{\chi}^M(t)|\Phi_\lambda\rangle=\langle\Phi_\lambda|\boldsymbol{\chi}^S(t)|\Phi_\lambda\rangle. \tag{3.1}$$

Physically this result is a consequence of the helicity state's being a linear superposition of all the eigenstates of any given component of $\mathbf{S}$ perpendicular to $\mathbf{P}$. Thus the direction, $\mathbf{P}\times\mathbf{S}$, in which the displacement of the center of mass and the center of spin should occur is indeterminate. If the spin is diagonalized along some fixed axis rather than the momentum, then the expectation value of $\mathbf{P}\times\mathbf{S}$ does not vanish and a net separation of the positions does occur.

While the direction of the displacements is indeterminate for helicity eigenstates, the magnitudes are not. In fact, upon squaring the difference between any two of the position vectors, an operator is obtained which is diagonalized for momentum-helicity eigenstates. For

[13] In other words, the center of mass at a definite time does not lie on the same world line when viewed from different frames. The coupling of spinning and translational motion displaces the center of mass from its position in the rest frame. See Fig. 3 in G. N. Fleming, Ref. 1.

example,

$$[\chi^M(t)-\chi^I(t)]^2=((\mathbf{P}\times\mathbf{S})^2/M^2c^2P_0^2)$$
$$=(\mathbf{P}^2\mathbf{S}^2-(\mathbf{P}\cdot\mathbf{S})^2/M^2c^2P_0^2). \quad (3.2)$$

If one defines $r_{AB}^2=[\chi^A(t)-\chi^B(t)]^2$ then,

$$r_{MI}^2|\mathbf{p},\lambda\rangle=\hbar^2[s(s+1)-\lambda^2](\mathbf{p}^2/m^2c^2p_0^2)|\mathbf{p},\lambda\rangle, \quad (3.3a)$$

$$r_{SI}^2|\mathbf{p},\lambda\rangle=\hbar^2(s(s+1)-\lambda^2)\times(\mathbf{p}^2/m^2c^2(p_0+mc)^2)|\mathbf{p},\lambda\rangle, \quad (3.3b)$$

and

$$r_{MS}^2|\mathbf{p},\lambda\rangle=\hbar^2(s(s+1)-\lambda^2)\times(\mathbf{p}^2/p_0^2(p_0+mc)^2)|\mathbf{p},\lambda\rangle. \quad (3.3c)$$

The following characteristics may be noted:

(i) The separations decrease with increasing helicity squared since large λ^2 means the component of $\mathbf{S}$ perpendicular to $\mathbf{p}$ is small and it is the orthogonal part of the spinning motion which, classically, results in the redistribution of mass.

(ii) r_{MI}^2 and r_{SI}^2 continue to increase with energy to the same limit, $\hbar^2[s(s+1)-\lambda^2]/m^2c^2$, while r_{MS}^2 eventually decreases back to zero after an initial increase.

The values of $r_{MI}(\max)$ for the proton and electron, are

$$\text{proton}\sim 1.50\times 10^{-1}\text{ F},$$
$$\text{electron}\sim 2.75\times 10^{-11}\text{ cm}.$$

These values put an absolute lower limit on the transverse dimensions of the energy distributions associated with these particles when free. In the case of the proton the electromagnetic self-field is not expected to contribute substantially to the proton mass and the value may be regarded as a lower limit on the extension of a nucleon core which is in the center of the mesonic cloud.[14] In the case of the electron the infinite radius of the Coulomb potential obscures the interpretation. This is particularly true if the bare mass of the electron vanishes, as seems to be the case.[15] The discrepancy with the classical electron radius is simply a result of the classical calculation taking no account of the electron spin.

