Bootstrap and $SU(3)$ Assignment of 2^+ Mesons

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For meson-meson scattering special properties of some submatrices of the total crossing matrix are derived. These properties are applied to the study of the bootstrap of 2^+ mesons in the scattering system of two pseudoscalar octets. We find that self-consistency can be sustained by a 1^- octet, a 2^+ singlet, and a 2^+ octet. We assign $f^0(1250)$ to the 2^+ singlet, and $A_2(1310)$ and $K^*(1415)$ to the 2^+ octet, leaving an $I=0$, 2^+ particle yet to be found. Consequences of this assignment are discussed.

I. INTRODUCTION

FOR some years now the importance of the crossing matrix has been recognized,[1,2] and some of its properties, e.g., its diagonalizability and that its eigenvalues are ± 1, have been found and exploited in the consideration of various dynamical problems.[3,4] However, these properties refer to the total crossing matrix, which oftentimes couples too many states to be of much use in actual calculations for problems where only a subset of all the possible states is involved in the scattering process. It is then of interest to investigate whether some submatrix of the total crossing matrix may possess special properties which can shed some light on these problems. In this paper we find these properties and show that they can effectively be used to facilitate the understanding of scattering problems where only a subset of all possible scattering states is involved.

More specifically, we consider meson-meson scattering where representations of the internal symmetry are either symmetric or antisymmetric under the interchange of the scattering particles. The crossing matrix can then be written in block form as shown in Eq. (2.1). We show on general ground that S and A can be diagonalized separately and that the columns of M form eigenvectors of S.

We then use these properties to study the scattering of two pseudoscalar octets in the $J^P=2^+$ state. Our aim there is to determine which multiplets in the $SU(3)$ symmetry the 2^+ mesons are most likely to form. To this end we adopt a crude dynamical model which preserves the qualitative features of the problem. We find that a singlet and an octet in the 2^+ state together with an octet in the 1^- state bootstrap themselves.

Assigning the experimentally observed[5,6] $f^0(1250)$, $A_2(1310)$, and the newly discovered[7] $K^*(1415)$ to the 2^+ singlet and octet leaves an $I=0$, 2^+ resonance yet to be found. We then consider some of the consequences of this assignment, such as masses and decay widths, and find them to be in good agreement with the experimental results. The assignment of these 2^+ mesons in the $SU(6)$ symmetry is also discussed.

II. THE CROSSING MATRIX

We consider the scattering of meson by meson, where the particles involved all belong to the same representation of some internal symmetry. Our attention is on those problems where the scattering states can be separated into purely symmetric and purely antisymmetric parts. In this section we derive some general properties of the crossing matrices that relate the scattering amplitudes in the three s, t, and u channels of such meson systems.

Since Bose statistics requires that particles in a symmetric (antisymmetric) state can only be in even (odd) angular-momentum states, we can deduce some relations between the crossing matrices by studying the relationship between angular-momentum states of the three channels. Let us start by writing down the scattering angles in terms of the usual scalar variables s, t, and u:

$$\cos\theta_s = 1+2t/(s-4) = -1-2u/(s-4),$$
$$\cos\theta_t = 1+2s/(t-4) = -1-2u/(t-4),$$
$$\cos\theta_u = 1+2t/(u-4) = -1-2s/(u-4),$$

where the masses of all the particles are taken to be unity, and the subscript of the angle θ refers to the channel in which the scattering angle is defined. We note that with the above definition of the angles there exist the following relationships under the interchange of any two variables:

$$t\leftrightarrow u:\quad \cos\theta_s \to -\cos\theta_s,\quad \cos\theta_t \leftrightarrow -\cos\theta_u,$$
$$s\leftrightarrow u:\quad \cos\theta_t \to -\cos\theta_t,\quad \cos\theta_s \leftrightarrow \cos\theta_u,$$
$$s\leftrightarrow t:\quad \cos\theta_u \to -\cos\theta_u,\quad \cos\theta_s \leftrightarrow \cos\theta_t.$$

Associated with the scattering amplitude in the angular momentum state l_i of the ith channel, $i=s$, t, u, is the Legendre polynomial $P_{l_i}(\cos\theta_i)$, which changes in a definite way under the interchange of any two variables, according to the above relationships. If we write the amplitudes in the form $\begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_i$, where the ones in the

* Work supported by the National Science Fundation.
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[3] R. C. Hwa and S. H. Patil, Phys. Rev. **138**, B933 (1965).
[4] P. Babu, Nuovo Cimento **34**, 770 (1964).
[5] W. Selove, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters **9**, 272 (1962).
[6] S. U. Chung, O. I. Dahl, L. M. Hardy, R. I. Hess *et al.*, Phys. Rev. Letters **12**, 621 (1964); M. Aderholz, L. Bondar, W. Brauneck, H. Lengeler *et al.*, Phys. Letters **10**, 226 (1964).
[7] D. Miller (Berkeley group) and British group, communicated by A. Rosenfeld.

symmetric states $\mathcal{S}$ (even l_i) are partitioned from the ones in the antisymmetric states $\mathcal{A}$ (odd l_i), then we have

$$t\leftrightarrow u:\quad \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_s \to \begin{pmatrix}\mathcal{S}\\ -\mathcal{A}\end{pmatrix}_s,\quad \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_t \leftrightarrow \begin{pmatrix}\mathcal{S}\\ -\mathcal{A}\end{pmatrix}_u,$$

$$s\leftrightarrow u:\quad \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_t \to \begin{pmatrix}\mathcal{S}\\ -\mathcal{A}\end{pmatrix}_t,\quad \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_s \leftrightarrow \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_u,$$

$$s\leftrightarrow t:\quad \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_u \to \begin{pmatrix}\mathcal{S}\\ -\mathcal{A}\end{pmatrix}_u,\quad \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_s \leftrightarrow \begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_t.$$

Let the crossing matrix C_{st} be defined as follows:

$$\begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_s = C_{st}\begin{pmatrix}\mathcal{S}\\ \mathcal{A}\end{pmatrix}_t;\quad C_{st}=\begin{pmatrix}S & M\\ N & A\end{pmatrix}. \tag{2.1}$$

Crossing symmetry requires that $C_{st}^2=1$; consequently, we have $C_{st}=C_{ts}$. To get C_{su} from C_{st}, we merely have to interchange t and u, and in view of the above relations among amplitudes under such an interchange, we obtain

$$C_{su}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}C_{st}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}S & -M\\ -N & A\end{pmatrix}. \tag{2.2}$$

It follows immediately that $C_{su}^2=1$ and $C_{su}=C_{us}$. In a similar way we find that

$$C_{tu}=\begin{pmatrix}S & M\\ -N & -A\end{pmatrix},\quad C_{ut}=\begin{pmatrix}S & -M\\ N & -A\end{pmatrix}. \tag{2.3}$$

Note that although $C_{tu}C_{ut}=C_{ut}C_{tu}=1$, C_{tu}^2 and C_{ut}^2 are not equal to unit matrix (unless $M=N=0$, which is unacceptable). This is a feature of the crossing matrix that is not generally recognized.

We now proceed to show several properties of the submatrices S, A, M, and N of the total crossing matrix.

Proposition 1. S and A are separately diagonalizable. From the condition $C_{st}^2=C_{su}^2=1$, we have

$$S^2+MN=1,\quad A^2+NM=1, \tag{2.4a}$$

$$SM+MA=0,\quad AN+NS=0. \tag{2.4b}$$

On the other hand, from $C_{tu}=C_{ts}C_{su}$ we obtain

$$S^2-MN=S,\qquad A^2-NM=-A, \tag{2.5a}$$

$$SM-MA=-M,\quad AN-NS=N. \tag{2.5b}$$

Thus, S and A satisfy the following polynomial equations:

$$2S^2-S-1=0, \tag{2.6a}$$

$$2A^2+A-1=0. \tag{2.6b}$$

Following the procedure given in Ref. 3, it can then easily be established that S and A can separately be diagonalized. The eigenvalues are

$$S:\quad \lambda_s=1,\,-\tfrac{1}{2} \tag{2.7a}$$

$$A:\quad \lambda_a=-1,\,\tfrac{1}{2}. \tag{2.7b}$$

Proposition 2. Column vectors in M are eigenvectors of S with eigenvalue $-\frac{1}{2}$.

This follows immediately from (2.4b) and (2.5b) which give

$$SM=-\tfrac{1}{2}M,\quad AN=\tfrac{1}{2}N. \tag{2.8}$$

Thus we also have

Proposition 2a. Column vectors in N are eigenvectors of A with eigenvalue $\frac{1}{2}$.

In the reduction of the direct product of two regular representations of $SU(n)$, the number of antisymmetric states is never greater than the number of symmetric states. If we denote the dimension of S by p and the dimension of A by q, then we have $p\geq q$. While it is possible that all column vectors of M are linearly independent, in the case $p>q$ not all the column vectors of N can be linearly independent, on account of proposition 2a.