The reader should be careful not to confuse the quantity, r_{MI}^2, with the second moment of the mass distribution relative to the center of inertia or with the square of the rms deviation in the location of the center of mass. A second moment of the mass distribution would be interesting but does not seem to be expressible in terms of the Poincaré generators alone. The quantity r_{MI}^2 is the square of the distance, *in one frame*, between two "points" in the system which are determined by averages over the mass distribution calculated *in two different frames*. From the definitions,

$$r_{AB}{}^2{}_{\text{av}}=\langle\Phi|[\chi^A(t)-\chi^B(t)]^2|\Phi\rangle,$$
$$\Delta\chi_i{}^A=\langle\Phi|[\chi_i{}^A(t)-\chi_{\text{av}}{}^A(t)_i]^2|\Phi\rangle,$$

and

$$\Delta\chi^A=\hat{e}_1\Delta\chi_1{}^A+\hat{e}_2\Delta\chi_2{}^A+\hat{e}_3\Delta\chi_3{}^A, \quad (3.4)$$

where $(A, B=I, M, S)$ and $\hat{e}_i$ are unit vectors along the coordinate axes, one can obtain the inequalities

$$|\Delta\chi^A|\gtrless|(r_{AB}{}^2)_{\text{av}}-(\chi_{\text{av}}{}^A(t)-\chi_{\text{av}}{}^B(t))^2|^{1/2}\pm|\Delta\chi^B|. \quad (3.5)$$

In the case of helicity eigenstates one obtains

$$|\Delta\chi^I|<(\hbar/mc)[s(s+1)-\lambda^2]^{1/2}+|\Delta\chi^S| \quad (3.6a)$$

and

$$|\Delta\chi^M|<(\hbar/mc)[s(s+1)-\lambda^2]^{1/2}\times[\sqrt{5}-1/7+2\sqrt{5}]^{1/2}|\Delta\chi^S|, \quad (3.6b)$$

where

$$\text{Max}[\mathbf{P}^2/P_0^2(P_0+mc)^2]=(m^2c^2)^{-1}[\sqrt{5}-1/7+2\sqrt{5}].$$

The limits (3.6) indicate that even if $\chi^S(t)$ turns out not to be an observable in the sense of the theory of measurement, the operators $\chi^M(t)$ or $\chi^I(t)$ remain reasonable candidates.

4. PARTICLES WITH VANISHING REST MASS

Ever since the appearance of the Newton-Wigner paper on localized states, it has been known that in general a local position operator does not exist for a particle with vanishing rest mass. The exceptions to this are provided by spinless massless particles and spin $\frac{1}{2}$ massless particles possessing both helicity states. Neither type of particle appears to be realized in nature. In the remaining cases it is known that the source of the difficulty in trying to construct a local position operator is the nonexistence of a complete set of helicity states,[4] the helicity being restricted to $\lambda=\pm s$. In terms of the discussion presented in Sec. 2, this situation is easily understood. The results, (2.1) to (2.5), apply unaltered and the center-of-mass position operator exists notwithstanding the unbounded character of P_0^{-1}. To construct the local center of spin from it, however, requires the addition of a term proportional to $\mathbf{P}\times\mathbf{S}$ and this operator, being a combination of helicity raising and lowering operators, requires a complete set of helicity states for its existence.

The center-of-inertia operator also does not exist for massless particles for the more elementary reason that it refers to the rest frame of the particle for its definition. A strong argument, then, for admitting the center-of-mass position operator to candidacy as a position observable, is that without it or something very much like it quantum mechanics seems unable to describe simply the macroscopic localizability of such familiar

[14] A simple classical model of the proton consisting of a spin $\frac{1}{2}$ nucleon core surrounded by a pseudoscalar field results in less of a displacement of the center of mass for the whole system than for the nucleon core alone. In other words

$$r_{\text{MI}}{}^{(\text{proton})}<r_{\text{MI}}{}^{(\text{nucleon core})}.$$

[15] K. Johnson, M. Baker, and R. Willey, Phys. Rev. **136**, B1111 (1964).

things as flashlight beams or the CERN neutrino experiment.

A careful calculation shows that (2.6), with $M=0$, still holds in the massless case[16] where now,

$$\mathbf{S}=\hbar\lambda\mathbf{P}/P_0\,, \tag{4.1}$$

λ being the helicity operator. Therefore, the uncertainty relation

$$\Delta\chi_i{}^M\Delta\chi_j{}^M\geq\tfrac{1}{2}\hbar^2 s\,|\,\epsilon_{ijk}\langle\Phi_\lambda|P_k/P_0{}^3|\Phi_\lambda\rangle| \tag{4.2}$$

applies for helicity eigenstates. For photons ($s=1$) this relation is not useful for anything like a columnated beam ($\Delta p_i\ll p_0$) since the Heisenberg uncertainty relation

$$\Delta\chi_i{}^M\Delta\chi_j{}^M\geq \hbar^2/2\Delta p_i\Delta p_j\gg\hbar^2/2p_0$$

dominates.