Proposition 3. All the eigenvalues of A are $\frac{1}{2}$ *if and only if all the column vectors in M are linearly independent.*

From (2.4b) and (2.5b) we can also obtain $MA=\frac{1}{2}M$. Clearly, if all the column vectors in M are linearly independent, A must be diagonal and each diagonal element must be $\frac{1}{2}$. Conversely, if all eigenvalues of A are $\frac{1}{2}$, A must already be diagonal itself. From (2.4a) we have $NM=\frac{3}{4}$. Suppose that the kth and lth column of M are linearly dependent, i.e., $M_{il}=cM_{ik}$, $1\leq i\leq p$, $k\neq l$, where c is a constant. Then we have $(NM)_{kl}=c(NM)_{kk}$, a contradiction to NM being diagonal, if $c\neq 0$. If $c=0$, then $(NM)_{ll}=0$, also a contradiction. Generalization of the proof to more than two linearly dependent column vectors in M is straightforward.

From the second equation of (2.8) we can obtain a companion proposition on the row vectors of N.

Proposition 3a. All the eigenvalues of A are $\frac{1}{2}$ *if and only if all the row vectors in N are linearly independent.*

There seems to be no reason on general grounds, i.e., based on the kind of considerations made here, that A must be diagonal with eigenvalue $\frac{1}{2}$ only, although this is, in fact, the case for the crossing matrix[8,9] in $SU(n)$. Presumably one must go into the detailed properties of the crossing matrix to derive this result.

III. BOOTSTRAP IN $SU(3)$

We now apply the results of the preceding section to the problem of the scattering of two pseudoscalar octets in $SU(3)$ symmetry. The decomposition of the direct product is

$$8\times 8=1+8_s+27+8_a+10+10^*,$$

where the first three states on the right-hand side are symmetric under the interchange of the two scattering particles, while the last three are antisymmetric. According to Bose statistics the former can only occur in

[8] D. E. Neville, Phys. Rev. **132**, 844 (1963).

[9] See also the discussion in the beginning of Sec. III.

even-angular-momentum states $0^+, 2^+, 4^+, \cdots$, the latter in odd-angular-momentum states $1^-, 3^-, 5^-, \cdots$.

The crossing matrix C_{su} for octet-octet scattering has previously been calculated by Cutkosky.[10] It is a 7-dimensional matrix because there exists an amplitude Q which couples 8_s to 8_a in meson-baryon scattering.[3] In our present problem of meson-meson scattering, no such transition is possible. Moreover, in the absence of the amplitude Q, the 10 and 10* rows and columns in the crossing matrix are identical, since the two representations are the same by charge-conjugation invariance. Thus, we may delete the 10* row and column, provided that we double the values of the elements in the 10 column.[11] We then obtain

$$C_{st}=\left(\begin{array}{ccc|cc} \frac{1}{8} & 1 & 27/8 & 1 & \frac{5}{2} \\ \frac{1}{8} & -3/10 & 27/40 & \frac{1}{2} & -1 \\ \frac{1}{8} & \frac{1}{5} & 7/40 & -\frac{1}{3} & -\frac{1}{6} \\ \hline \frac{1}{8} & \frac{1}{2} & -9/8 & \frac{1}{2} & 0 \\ \frac{1}{8} & -\frac{2}{5} & -9/40 & 0 & \frac{1}{2} \end{array}\right), \tag{3.1}$$

where the rows and columns correspond to the states labeled by 1, 8_s, 27, 8_a, and 10, in that order. The dashed lines in (3.1) partition the matrix according to (2.1). It can be verified that this matrix satisfies all the properties of the crossing matrix discussed in the preceding section.

Consider now the scattering of two pseudoscalar octets in the $J^P=2^+$ state. The Born-term potential arising from the exchange of a 1^- octet and a 2^+ multiplet is

$$T_s^B(s)=f_1(s)M_{8'}+f_2(s)S\Gamma_s, \tag{3.2}$$

where

$$M_{8'}=\begin{pmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{3} \end{pmatrix}, \quad \Gamma_s=\begin{pmatrix} g_1^2 \\ g_8^2 \\ g_{27}^2 \end{pmatrix},$$

$$S=\begin{pmatrix} \frac{1}{8} & 1 & 27/8 \\ \frac{1}{8} & -3/10 & 27/40 \\ \frac{1}{8} & \frac{1}{5} & 7/40 \end{pmatrix}.$$

Here g_1^2, g_8^2, g_{27}^2 are the coupling constants of the 2^+ multiplets with the pseudoscalar mesons. In writing (3.2), we have assumed that the masses of the particles exchanged in the 2^+ state are approximately equal so that the same functional form $f_2(s)$ may be used for all of the 2^+ multiplets. If there is no particle in a particular multiplet, or if the mass of the particle in that multiplet is much larger than the others, then the corresponding coupling constant should be made essentially zero.

One may now proceed with the Born terms given in (3.2) and calculate the scattering amplitudes. However, our interest here is only in the qualitative features of the problem, e.g., the representations of the particles that can be sustained self-consistently. Thus, if we do not ask for the masses of the particles in our consideration, we may forego the dynamical details and adopt an approximation which preserves the property that the strengths of the forces are reflected in the magnitudes of the coupling constants. The gross features of the problem are then maintained, if we approximate (3.2) by the following relation between the coupling constants, which is consistent with the result obtained by use of the determinantal method,[12]

$$\Gamma_s=\alpha_1 M_{8'}+\alpha_2 S\Gamma_s, \tag{3.3}$$

where α_1 and α_2 are positive constants.

The solution of the above inhomogeneous matrix equation is readily obtained in view of proposition 2 in the preceding section. We have learned that $M_{8'}$ is an eigenvector of S with eigenvalue $-\frac{1}{2}$. Thus if we let $\Gamma_s=\alpha M_{8'}$, (3.3) is satisfied with $\alpha=2\alpha_1(2+\alpha_2)^{-1}$, a positive constant. Because the eigenvalue of S for this vector is $-\frac{1}{2}$, it is not a vector that bootstraps itself. The main force leading to the 2^+ multiplets as given by $\alpha M_{8'}$ is provided by the exchange of the vector meson.

Since S also has an eigenvalue 1, we ask what the corresponding eigenvector is, for it is the bootstrap vector[3] (when $\alpha_2=1$) in the 2^+ state in the absence of the exchanged 1^- meson. We find that the eigenvector is $(1,\frac{1}{5},\frac{1}{5})$. The general solution of the inhomogeneous equation (3.3) is then a linear combination of this homogeneous solution and the particular solution $\alpha M_{8'}$ found above. In both vectors we see that the singlet component is dominant. Thus we conclude that among the 2^+ multiplets the singlet is most favored to exist. The octet is also likely to exist, but because of its weaker coupling to the pseudoscalar mesons its mass is expected to be higher. Here we see that the result is inconsistent with the approximation made concerning the equality of masses of the 2^+ multiplets. However, this is not unexpected. Our main point in this section is to show that the singlet and octet 2^+ mesons are most likely to exist, and this property, we believe, will not be altered by a more refined calculation. These conclusions are similar to the results of Pignotti[12a], and Chan, DeCelles, and Paton[13] but disagree with the assignment of Suzuki.[14]

To complete the bootstrap cycle, we must investigate the production of the 1^- octet from the exchange of the 1^- octet itself and the 2^+ singlet and octet. From (3.1) we see that the inhomogeneous equation corresponding to (3.3) is

$$\begin{bmatrix} g_{8'}^2 \\ g_{10}^2 \end{bmatrix}=\beta_1\begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}+\beta_2\begin{bmatrix} \frac{1}{2} \\ -\frac{2}{5} \end{bmatrix}+\beta_3\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} g_{8'}^2 \\ g_{10}^2 \end{bmatrix}, \tag{3.4}$$

where β_1, β_2, and β_3 are positive constants. The solution is clearly proportional to the inhomogeneous term. If we

[10] R. E. Cutkosky, Ann. Phys. **23**, 415 (1963).
[11] This procedure is analogous to the one used by G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).
[12] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
[12a] A. Pignotti, Phys. Rev. **134**, B630 (1964).
[13] Chan Hong-Mo, P. DeCelles, and J. Paton, Nuovo Cimento **33**, 70 (1964).
[14] M. Suzuki (to be published).

use the qualitative result obtained earlier that g_1^2 is roughly twice g_8^2, we put $\beta_1 \approx 2\beta_2$ and get

$$\begin{bmatrix} g_{8'}^2 \\ g_{10}^2 \end{bmatrix} \approx \beta \begin{bmatrix} 1 \\ -\frac{1}{5} \end{bmatrix}, \tag{3.5}$$

where β is some positive constant. Thus we may conclude from (3.5) that only the octet vector meson is preferred to exist. This completes the bootstrap cycle.