5. NONLOCAL CORRECTIONS TO POTENTIAL INTERACTIONS

In the preceding sections the discussion has been restricted to closed systems and free particles. If these considerations are to lead to anything more than qualitative pictures of model-independent particle structure, then some specific statements relating the position operators to the "forces" acting on the particle must be made. In fact, the nonlocal character of the center of mass and the center of inertia suggest that such efforts may lead to natural ways of introducing nonlocal modifications into present-day local field theory.[17] Such modifications cannot be ruled out as possible answers to some of the difficulties in conventional relativistic quantum theory.

In general, the influence of external forces on a system with the symmetric energy-momentum tensor, $\theta_{\mu\nu}(x)$, may be described in terms of the quantity, $\partial^\mu\theta_{\mu\nu}(x)$, which vanishes everywhere only if the system is closed. Assuming the equal time commutation relations of the $\theta_{\mu\nu}(x)$, among themselves, to have the same form as for a closed system,[18] it follows that the algebraic relations between the observables previously introduced do not change. Thus all the equations in Sec. 2 remain valid with the modification that all the operators appearing in them become time dependent. Equations involving derivatives with respect to the time are, of course, changed by the presence of interactions and one of the most interesting of these is the relation between the velocity of the center of mass, $d\chi^M(t)/cdt$, and $\mathbf{P}(t)/P_0(t)$. From (2.1) and (2.2) and liberal use of integration by parts with the neglect of surface integrals at infinity, one can derive the general result,

$$P_0(t)\!:\!\frac{d\chi^M(t)}{cdt}-\mathbf{P}(t)=\int d^3x[\mathbf{x}-\chi^M(t)]\partial^\mu\theta_{\mu 0}(x)\,. \tag{5.1}$$

The free-particle result would remain valid if the particle looked like a "point" to the interaction source,

$$\partial^\mu\theta_{\mu0}(x)\propto\delta^3[\mathbf{x}-\chi^M(t)]$$

or if the interaction were proportional to the energy density,[19]

$$\partial^\mu\theta_{\mu0}(x)=g\theta_{00}(x)\,.$$

The last possibility would yield

$$dP_0(t)/cdt=gP_0(t)$$

with unphysical solutions. In general then, one cannot expect the free-particle expression for the velocity of the center of mass to be valid in the presence of interactions and if the total Hamiltonian $P_0{}^{\mathrm{tot}}$ is written as $P_0{}^{\mathrm{tot}}=P_0(t)+V(t)$ then the Heisenberg equations of motion,

$$i\hbar d\chi^M(t)/cdt=[\chi^M(t),P_0{}^{\mathrm{tot}}]\,, \tag{5.2}$$

requires

$$[\chi^M(t),V(t)]\neq 0\,. \tag{5.3}$$

The equations for the center of inertia and center of spin which are analogous to (5.1) are

$$P_0(t)\!:\!\frac{d\chi^I(t)}{cdt}-\mathbf{P}(t)=\int d^3x[\mathbf{x}-\chi^I(t)]\partial^\mu\theta_{\mu0}(x) -(d/cdt)(\mathbf{P}(t)\times\mathbf{S}(t)/mc) \tag{5.4}$$

and

$$P_0(t)\!:\!\frac{d\chi^S(t)}{cdt}-\mathbf{P}(t)=\int d^3x[\mathbf{x}-\chi^S(t)] \times\partial^\mu{}_{\mu0}(x)-\frac{d}{cdt}\left(\frac{\mathbf{P}(t)\times\mathbf{S}(t)}{P_0(t)+mc}\right) \tag{5.5}$$

and again the free-particle relation is not generally true.

The simplest form of interaction is that due to a static potential which may be expressed in terms of the position operator of the particle. For a relativistic particle, however, the important question arises of which position operator should be used. The conventional approach, in which the position operator is assumed to be local, forces the choice of the center of spin. This immediately results in

$$d\chi^S(t)/cdt=\mathbf{P}(t)/P_0(t)$$

since

$$[\chi^S(t),V(\chi^S(t))]=0$$

[16] The massless case is not completely equivalent to the continuous $m=0$ limit of the massive case because this limit yields a reducible representation from a massive irreducible one. With $m=0$ the square of the helicity operator commutes with all the elements of the Poincaré group and must be a multiple of the identity for an irreducible representation, thereby eliminating some of the helicity states needed to construct **S**. For a discussion of the *limit* $m=0$ see F. Coester, Phys. Rev. **129**, 2816 (1963).