We also note that the forces in the 1^- octet channel due to the exchange of 2^+ singlet and octet are positive, and should therefore help to reduce the width of the ρ meson, say, which is notoriously large in most calculations where the exchange of 2^+ multiplets is not considered.

IV. MASSES AND DECAY WIDTHS

We now examine the experimental situation and find strong indications for the existence of a 2^+ singlet and a 2^+ octet at a higher mass than the singlet, in agreement with the conclusions of the preceding section. We identify the well-known $f^0(1250)$ with the singlet; $A_2(1310)$ and the newly found $K^*(1414)$ can be identified with seven of the components of the octet. In the following we consider some of the predictions of such an assignment.

Using any of the many models, e.g., the quark model, one can relate the masses of the 2^+ octet with the 0^- octet assuming the mass splittings within the octets to be due to mass difference between the doublet and singlet within the fundamental triplet. Then we get

$$m_{K^*(2^+)}^2 - m_{A_2}^2 = m_K^2 - m_\pi^2. \tag{4.1}$$

If we set $m_{A_2} = 1310$ MeV, we find $m_{K^*(2^+)} = 1400$ MeV, which is in good agreement with the experimental value.

The prediction of the mass of $I=0$ component of the octet is complicated by the possibility of mixing with the singlet. If we assume that there is no mixing, then we can write down a mass formula similar to (4.1), but with $K^*(2^+)$ replaced by the $I=0$ component and K by η. In this way we get the mass of the $I=0$ component to be approximately 1415 MeV. On the other hand if there is mixing, and if we use the mixing formula of Schwinger[15] in mass squared

$$(\omega-\rho)(\phi-\rho) = \tfrac{4}{3}(K^*-\rho)(\omega+\phi-2K^*)$$

with the vector particles replaced by the corresponding 2^+ particles, we get the mass of the $I=0$ component to be about 1480 MeV. Thus we see that the mass of the $I=0$ component depends very much on mixing. The experimental situation regarding the $I=0$ component is as yet unclear.

One can also calculate the various decay widths of A_2 and $K^*(2^+)$. For example, if we take $\Gamma_{A_2\to K\bar{K}} = 20$ MeV (the experimental value being 18–30 MeV),[7] we obtain $\Gamma_{K^*(2^+)\to K\pi} \approx 140$ MeV, which is in good agreement with the experimental value of 100–160 MeV. We also find that

$$\frac{\Gamma_{A_2\to\pi\eta}}{\Gamma_{A_2\to K\bar{K}}} \approx 2,$$

and

$$\frac{\Gamma_{K^*(2^+)\to K\pi}}{\Gamma_{K^*(2^+)\to K\eta}} \approx 35.$$

The decay rates of the $I=0$ component depend upon its mixing with the singlet. If there is no mixing, then its mass is about 1415 MeV, and the width of its decay into 2π is about 100 MeV and those for $K\bar{K}$ and $\eta\eta$ about 5 MeV each. This is somewhat larger than the 60-MeV total width found for the $\bar{K}K^*$ resonance[16] at 1410 MeV, which is therefore disfavored as a candidate. If we take the mass of f_0 at 1250 MeV and that of the $I=0$ component of the octet to be at 1415 MeV without mixing but at 1480 MeV with mixing, then we can calculate the mixing parameters according to the procedure of Dashen and Sharp.[17] From this mixing we find that both the decay of f_0 into $K\bar{K}$ and the decay of the $I=0$ component of the octet into 2π can simultaneously be suppressed.

We now look for the $SU(6)$ irreducible representation to which the 2^+ multiplets could belong. Since we have found the 2^+ multiplets as resonance states of two pseudoscalar octets, we expect these 2^+ multiplets to belong to one of the symmetric irreducible representations contained in the reduction of 35×35; they are[18]: 1, 35, 189, and 405. Of these, only 189 and 405 contain 2^+ multiplets. Now 189 contains only (1, 5) and (8, 5), while 405 contains (1, 5), (8, 5), and (27, 5). Since in our model calculation (27, 5) is rather discouraged, we tentatively assign the singlet and octet of 2^+ to the 189 representation.

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[15] J. Schwinger, Phys. Rev. **135**, B816 (1964).

[16] R. Armenteros, *Proceedings of the Sienna International Conference on Elementary Particles, 1963*, edited by G. Bernadini and G. P. Puppi (Società Italiana di Fisica, Bologna, 1963).

[17] R. F. Dashen and D. H. Sharp, Phys. Rev. **133**, B1585 (1964).

[18] M. Bég and V. Singh, Phys. Rev. Letters **13**, 418 (1964).