[17] Some recent examples of investigations of nonlocal field theories are B. Schroer, J. Math. Phys. **5**, 1361 (1964) and R. F. Streater, Phys. Rev. **136**, B1748 (1964).

[18] This assumption is based upon the boundary condition of free motion in the distant past or distant future.

[19] Such a relation could only hold in one inertial frame, of course since the two sides of the equation transform differently.

for such a choice, a rather special case in view of (5.5). Furthermore, the physical interpretation of the various position operators which has been presented in this paper and A makes the center of spin seem a rather *ad hoc* choice. From the point of view of one who is trying to provide an approximate static source description of the interaction via a potential, it would seem more reasonable to regard the net force on the particle as behaving as though it were acting at the center of mass. In any case the nonlocality that is introduced into the interaction by such an hypothesis is interesting to consider.

Assume then, that the potential is a spherically symmetric function of the center-of-mass position operator, i.e.

$$V(t)=V(|\chi^M(t)|)=V(r^M(t)). \tag{5.6}$$

The mathematical difficulties of solving a scattering problem, say, with such a nonlocal potential are unfamiliar and it is sensible to expand the potential about the center of spin which is local. Since the difference between the center of mass and the center of spin is small compared to the range of typical interactions for typical particles, it should be a reasonable approximation to retain only the first two terms of the expansion if the energy of the particle is low. The result of the expansion is

$$\begin{aligned} V(r^M) &\simeq V(r^S)+(\chi^M-\chi^S):\nabla^S V(r^S) \\ &= V(r^S)+\frac{\mathbf{P}\times\mathbf{S}}{P_0(P_0+mc)}:\frac{\chi^S}{r^S}\frac{d}{dr^S}V(r^S) \\ &= V(r^S)+\frac{1}{P_0(P_0+mc)}:\mathbf{L}\cdot\mathbf{S}\frac{1}{r^S}\frac{d}{dr^S}V(r^S); \end{aligned} \tag{5.7}$$

and the low-energy limit to order $(mc)^{-4}$ is

$$V(r^M)\simeq V(r^S)+\frac{\mathbf{L}\cdot\mathbf{S}}{2m^2c^2}\frac{1}{r^S}\frac{d}{dr^S}V(r^S) -\frac{3\mathbf{L}\cdot\mathbf{S}}{8m^4c^4}\frac{1}{r^S}\frac{d}{dr^S}V(r^S):\mathbf{P}^2, \tag{5.8}$$

where $\mathbf{L}=\chi^S\times\mathbf{P}$ is the orbital angular momentum of the particle and the time dependence of the operators has been suppressed. The first correction term to a spherically symmetric local potential will be recognized as the spin-orbit coupling that Thomas[11] derived many years ago as a consequence of classical relativity and which appears in the nonrelativistic limit of the Dirac equation for spin $\frac{1}{2}$ particles.[20] Indeed the only indication of a purely quantum effect reminiscent of the nonlocality of the center of mass itself is the symmetrized product appearing in the third term which gives the dominant energy dependence at low energies. It may be noted in passing that if the original potential had been spherically symmetric with respect to the center of inertia rather than the center of mass, then the corresponding expansion would have again yielded the Thomas spin-orbit coupling but with the opposite sign.

The appearance of the Thomas term gives credence to the notion that sensible physical results may be obtained by considering the center of mass operator on the same plane with the local center of spin. On the other hand the essentially classical nature of the expansion [e.g., the Darwin term appearing in the nonrelativistic limit of the Dirac equation and of order $(mc)^{-2}$ does not appear] suggests that consequences of the quantum mechanical nonlocality of χ^M are probably subtle and not too easily displayed. The spin orbit interaction, of course, already displays some nonlocality as a result of the nonlocal properties of **L**. Finally if one is to exploit the extended structure of elementary particles, implied by the position operators, in high-energy phenomena, it is probably necessary to eliminate the restriction to a potential interaction language.

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[20] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., 1955), 2nd ed. pp. 332–333.